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Credible Monetary Policy in an Infinite Horizon Model: Recursive Approaches

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This paper develops recursive methods to study optimal and time consistent policy in dynamic models. We analyze a version of Calvo's 1978 monetary model and show that its time consistent outcomes can be completely characterized as the largest fixed point of either of two operators. Recursive application of these operators provides a computing algorithm which always converges to the set of time consistent outcomes. Finally, we obtain valuable information about the nature of time consistent outcomes: It is discovered, in particular, that all such outcomes are Markovian. The methods obtained are intuitive and should be useful for many applications. *Journal of Economic Literature* Classification Numbers: E61; E52; C61.

1. INTRODUCTION

Studies of macroeconomic policy in models of long lived agents are of utmost importance for both theoretical and practical reasons. Key examples are the taxation of capital and labor in an infinite horizon growth model[1] and the optimal conduct of monetary policy in a Sidrauski or cash-in-advance framework.[2] These models are considerably complex, partly because they typically involve solving for infinite horizon competitive equilibria for each of a (sometimes large) set of government policies. Progress has been achieved, by and large, by assuming that the government can commit at the beginning of time to a policy specifying its actions for all current and future dates and states of nature. With this assumption, impressive advances have been made recently, in particular, in characterizing optimal macroeconomic policy.[3]

However, the significance of the results thus obtained is unclear if governments cannot commit to date-state contingent policies. If instead governments are assumed to choose policies sequentially, optimal policies

[1] The extensive literature examining this problem includes [5, 7, 14, 17].

[2] See in particular [4, 8, 13, 26].

[3] In particular, see [8].

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under commitment may be time inconsistent, as first pointed out by Kydland and Prescott [15] and Calvo [4]. As a consequence, it would seem urgent to check whether policies derived under the assumption of perfect commitment are time consistent and, more generally, to characterize the set of time consistent outcomes. But this goal has proven to be very elusive in models with long-lived agents, presumably due to the difficulty of the issues involved.

This paper suggests a way to deal with all these issues that can be applied to a wide class of models with long lived agents, including the capital–labor taxation and the optimal money supply problems mentioned at the beginning. Its key insight is that one may completely characterize the set of all time consistent outcomes in a recursive fashion. To develop this idea the paper analyzes a version of Calvo's [4] model. In that context we show that the set of all time consistent outcomes is the fixed point of two related but different operators, inspired by the work of Abreu, Pearce, and Stachetti [2], Cronshaw and Luenberger [10], and Kydland and Prescott [16]. The approach in the paper is in the spirit of dynamic programming [3] and yields valuable insights about the time consistency problem; it is discovered, for example, that all time consistent outcomes have a Markovian structure. In addition, the approach yields algorithms that always converge to the set of time consistent outcomes. Hence, the recursive methods developed in this paper amount to an essentially complete solution of the time consistency problem in [4].

In order to understand the intuition for the recursive methods of this paper, it is natural to ask first why it is that characterizing time consistent outcomes is so difficult in [4] and, more generally, in models of long-lived agents. The short answer is that there are "too many infinities" to take care of. Somewhat more precisely, a time consistent solution[4] must include a description of government behavior and market behavior such that the continuation of such behavior after any history is a competitive equilibrium and is optimal for the government. Hence, given any history, checking for time consistency involves solving for a nontrivial infinite horizon competitive equilibrium problem; moreover, this has to be done for every one of an infinite number of histories.

The approach in this paper exploits two key ideas that help reduce the problem of "too many infinities" to more manageable dimensions. The first is that the need to check for time consistency after each of an infinity of histories can be managed more effectively by introducing as a (fictional) state variable the continuation value of the equilibrium. A similar insight has been useful to obtain recursive solutions of repeated games and dynamic

[4] As described below, the concept of time consistency employed in this paper is the appropriate generalization of the "sustainable plans" concept developed by Chari and Kehoe [6] and Stokey [24].

principal agent problems, as shown by Abreu, Pearce and Stachetti [2], Spear and Srivastava [23], Green [12], and Thomas and Worrall [25].

