

**QR43, Introduction to Investments**  
**Class Notes, Fall 2003**  
**IV. Portfolio Choice**

**A. Mean-Variance Analysis**

**1. The variance of a portfolio.**

Consider the choice between two risky assets with returns  $R_1$  and  $R_2$ . Form a portfolio with weights  $w_1$  and  $w_2 = (1 - w_1)$ .

$$R_p = w_1R_1 + w_2R_2.$$

$$\begin{aligned}\sigma_p^2 &= \text{Var}(w_1R_1 + w_2R_2) \\ &= \text{Var}(w_1R_1) + \text{Var}(w_2R_2) + 2\text{Cov}(w_1R_1, w_2R_2) \\ &= w_1^2\text{Var}(R_1) + w_2^2\text{Var}(R_2) + 2w_1w_2\text{Cov}(R_1, R_2) \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} \\ &= w_1^2\sigma_1^2 + (1 - w_1)^2\sigma_2^2 + 2w_1(1 - w_1)\sigma_{12}.\end{aligned}$$

The variance of the portfolio return is a quadratic function of  $w_1$ . A plot of variance against the portfolio weight is a parabola; a plot of standard deviation against the portfolio weight is a hyperbola.

Recall that  $\sigma_{12} = \rho_{12}\sigma_1\sigma_2$ , where  $\rho_{12} \equiv \text{Corr}(R_1, R_2)$ . Thus

$$\sigma_p^2 = w_1^2\sigma_1^2 + (1 - w_1)^2\sigma_2^2 + 2w_1(1 - w_1)\sigma_1\sigma_2\rho_{12}.$$

Special cases:

1. When  $\rho_{12} = 1$ , the largest possible correlation,

$$\sigma_p^2 = (w_1\sigma_1 + (1 - w_1)\sigma_2)^2 .$$

In this case,  $\sigma_p^2$  can be set to zero by choosing  $w_1 = -\sigma_2/(\sigma_1 - \sigma_2)$ .

