2 Some Models That Work

By a model is meant a mathematical construct which, with the addition of certain verbal interpretations, describes observed phenomena. The justification of such a mathematical construct is solely and precisely that it is expected to work.

—John von Neumann

Bachelier's Story

In the year 1900, a young polytechnicien named Pierre Bachelier submitted to the University of Paris one of the most remarkable Ph.D. theses ever written in economics. It was the first of two short theses submitted as part of the candidate's degree requirements, and it ran to a mere sixty-six pages.

Bachelier's thesis is entitled "Theory of Speculation," and it begins by proposing a model for the evolution of stock prices over time. Given today's price, what can we say about tomorrow's price? If we describe tomorrow's price as a random variable, then the expected price tomorrow must coincide with today's price (plus a small correction for the one-day rate of interest); for otherwise, it would be profitable either to buy (or to sell) at today's price, thereby making a positive expected profit on the transaction—and so today's price will rise (or fall) until this condition is satisfied. In other words, stock prices from day to day will follow a "random walk," being equally likely to rise or fall in value (relative to the return obtained on a riskless interest-bearing asset). If we model this kind of process, not in discrete time periods, but in continuous time, then we obtain what is nowadays usually referred to as a "Wiener process".[1] Bachelier's first achievement was to develop the mathematics of this process.[2,3] Armed with these results, Bachelier proceeded to ask the question: how can we place a value on a stock option?

A ("call") option on a stock is a financial instrument that permits its owner to exercise the right to buy a unit of stock at some predetermined price at some future date. If the price of the stock rises to a level exceeding this predetermined price, then the owner of the option will exer-

1. In honor of the MIT mathematician, Norbert Wiener, who studied these processes.
2. Bachelier's thesis predates by five years the paper by Einstein, who independently developed the analysis of such a process and applied it to the modeling of Brownian motion (the movement of small particles in a liquid), thus supplying one of the classic demonstrations of the atomicity of matter.
3. The diffusion equation that defines Bachelier's process admits more than one form of solution. The solution that Bachelier obtained is the only one that satisfies some basic differentiability properties and which has finite mean and variance (See below).

cise this right, and will earn a profit equal to the difference between the stock's actual price and the exercise price at which he or she is allowed to buy it. (A "put" option works the other way around, allowing the holder to sell at some predetermined price. These options will be exercised if the stock price is sufficiently low.) In what follows, we simplify by assuming away the payment of dividends during the period when the option is held.

Now, the value of such an option will clearly depend upon the volatility of the stock price. Intuitively, what we are looking at is the weight in the tails of the probability distribution of the stock price at some future date (figure 2.1).

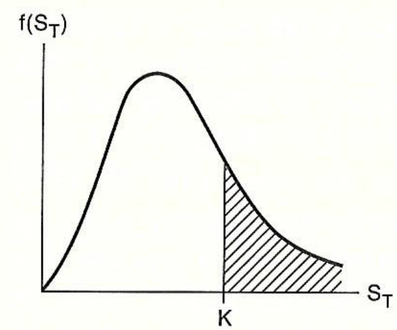


Figure 2.1
The probability density function $f(S_T)$ of the stock price at time $t = T$, given the price at time $t = 0$. The exercise price, or "strike price," labeled K, is the price above which the option will be exercised. Our focus of interest lies in the weight of the tail of the distribution, as indicated by the shaded area.

Bachelier assumed, implicitly, that the agents in the market were risk neutral. The value at time $t = 0$ of a (European call) option, could then simply be expressed

$$C = e^{-rT}E(\max(S_T - K, 0)), \tag{1}$$

where the random variable S_T denotes the stock price at the expiration date $t = T$, K is the exercise price, and r is the rate of interest (discount rate).

Now Bachelier's solution indicated that S_T could be described by a normal distribution whose standard deviation increased proportionally with the square root of T, viz.

$$\sigma = a\sqrt{T}.$$

So, to value the option, subject to this model of the evolution of prices, we need to pin down the value of a. This can be estimated, albeit crudely, by calculating the standard deviation of the "day to day" changes in the stock price over some period in the recent past. By doing this, Bachelier was able to calculate predicted values for options on French government bonds. He was also able to predict the probabilities that these options would be exercised. In both cases, his predicted values conformed closely to the actual prices observed, and to the fraction of options exercised.

Bachelier's thesis did not impress his examiners. The custom was to award a thesis one of two grades (*avec mérite*, and *avec grand mérite*). The higher grade was, in practice,

essential if the candidate was to obtain a university teaching post. Bachelier's passed *avec mérite*, and he made no further contribution to economics. To contemporary economists, the problem was perhaps too far removed from their usual interests. To mathematicians, the analysis was not sufficiently rigorous. Yet Bachelier's thesis anticipated one of the most fruitful literatures in economics by half a century. It was not until the 1960s that economists would turn to these issues and retrace Bachelier's footsteps.

Option Pricing Today

To calculate an option price, we must first decide on some true model that describes the evolution of the stock price; second, we need to estimate a number that measures the degree of volatility in the stock price; and finally, we need a formula that allows us to calculate, given the true model, the price of the option as a function of the estimated volatility parameter.