The second key idea underlying my approach is that, in checking that the continuation of a candidate for a time consistent solution is consistent with an infinite horizon competitive equilibrium, one can often exploit the fact that the set of competitive equilibria can itself be expressed recursively. The crucial observation is that, for a wide class of models, competitive equilibria can be expressed as the solution of a sequence of Euler-type equations. Although there are an infinity of such equations, each one connects only a small number of periods (say, today and tomorrow); a plausible guess, then, is that infinite horizon competitive equilibria can be characterized very simply by introducing an adequate state variable. This variable turns out to be the "right-hand side" of the Euler equation, a conjecture suggested first by Kydland and Prescott [16] in the context of capital–labor taxation with commitment.[5]

Although this paper is restricted to Calvo's model, it will hopefully become clear that its approach should be applicable to many dynamic models. While Calvo's model is very simple, solving for its competitive equilibria is a nontrivial infinite horizon problem. Hence Calvo's model presents the crucial difficulties associated with characterizing time consistency in models with long-lived agents. As a consequence, the intuition, the power, and the possible limitations of the methods proposed below are well illustrated in Calvo's setup. The price may be that Calvo's model is not very "realistic;" in particular, there is no physical state variable. However, it should become clear that our methods can be readily adapted to more complicated and "realistic" models, which may include physical capital, uncertainty, and so on, as long as their competitive equilibria can be expressed as a system of (possibly stochastic) Euler equations.

This paper is, or course, related to a very large literature. In particular, the concept of time consistency employed in this paper is the appropriate generalization of the "sustainable plans" concept proposed by Chari and Kehoe [6] and Stokey [24]. Both the Chari–Kehoe and Stokey papers adapt results from Abreu [2] to characterize the set of all the sustainable plans of their models. Abreu's method involves finding the worst continuation time consistent outcome, which may be very difficult in many models of interest. In contrast, the recursive methods developed in this paper do not require finding the worst continuation; in fact, they yield the worst and the best continuations as part of the solution.[6]

[5] The Kydland–Prescott approach has been recently been extended by [18].

[6] Pearce and Stachetti [21] have recently applied the methods of [2] to a time consistency problem. Since Pearce and Stachetti assume that there is no borrowing or lending of any kind, their problem is essentially a repeated game between the government and the public.

A very similar recursive approach to time consistency in models with long-lived agents has been independently developed by Phelan and Stachetti [22] in the context of a capital–labor taxation problem. As in my analysis, they combine the work of Abreu, Pearce, and Stachetti, and Kydland and Prescott to arrive to a recursive characterization of the set of sustainable plans. Our papers are obviously complementary.

Two recent noteworthy attempts at characterizing time consistent monetary policy in infinite horizon models are Obstfeld [20] and Ireland [13]. Obstfeld [20] studies a dynamic seigniorage problem and, in characterizing time consistent outcomes, focuses on the Markov perfect equilibria of the model, taking as state variables the previous real quantity of money and the inherited government debt. Hence this approach only provides a partial characterization of the set of time consistent outcomes. Our analysis emphasizes that, in a similar problem, a small enlargement of the set of state variables yields the set of all time consistent outcomes, and that all such outcomes are Markovian.

[13] analyzes a cash-in-advance model and is able to characterize all of its time consistent equilibria. This is achieved by showing that the the worst allowable hyperinflation is a time consistent outcome, which is then used to support all other time consistent outcomes as in [1, 6, 24]. As previously emphasized, Ireland's paper is insightful but his solution method depends on very special features of his environment, in particular that the worst possible hyperinflation is a dominant strategy for the government. Hence his arguments are not generally useful in finding the worst time consistent outcome in other models. In contrast, the recursive approach pursued below can probably be extended to a wide class of models and makes it unnecessary to look for the worst time consistent outcome.

The paper proceeds as follows. Section 2 sets up the economic environment under study. Section 3 discusses competitive equilibria; it is emphasized, in particular, that the set of competitive equilibria is recursive in a precise sense. Section 4 examines optimal government policy under commitment. Following arguments of [16], we show that the Ramsey problem can be written as a dynamic programming problem after the introduction of a fictional state variable. More importantly, the set of possible fictional states is shown to be the largest fixed point of a particular operator and can be computed recursively. Section 5 discusses the solution concept, sustainable plans, that is used later to characterize time consistency. Sections 6 and 7 contain the paper's main results. Section 6 studies an operator inspired by [2] and [16], whose largest fixed point yields the set of all sustainable outcomes. It is also shown there that the repeated application of that operator yields a sequence of sets that converges to the sustainable set. Section 7 studies a second operator, motivated by [10] and [16], whose largest fixed point also yields the set of sustainable outcomes, and whose repeated application also converges to that set. To demonstrate the computational feasibility of the theory, Section 8 computes and discusses the solutions for a parametric version of the model. Section 9 concludes. Some proofs are delayed to an Appendix.