2. When  $\rho_{12} = -1$ , the smallest possible correlation,

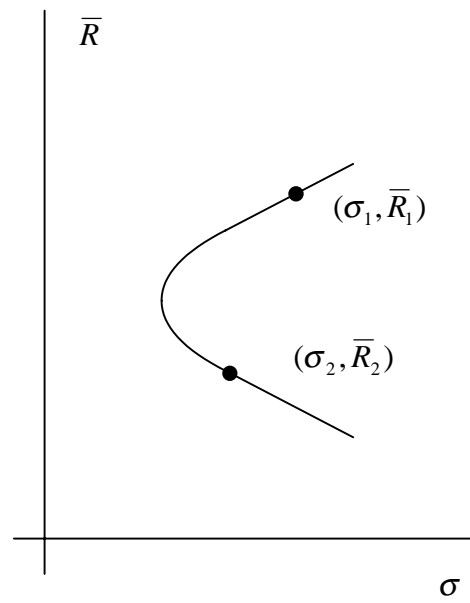
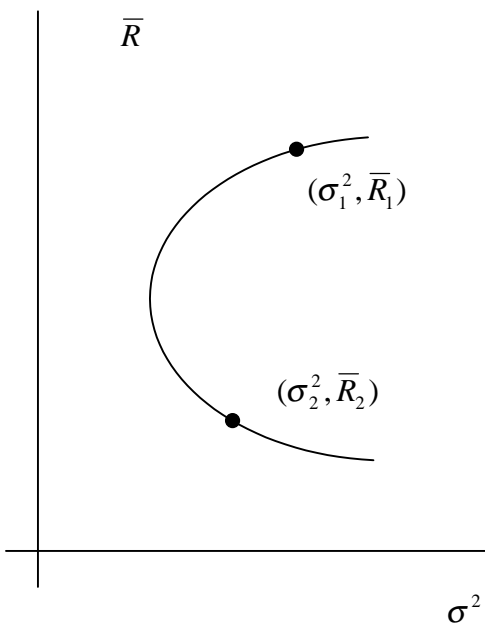
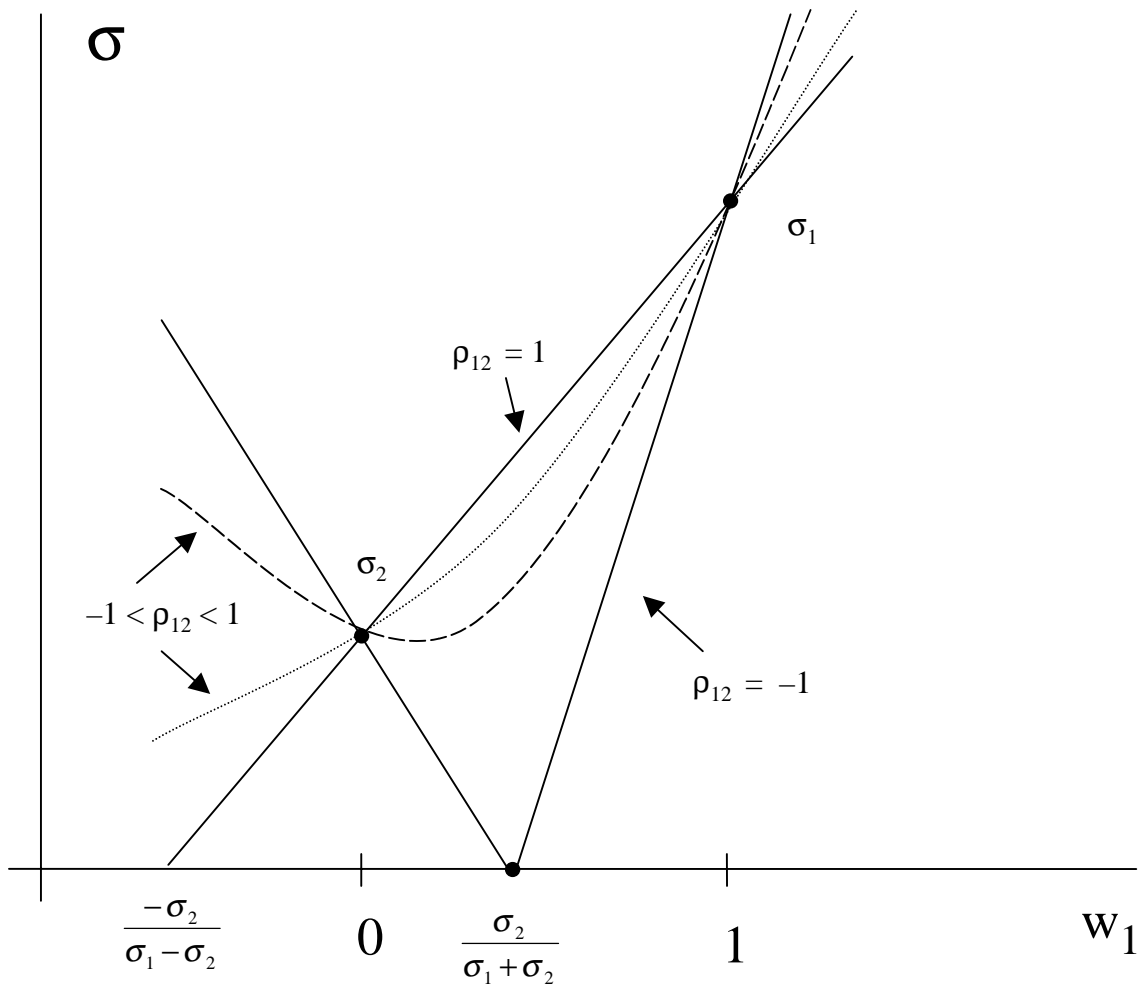
$$\sigma_p^2 = (w_1\sigma_1 - (1 - w_1)\sigma_2)^2 ,$$

which can be set to zero by choosing  $w_1 = \sigma_2/(\sigma_1 + \sigma_2)$ .