Modern treatments of the problem model the evolution of the underlying stock price using a lognormal, rather than a normal distribution:[4] writing the current price as S_0, the

4. Consider a random variable X, defined on $0 < X < \infty$, such that $Y = \ln X$ is normally distributed with mean μ and variance σ^2. Then X is lognormally distributed, with parameters μ and σ^2, its distribution function being written $\Lambda(x|\mu, \sigma^2) = \text{Prob}\{X \leq x\}$. The associated density function is given by

$$d\Lambda(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2}(\ln x - \mu)^2\right\} dx \qquad (x > 0). \tag{2}$$

ratio S_T/S_0 is described by a normal distribution with mean μT and variance $\sigma^2 T$.

The centerpiece of the modern theory lies in moving from Bachelier's restrictive setting of risk-neutral agents, where equation (1) holds, to the more general setting in which agents may be risk averse (Black and Scholes 1973; Merton 1973; Cox, Ross, and Rubinstein 1979). A full treatment is beyond the scope of this chapter; for an introductory account, see for example Gemill 1993. The point of departure involves a simple idea; We consider an investor who buys Δ units of stock, and sells 1 unit of the option, where the value of Δ is chosen so that the gain he makes from a rise in the price of the stock is exactly offset by the loss he incurs in his ("short") holding of the option. In other words, he forms a portfolio that yields a constant return, independently of the movement of the stock price. (As p

The j-th moment of the distribution is given by

$$\lambda_j' = \int_0^{\infty} x^j \, d\Lambda(x) = \int_{-\infty}^{+\infty} e^{jy} \, dN(y) = e^{j\mu + \frac{1}{2} j^2 \sigma^2}.$$

In particular, the mean of the distribution equals $\exp(\mu + \frac{1}{2}\sigma^2)$. The j-th moment distribution is defined by

$$\Lambda_j(x|\mu, \sigma^2) = \frac{1}{\lambda_j'} \int_0^x u^j \, d\Lambda(u|\mu, \sigma^2). \tag{3}$$

Two nice properties of the lognormal distribution are, (i) if X is lognormal with parameters μ and σ^2, then $1/X$ is lognormal with parameters $-\mu$ and σ^2, (ii) the j-th moment distribution function can be expressed in terms of the distribution function itself: a direct calculation (Aitcheson and Brown 1966, p. 12) yields

$$\Lambda_j(x|\mu, \sigma^2) = \Lambda(x|\mu + j\sigma^2, \sigma^2).$$

(These formulae hold for all j; the special case where j is a positive integer is of special interest, as the λ_j are then the ordinary moments.)

changes, Δ must be changed too, in order to maintain this property.) This portfolio, being risk free, will command a price that coincides with its expected value—irrespective of investors' attitudes to risk. It turns out that, building on this insight, it is possible to develop the "Black-Scholes" formula, which gives the value of the option as a function of the initial stock price S_0, the exercise price K, and the variance parameter σ, together with the interest rate r and the time to exercise T.

While this formula looks complex at first glance, it is in fact no more than an explicit form of equation (1), where the probability distribution of the stock price at time T is lognormal and the mean return on the stock coincides with the rate of interest r (risk neutrality); see box 2.1. In a risk-neutral world, then, matters are relatively simple. The deep result is that this formula continues to hold good in a setting where agents are risk averse. (For a discussion, see for example Hull 1989.)

How Well Does It Work?

Bachelier's data is illustrated in figure 2.2. It suggests that the basic model that he developed worked remarkably well in predicting option prices for French government securities on the Paris Bourse in the closing years of the nineteenth century.[5] The modern empirical literature,

5. Bachelier's formula defines a relationship between the exercise price K and the price of the option C, for a given date of exercise T. So, for a given T, we could test this by looking at the predicted value of C given K, or at the predicted value of K given C. Bachelier uses the latter procedure (figure 2.2).

Box 2.1
From Bachelier to Black-Scholes

To go from formula (1) of the text to the Black-Scholes formula, we need to invoke a couple of basic properties of the lognormal distribution (footnote 4, properties (i) and (ii)). Denoting by S_T the stock price at time T, K the exercise price, T the time to maturity, and r the discount rate, we have

$$C = e^{-rT}E(\max(S_T - K, 0)). \qquad (1)$$

To ease notation, we normalize by writing the current stock price S_0 as unity, so we measure everything "per \$1 worth of current stock." The random variable $S_T/S_0 \equiv S_T$ is described by a lognormal distribution with parameters μT and $\sigma^2 T$. It follows that the random variable $s = 1/S_T$ is lognormal with parameters $-\mu T$ and $\sigma^2 T$ (property (i)), so (1) can be written as

$$e^{rT}C = E\left(\max\left(\frac{1}{s} - K, 0\right)\right),$$

where s has distribution function $\Lambda(-\mu T, \sigma^2 T)$, whence

$$e^{rT}C = \int_0^{1/K}\left(\frac{1}{s} - K\right) d\Lambda(s\,|-\mu T, \sigma^2 T)$$

$$= \int_0^{1/K}\frac{1}{s}\, d\Lambda(s\,|-\mu T, \sigma^2 T) - K\int_0^{1/K} d\Lambda(s\,|-\mu T, \sigma^2 T)$$

$$= \lambda'_{-1}\Lambda_{-1}\left(\frac{1}{K}\,\middle|\,-\mu T, \sigma^2 T\right) - K\Lambda\left(\frac{1}{K}\,\middle|\,-\mu T, \sigma^2 T\right),$$

where $\lambda'_{-1} = e^{\mu T + \frac{1}{2}\sigma^2 T}$. Under risk neutrality, the mean return $\mu + \frac{1}{2}\sigma^2$ must coincide with the rate of interest r, and invoking property (ii), footnote 4, this becomes