2. THE MODEL

We will analyze a discrete time version of a model first proposed by Calvo [4]. Before proceeding, it must be stressed again that Calvo's model is not chosen because of its realism: That model is clearly too simple to be "realistic." However, for our purposes its simplicity is a virtue: While Calvo's model is fairly manageable, it is a truly infinite horizon dynamic model. We believe that the intuition and the power of the recursive methods proposed in this paper are best illustrated in this setup.

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. In each period there is only one consumption good and currency is the only asset. The economy is populated by a large number of identical households and a government. The representative household lives forever and has preferences over consumption and real money holdings given by:

$$\sum_{t=0}^{\infty} \beta^t [u(c_t) + v(m_t)] \tag{1}$$

where c_t denotes consumption in period t, $m_t \equiv q_t M_t$ the real value of money holdings, M_t currency holdings at the end of period t, and q_t the price of currency in terms of the consumption good (the inverse of the price level). The functions u and v satisfy:

[A1] $u : \mathbb{R}_+ \to \mathbb{R}$ is C^2, strictly concave, and strictly increasing.

[A2] $v : \mathbb{R}_+ \to \mathbb{R}$ is C^2, and strictly concave.

[A3] $\lim_{c \to 0} u'(c) = \lim_{m \to 0} v'(m) = \infty$.

[A4] There is a finite $m = m^f > 0$ such that $v'(m^f) = 0$.

The assumptions [A1]–[A3] are fairly standard.[7] [A4] defines m^f as the satiation level of money. It will become clear that these four assumptions can be generalized substantially, as long as the model has a recursive structure and some boundedness conditions hold.

[7] Note that [A2] implies that $v(0)$ is finite, which in turn implies that $mv'(m) \to 0$ as $m \to 0$; see [19].

The household will maximize (1) subject to c_t, $M_t \geqslant 0$ and

$$q_t M_t \leqslant y_t - x_t - c_t + q_t M_{t-1}, \tag{2}$$

$$q_t M_t \leqslant \bar{m}, \tag{3}$$

for all $t \geqslant 0$, where y_t denotes a period t endowment of the consumption good, x_t is a lump sum tax (or transfer, if negative), and $\bar{m} \geqslant m^f$ is an exogenously given constant. The household takes the sequences $\{q_t\}$, $\{x_t\}$, $\{y_t\}$, and its initial currency holdings M_{-1} as given.

(2) is a typical accumulation equation that defines the real value of money holdings at the end of period t. (3) is less standard. The main reason to impose it is that, as discussed later, our analysis will require that real money balances be bounded above in equilibrium. (3) is one way to ensure that a bound exists; there may be other ways, if stronger assumptions are placed on preferences for instance. However, (3) is simpler and could be motivated by assuming that managing large amounts of cash is costly.[8] Note also that the bound $\bar{m}$ can be taken to be arbitrarily large. In such case one would expect (3) not to bind in well-behaved equilibria, given that large money holdings are associated with negative marginal utility. Consequently we see (3) as fairly harmless.

The government chooses how much money to create or to withdraw from circulation. Its choices completely determine the path of the money supply, given M_{-1}. We shall express money growth in terms of the inverse growth rates, $h_t \equiv M_{t-1}/M_t$, and assume that:

[A5] For some $\underline{\pi}$, $\bar{\pi}$ such that $0 < \underline{\pi} < 1 < 1/\beta \leqslant \bar{\pi}$, $h_t \in [\underline{\pi}, \bar{\pi}] \equiv \Pi$.

[A5] bounds admissible rates of money creation and, like (3), is needed for technical reasons. It is probably uncontroversial to impose that the supply of money be positive; imposing that M_t/M_{t-1} be not less than $1/\bar{\pi} > 0$, where $1/\bar{\pi}$ can be arbitrarily small, is only a mild strengthening of that requirement. A stronger restriction is that money growth must be bounded above by some (arbitrarily large) number $1/\underline{\pi}$. Although this assumption can probably be defended on the basis of realism, it may be interesting to see what happens if it is dropped. This is left for future research.