In between the extreme values,  $\sigma_p^2 > 0$  for  $-1 < \rho_{12} < 1$ . Since 1 is the largest possible correlation,

$$\sigma_p^2 \leq (w_1\sigma_1 + (1 - w_1)\sigma_2)^2 \text{ if } 0 < w_1 < 1 .$$

3. When  $\sigma_1^2 = \sigma_2^2$ , then the minimum variance is achieved by setting  $w_1 = w_2 = 1/2$ .



## 2. Trading off mean and variance.

Suppose you judge a portfolio by the mean and variance (or equivalently, standard deviation) of its return. What combinations of mean and variance (or standard deviation) can you get?

$$\bar{R}_p = w_1 \bar{R}_1 + (1 - w_1) \bar{R}_2.$$

$$\bar{R}_p = w_1 \bar{R}_1 + (1 - w_1) \bar{R}_2 = \bar{R}_2 + w_1 (\bar{R}_1 - \bar{R}_2).$$

The mean portfolio return is a linear function of  $w_1$ . Hence a target mean return determines  $w_1$ . Given a target  $\bar{R}_p$ ,

$$w_1 = \frac{\bar{R}_p - \bar{R}_2}{\bar{R}_1 - \bar{R}_2}.$$

But we already know that the variance of the portfolio return is a quadratic function of  $w_1$ . Hence it is also a quadratic function of the mean portfolio return  $\bar{R}_p$ . A plot of mean against variance is a parabola; a plot of mean against standard deviation is a hyperbola.

### 3. What if one asset is riskless?

In this case  $\sigma_2^2 = 0$  and  $R_2 = R_f$ , the riskless interest rate. We have

$$R_p - R_f = w_1(R_1 - R_f).$$

$$\bar{R}_p - R_f = w_1(\bar{R}_1 - R_f).$$

$$\sigma_p^2 = w_1^2 \sigma_1^2.$$

$$\sigma_p = w_1 \sigma_1.$$

$$w_1 = \frac{\sigma_p}{\sigma_1}.$$

$$\bar{R}_p - R_f = \sigma_p \left( \frac{\bar{R}_1 - R_f}{\sigma_1} \right).$$

This defines a straight line, called the *capital allocation line* (CAL), on a mean-standard deviation diagram. The slope

$$S_1 = \left( \frac{\bar{R}_1 - R_f}{\sigma_1} \right)$$

is called the *reward-to-variability ratio* or *Sharpe ratio* of the risky asset. Any portfolio that combines a single risky asset with the riskless asset has the same Sharpe ratio as the risky asset itself.

#### 4. How much risk should you take?

If there is a riskless asset and a single risky asset, then the problem can be written as

$$\begin{aligned} \text{Max} \quad & \bar{R}_p - \frac{1}{2}A\sigma_p^2 \\ & = R_f + w_1(\bar{R}_1 - R_f) - \frac{1}{2}Aw_1^2\sigma_1^2, \end{aligned}$$