Box 2.1 (continued)

$$= e^{rT}\Lambda\left(\frac{1}{K}\,\middle|\,-\mu T - \sigma^2 T, \sigma^2 T\right) - K\Lambda\left(\frac{1}{K}\,\middle|\,-\mu T, \sigma^2 T\right)$$

$$= e^{rT}N\left(\ell n\frac{1}{K}\,\middle|\,-\mu T - \sigma^2 T, \sigma^2 T\right) - KN\left(\ell n\frac{1}{K}\,\middle|\,-\mu T, \sigma^2 T\right)$$

$$= e^{rT}N\left(\frac{\ell n\frac{1}{K} + \mu T + \sigma^2 T}{\sigma\sqrt{T}}\,\middle|\,0,1\right) - KN\left(\frac{\ell n\frac{1}{K} + \mu T}{\sigma\sqrt{T}}\,\middle|\,0,1\right).$$

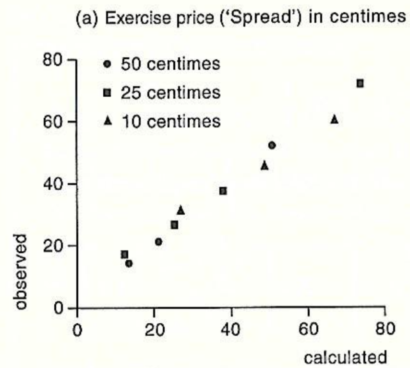
Writing the standard normal distribution $N(x|0,1)$ as $N(x)$ to ease notation, multiplying across by e^{-rT} and once again invoking the "risk neutrality" equation, we have

$$C = N\left(\frac{\ell n\frac{1}{K} + r + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - Ke^{-rT}N\left(\frac{\ell n\frac{1}{K} + rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right).$$

This coincides with the Black-Scholes theorem, when the initial price S_0 is written as unity.

which is based on the Black-Scholes model, has focused heavily on problems surrounding the measurement of σ. The easiest approach here is the one followed by Bachelier, which involves estimating the standard deviations of period-to-period changes in the stock price over some recent period (20 weeks, say).[6] An alternative method can be used in cases where there are several different options (of different durations, say) on each stock. Here we can use the observed price of one option to infer a value of

6. See, for example, Black and Scholes 1973, Finnerty 1978.



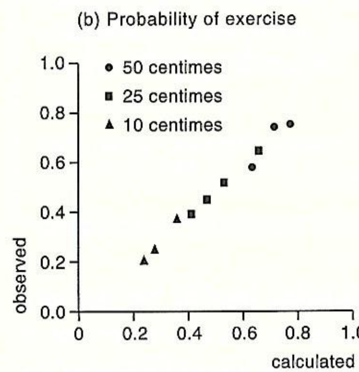


Figure 2.2
Bachelier's results. Panel (a) shows the observed exercise price versus the predicted exercise price (each measured relative to the futures price) for options of different prices (50, 25, and 10 centimes). Panel (b) relates to the predicted probability of exercise.

σ, and then use this to predict the values of the other options.[7] Studies that use such indirect methods to infer a value of σ generally achieve somewhat better predictions of option prices.

A deeper issue surrounds the validity of the underlying model of stock price movements. It is generally agreed that the conventional model, according to which the (log) stock price T periods hence is described by a normally distributed random variable with mean 0 and standard deviation $\sigma\sqrt{T}$ provides a good approximation to empirical data. It is also agreed, however, that it fails systematically in one respect: the frequency with which very large (positive or negative) changes are observed is somewhat greater than that predicted on the basis of this conventional model. The effect of placing greater weight in the tails of the distribution is that the Black-Scholes formula will slightly undervalue call options whose exercise price is far above the current price.[8] Practitioners make a small adjustment for this on a rule of thumb basis. For the theorist, however, it poses an important potential challenge to the conventional model.

7. A more usual procedure is to combine information on all options to obtain a "best estimate" of σ, and then apply this to obtain estimates of individual option prices. See, for example, Latané and Rendleman 1976.

8. It will also undervalue put options whose exercise price is far below the current price. The empirical evidence presented by Black and Scholes 1973 suggests however, that the model tends to overprice (resp. underprice) options with high (resp. low) standard deviation estimates, and they suggest that this may be attributable to an error-in-variables problem (Whaley 1982, p. 33).