The government uses the money newly printed in period t to finance the transfers or taxes to households according to its budget constraint:

$$q_t(M_t - M_{t-1}) = -x_t. \tag{4}$$

[8] Alternatively, (3) could just be assumed to represent a storage constraint.

Taxes or transfers are assumed to be distortionary. One can introduce, for example, a model of production and distortionary labor taxation. But since I am trying to formalize things as simply as possible, I will assume as in [4] that the household's endowment, y_t, is a function $f(x_t)$ of the taxes collected in period t. Now, $f : \mathbb{R} \to \mathbb{R}$ is at least C^2 and is assumed to satisfy:

[A6] $f(0) > 0$, $f'(0) = 0$, $f''(x) < 0$.

[A7] f is symmetric about zero: $f(x) = f(-x)$, all $x \in \mathbb{R}$.

[A6] ensures that imposing taxes or giving subsidies is increasingly costly in terms of the consumption good. This is an admittedly simple way to model the idea that taxation is distortionary. Symmetry [A7] is imposed only to simplify notation.

The government budget constraint (4) can now be rewritten as:

$$-x_t = q_t M_t(1 - h_t) = m_t(1 - h_t). \tag{5}$$

Since $m_t \in [0, \bar{m}]$ and $h_t \in \Pi$, x_t must belong to the interval $[(\underline{\pi} - 1)\,\bar{m}, (\bar{\pi} - 1)\,\bar{m}] \equiv X$. Equation (5) emphasizes that h_t, the inverse of the money growth rate in period t, can be thought of as the (gross) rate of the inflation tax.

Our last assumption will ensure that output is always strictly positive:

[A8] f is strictly positive on X.

Together with [A1] and [A6], [A8] ensures that given any sequence of taxes such that $x_t \in X$, all t, the marginal utility of consumption $u'[f(x_t)]$ is a uniformly bounded sequence. The importance of this fact will become apparent shortly.

In this model, it is clearly desirable to bring the quantity of money towards the satiation level m^f. However, in equilibrium this can only be achieved by steadily reducing the supply of money which, given [A6]–[A7], has negative effects on output. Hence it is clear that an optimal policy will imply some positive deflation, although not as fast as the rate of time preference.

3. COMPETITIVE EQUILIBRIA

In this section competitive equilibrium is defined in the usual way and, to prepare the ground for our main discussion, some facts about equilibria are collected. In particular, this section makes precise the idea that competitive equilibria have a recursive structure.

For the rest of the paper, bold letters will denote sequences, and a subscript (resp. superscript) will denote the first (resp. last) date of the sequence. Thus $\mathbf{x}_t^s = (x_t, x_{t+1}, ..., x_s)$. If the subscript is omitted, the first date is understood to be $t=0$, while an omitted superscript implies that the last date is $s=\infty$. Thus $\mathbf{x}^t = (x_0, ..., x_t)$, $\mathbf{x}_t = (x_t, x_{t+1}, ...)$ and $\mathbf{x} = (x_0, x_1, ...)$.

A *policy* is a sequence $\mathbf{h} = (h_0, h_1, ...)$ describing money growth, and a sequence of tax amounts, $\mathbf{x} = (x_0, x_1, ...)$, such that $h_t \in \Pi = [\underline{\pi}, \bar{\pi}]$ and $x_t \in \mathbb{R}$, all $t \geqslant 0$. An *allocation* is a set of nonnegative sequences of consumptions, $\mathbf{c}$, real money demands, $\mathbf{m}$, endowments, $\mathbf{y}$, and inverse price levels, $\mathbf{q}$. Given M_{-1}, a policy $(\mathbf{h}, \mathbf{x})$ and an allocation $(\mathbf{c}, \mathbf{m}, \mathbf{y}, \mathbf{q})$ form a *competitive equilibrium* if:

(i) Markets clear in every period $t \geqslant 0$: $m_t = q_t M_t$ and $y_t = f(x_t)$.

(ii) The government budget constraint (4) is satisfied and $M_t = M_{t-1}/h_t$.

(iii) The pair $(\mathbf{c}, \mathbf{M})$ solves the consumer's problem, given the sequence of prices $\mathbf{q}$, endowments $\mathbf{y}$, and taxes $\mathbf{x}$.