where  $A$  is the coefficient of risk aversion. The solution is

$$w_1 = \frac{\bar{R}_1 - R_f}{A\sigma_1^2} = \frac{S_1}{A\sigma_1}.$$

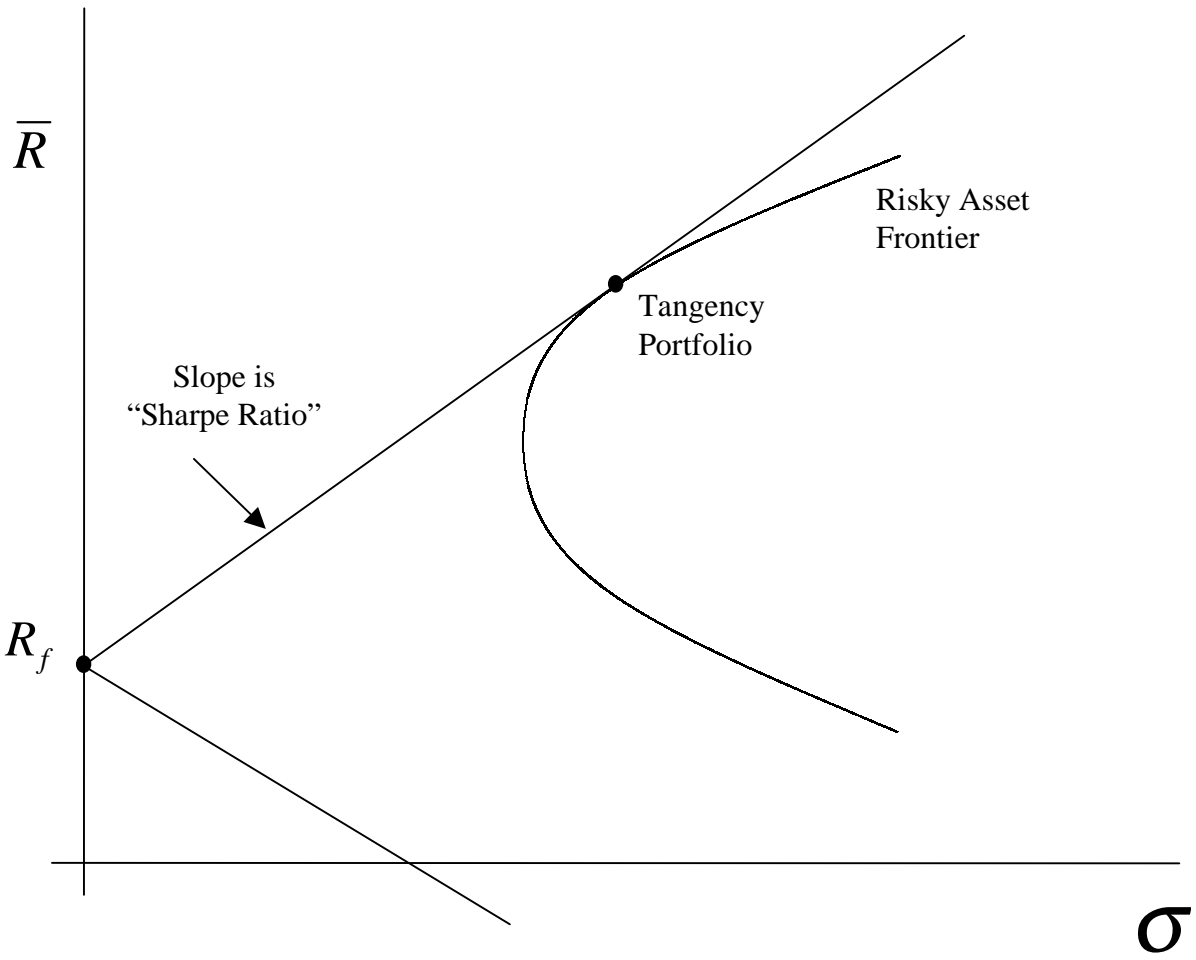
The optimal share in the risky asset is the risk premium divided by risk aversion times variance, or equivalently the Sharpe ratio of the risky asset divided by risk aversion times standard deviation.

## 5. What if there are many risky assets?

In the many-asset case:

- Not all portfolios minimize variance for a given mean. Portfolios that do minimize variance for a given mean are called *minimum-variance* portfolios and we say they lie on the *minimum-variance frontier*.
- The upper portion of the frontier also maximizes mean for a given variance. We call this the *mean-variance efficient frontier*. All investors who like mean and dislike variance should hold *mean-variance efficient* portfolios on this frontier.
- All minimum-variance portfolios can be obtained by mixing just two minimum-variance portfolios in different proportions.
- Thus if all investors hold minimum-variance portfolios, all investors hold combinations of just two underlying portfolios or “mutual funds”. This is the *mutual fund theorem* of mean-variance analysis.

The mutual fund theorem is particularly easy to understand in the case where we have a riskless asset. By combining the riskless asset with any risky asset, we can travel along a straight line on a plot of mean against standard deviation. The best available straight line is a tangency line from the riskless asset return on the vertical axis, to the minimum-variance set obtainable if one can only hold the risky assets and not the riskless asset. The slope of this line is the Sharpe ratio of the tangency portfolio. The tangency portfolio has the highest Sharpe ratio of any risky asset or portfolio of risky assets.





## 6. Diversification with many symmetrical assets.

Consider  $N$  assets that all have the same variance  $\sigma^2$  and correlation  $\rho$ . Then each asset has the same covariance  $\rho\sigma^2$  with each other asset. Since all the assets are the same, an equally-weighted portfolio has the lowest possible variance.

$$\begin{aligned} \text{Var}\left(\frac{1}{N}R_1 + \dots + \frac{1}{N}R_N\right) &= \left(\frac{1}{N^2}\right) \left( \sigma_1^2 + \sigma_{12} + \dots + \sigma_{1N} \right. \\ &\quad \left. + \sigma_{21} + \sigma_2^2 + \dots + \sigma_{2N} \right. \\ &\quad \left. \dots \right. \\ &\quad \left. + \sigma_{N1} + \sigma_{N2} + \dots + \sigma_N^2 \right) \\ &= \left(\frac{N}{N^2}\right) \sigma^2 + \left(\frac{N^2 - N}{N^2}\right) \rho\sigma^2 \\ &= \frac{1}{N}\sigma^2 + \left(1 - \frac{1}{N}\right) \rho\sigma^2. \end{aligned}$$

This is a weighted average of  $\sigma^2$  and  $\rho\sigma^2$ . As the number of assets  $N$  increases, the variance of the portfolio approaches  $\rho\sigma^2$ . By diversifying from one asset with variance  $\sigma^2$  to an equally-weighted portfolio with many assets, you can reduce variance by a factor  $(1 - \rho)$ .

- In the US stock market in recent decades,  $\sigma^2$  for individual stocks has risen,  $\rho$  across individual stocks has fallen, and the variance of the market index  $\rho\sigma^2$  has remained roughly constant. Thus the benefit of diversification has increased.
- In the global market in recent decades,  $\rho$  across individual countries has increased. Thus the benefit of international diversification has decreased (although it is still substantial).

## 7. Understanding diversification

It is important to understand how diversification does (and does not) work. *Averaging* bets reduces variance relative to mean, but *adding* them does not.

For simplicity, consider independent bets  $A$  and  $B$ , each with mean  $\mu$  and variance  $\sigma^2$ .

$$E\left[\frac{1}{2}A + \frac{1}{2}B\right] = \mu, \quad \text{Var}\left[\frac{1}{2}A + \frac{1}{2}B\right] = \frac{\sigma^2}{2}.$$

$$E[A + B] = 2\mu, \quad \text{Var}[A + B] = 2\sigma^2.$$

Related points:

- Insurance companies do two things: they add many risks, and then divide the risks among many people. The latter step is essential to gain the benefits of diversification.
- People sometimes talk about “time diversification”. But investing over a long horizon involves adding returns in successive periods rather than averaging them. Thus, if returns in successive periods are independent of each other, there is no such thing as time diversification. Any argument that long-horizon investment is less risky than short-horizon investment must rely on mean-reversion, the tendency for unusually high returns to be followed by unusually low returns and vice versa.

## B. The Capital Asset Pricing Model (CAPM)

### 1. Difficulties with mean-variance analysis.

There are several difficulties with the practical implementation of mean-variance analysis.

- Estimates of means are imprecise over short periods.
- Means may not be constant over long periods.
- To find the optimal portfolio of  $N$  assets you have to estimate  $N(N + 1)/2$  variances and covariances. This can be a very large number!
- With a large number of assets, historical data will tend to suggest that some combinations of assets are close to riskless. (In fact, if you have more assets than time periods in your historical dataset, the data will imply that some combinations of assets are literally riskless and offer an arbitrage opportunity.) This can lead to a highly leveraged portfolio.

These difficulties have motivated a search for shortcut methods to find optimal portfolios. The leading shortcut is the *Capital Asset Pricing Model* (CAPM).

The CAPM also has broader implications:

- Testable restrictions on asset returns.
- Capital budgeting (what discount rate to use in evaluating investment projects).
- Mutual fund performance evaluation (how large a return should one expect given the risk that a fund manager is taking).

## 2. Derivation of the CAPM.

Assumptions of the CAPM:

- All investors are price-takers.
- All investors care about returns measured over one period.
- There are no nontraded assets.
- Investors can borrow or lend at a given riskfree interest rate (this assumption can be relaxed).
- Investors pay no taxes or transaction costs.
- All investors are mean-variance optimizers.
- All investors perceive the same means, variances, and covariances for returns.

These assumptions imply that:

- All investors work with the same mean-standard deviation diagram.
- All investors hold a mean-variance efficient portfolio.
- Since all mean-variance efficient portfolios combine the riskless asset with a fixed portfolio of risky assets, all investors hold risky assets in the same proportions to one another.
- These proportions must be those of the *market portfolio* or *value-weighted index* that contains all risky assets in proportion to their market value.
- Thus *the market portfolio is mean-variance efficient*.

The implication for portfolio choice is that a mean-variance investor need not actually perform the mean-variance analysis! The investor can “free-ride” on the analyses of other investors, and use the market portfolio (in practice, a broad index fund) as the optimal mutual fund of risky assets (tangency portfolio). The optimal capital allocation line (CAL) is just the *capital market line* (CML) connecting the riskfree asset to the market portfolio.

### 3. What does the CAPM imply about asset pricing?

Suppose you start with an arbitrary portfolio  $p$ . You consider a new portfolio that adds weight  $\theta_i$  to asset  $i$  and finances it by subtracting weight  $\theta_i$  from the riskless asset. Assume that  $\theta_i$  is small, so you are considering only a small portfolio shift. What happens to the variance of the portfolio?

$$\begin{aligned}\text{Var}(R_p + \theta_i(R_i - R_f)) &= \sigma_p^2 + \theta_i^2 \sigma_i^2 + 2\theta_i \sigma_{ip} \\ &\approx \sigma_p^2 + 2\theta_i \sigma_{ip},\end{aligned}$$

since  $\theta_i^2$  is negligibly small relative to  $\theta_i$  when  $\theta_i$  is small. (This is the basic idea of calculus.) The change in variance is approximately  $2\theta_i \sigma_{ip}$ .

What happens to the mean excess return on the portfolio over the riskless rate?

$$E[R_p + \theta_i(R_i - R_f)] = \bar{R}_p + \theta_i(\bar{R}_i - R_f),$$

so the change in mean is  $\theta_i(\bar{R}_i - R_f)$ .



Now consider two simultaneous portfolio shifts, into asset  $i$  and into asset  $j$ . The effect on variance is  $2\theta_i\sigma_{ip} + 2\theta_j\sigma_{jp}$ . The effect on mean is  $\theta_i(\bar{R}_i - R_f) + \theta_j(\bar{R}_j - R_f)$ .

Given  $\theta_i$ , we can choose  $\theta_j$  so that the variance of the portfolio does not change:  $2\theta_i\sigma_{ip} + 2\theta_j\sigma_{jp} = 0$ . This requires that

$$\theta_j = -\frac{\theta_i\sigma_{ip}}{\sigma_{jp}}.$$

Then the effect of the portfolio shifts on the mean return is

$$\begin{aligned} & \theta_i(\bar{R}_i - R_f) - \frac{\theta_i\sigma_{ip}}{\sigma_{jp}}(\bar{R}_j - R_f) \\ &= \theta_i\sigma_{ip} \left[ \frac{\bar{R}_i - R_f}{\sigma_{ip}} - \frac{\bar{R}_j - R_f}{\sigma_{jp}} \right]. \end{aligned}$$

If the original portfolio  $p$  is mean-variance efficient, then it is impossible to increase the mean while keeping the variance constant. But this requires that

$$\frac{\bar{R}_i - R_f}{\sigma_{ip}} = \frac{\bar{R}_j - R_f}{\sigma_{jp}}.$$

If not, you could choose a positive or negative  $\theta_i$  to increase the mean portfolio return while leaving the variance unchanged.

Thus we know that for an efficient portfolio  $p$ , we must have

$$\frac{\bar{R}_i - R_f}{\text{Cov}(R_i, R_p)} = \frac{\bar{R}_j - R_f}{\text{Cov}(R_j, R_p)}.$$

This equation must hold for all assets  $j$ , including the original portfolio itself. Setting  $j = p$ , and noting that  $\text{Cov}(R_p, R_p) = \text{Var}(R_p)$ , we get

$$\frac{\bar{R}_i - R_f}{\text{Cov}(R_i, R_p)} = \frac{\bar{R}_p - R_f}{\text{Var}(R_p)},$$

$$\bar{R}_i - R_f = \frac{\text{Cov}(R_i, R_p)}{\text{Var}(R_p)}(\bar{R}_p - R_f) = \beta_{ip}(\bar{R}_p - R_f),$$

where  $\beta_{ip} \equiv \text{Cov}(R_i, R_p)/\text{Var}(R_p)$  is the regression coefficient of asset  $i$  on portfolio  $p$ .

Finally, we recall that according to the CAPM, the market portfolio  $m$  is efficient. Therefore this equation describes the market portfolio:

$$\bar{R}_i - R_f = \beta_{im}(\bar{R}_m - R_f),$$

where  $\beta_{im} \equiv \text{Cov}(R_i, R_m)/\text{Var}(R_m)$  is the regression coefficient of asset  $i$  on the market portfolio  $m$ .

In other words, if we consider the regression of excess returns on the market excess return,

$$\bar{R}_i - R_f = \alpha_i + \beta_{im}(\bar{R}_m - R_f) ,$$

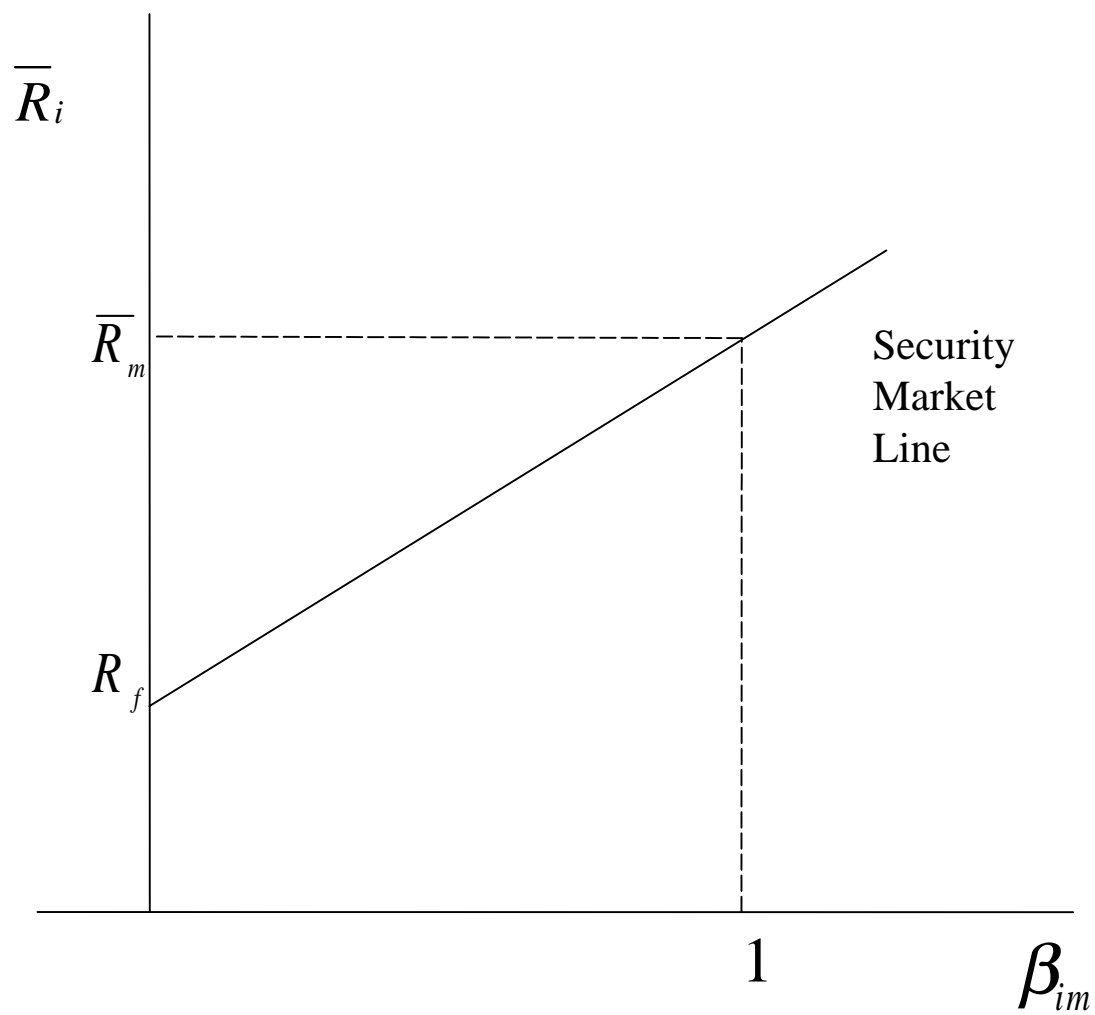
where  $\alpha_i \equiv \bar{R}_i - R_f - \beta_{im}(\bar{R}_m - R_f)$ ,  $\alpha_i$  should be zero for all assets.  $\alpha_i$  is sometimes called *Jensen's alpha*, *alpha*, or *risk-adjusted excess return*. The relationship

$$\bar{R}_i = R_f + \beta_{im}(\bar{R}_m - R_f)$$

is called the *security market line (SML)*, and  $\alpha_i$  measures deviations from this line.

Uses of alpha:

- Test the CAPM. Are there assets with measured alphas that are too far from zero to be the result of chance?
- Evaluate mutual fund managers. Does a mutual fund deliver risk-adjusted excess return?



#### 4. Empirical tests of the CAPM.

- At the broadest level, asset class returns are related to their cyclicalities. Asset classes that move with the overall level of wealth and economic prosperity, such as stocks, tend to have higher average returns than asset classes that do well in bad times, such as gold.
- Early work looked at stocks grouped by their estimated betas. High-beta stocks do tend to have higher returns than low-beta stocks, but the relationship is rather weak.
- Small stocks tend to have higher returns than large stocks. Although small stocks also have higher betas, the return difference is too high to be consistent with the CAPM.
- In fact, the size effect seems to explain a large part of the high average returns of high-beta stocks. If you group stocks by both size and beta, size predicts returns and beta does not.
- Stocks with high book-market, dividend-price, or earnings-price ratios (“value” stocks) also have high returns relative to the CAPM prediction.
- Stocks that have done well over the past year (“momentum” stocks) also have high returns relative to the CAPM prediction.
- Initial public offerings (IPO’s) do poorly relative to the CAPM prediction. This may be another manifestation of the value effect, since IPO’s tend to have extremely low book-market ratios.

Alternative reactions to these results:

- The anomalies are just the result of *data-snooping* or *data-mining*. If we test for many possible anomalies, some of them will always appear to be important in historical data, just as the result of chance.
- The CAPM is not testable because we do not know the composition of the true market portfolio, and so we use a broad stock index as an imperfect proxy. A rejection of the CAPM just tells us that our proxy is inadequate. (This is called the *Roll critique*.)
- The anomalies are concentrated in small, illiquid stocks that may have high average returns to compensate for their illiquidity.
- We need a more general model of risk and return, for example one that distinguishes between risks perceived by long-term investors and risks perceived by short-term investors.
- Markets are inefficient so no model of risk and return will fully resolve the anomalies.