Let $E = [0, \bar{m}] \times X \times \Pi$, and $E^\infty = E \times E \times E \times \cdots$. Under assumptions [A1]–[A8], one can prove that:

PROPOSITION 1. *A competitive equilibrium is completely characterized by a sequence* $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ *such that, for all* t, $m_t \in [0, \bar{m}]$, $h_t \in \Pi$, $x_t \in X$, *and*:

$$-x_t = m_t(1 - h_t) \tag{6}$$

$$m_t\{u'[f(x_t)] - v'(m_t)\} \leqslant \beta u'[f(x_{t+1})](m_{t+1} + x_{t+1}), \qquad \text{with equality if} \quad m_t < \bar{m}. \tag{7}$$

Proof. See Appendix.

Proposition 1 says that a sequence $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ is consistent with a competitive equilibrium if it belongs to E^∞ and if it satisfies the government budget constraint (6) and the household's Euler condition (7) in all periods. Hence the set of competitive equilibria can be described as the solution of an infinite sequence of conditions, each of which connects at most two periods. This observation is crucial for understanding our approach later.

The perceptive reader will have noted that Proposition 1 makes no mention of the household's transversality condition. The justification, as shown in the Appendix, is that the transversality condition is satisfied if $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ belongs to E^∞. Here is where [A8] plays an important role. Without it, equilibrium consumption could approach zero. Given [A3], the marginal utility of consumption would then be unbounded, invalidating my argument for ignoring the transversality condition. In such a case, the transversality issue must be dealt with in some other way in order to apply the methods of this paper.

Also, the need for imposing a bound on the household's real money holdings and for assuming that money growth rates belong to a compact set should be apparent. An important part of Proposition 1 is that, in any competitive equilibrium, m_t and x_t must belong to compact sets. This is ensured by imposing (3), [A5], and by the fact that, in equilibrium, x_t must satisfy the government budget constraint (5).

To proceed, note that E^∞ is compact when endowed with the product topology. Given Proposition 1, an element of E^∞ satisfying (6)–(7) will be called a *competitive equilibrium sequence*. The set of all such sequences will be denoted by $CE = \{(\mathbf{m}, \mathbf{x}, \mathbf{h}) \in E^\infty \mid (6) \text{ and } (7) \text{ are satisfied}\}$.

The following facts are now are easy to prove:

COROLLARY 1. *CE is not empty.*

Proof. There is a competitive equilibrium with a constant supply of money.

COROLLARY 2. *CE is compact.*

Proof. See Appendix.

COROLLARY 3. *The continuation of a competitive equilibrium is a competitive equilibrium. In other words, if* $(\mathbf{m}, \mathbf{x}, \mathbf{h}) \in CE$*, then* $(\mathbf{m}_t, \mathbf{x}_t, \mathbf{h}_t) \in CE$ *for all t.*

The proof of Corollary 3 follows immediately from Proposition 1 and is left to the reader. In spite of its simplicity, Corollary 3 is a crucial aspect of the model: It makes precise a sense in which the set of competitive equilibria has a recursive structure.

4. RECURSIVE TREATMENT OF THE RAMSEY PROBLEM

From now on we shall assume that the government's objective is to maximize the welfare of its representative citizen. The government's menu of choices to achieve its objective depends, however, on the "commitment technology" available to it. A natural starting point is to suppose that the government can fix the entire path of money growth rates once and for all at the beginning of time. This case of *perfect commitment* is the subject of this section.

The government's problem under commitment is to choose a policy $\mathbf{h}$ and an associated competitive equilibrium such that there is no other competitive equilibrium that results in higher consumer's welfare. This problem can be restated more precisely, given the results of the previous section, as that of choosing $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ in *CE* to maximize (1), with $c_t = f(x_t)$. Following previous authors,[9] this problem will be called the *Ramsey problem.*

Since (1) is continuous on E^∞, and *CE* is compact, we know that the Ramsey problem has a solution. Also, given Proposition 1, we know that the solution must solve:

$$\text{Max (1) subject to (6)–(7) and } c_t = f(x_t)$$

where the maximization is over sequences in E^∞.

The Ramsey problem, as stated above, can be solved (at least in principle) with a variety of methods. Since our objective is ultimately to look at recursive methods, next we describe a procedure that solves the Ramsey problem in a recursive way. My procedure is a variant of that originally proposed by [16].

The key to the procedure is to use a recursive description of competitive equilibria. From the perspective of any period t, a competitive equilibrium can be seen as the collection of a current policy and allocation, together with a "promise" of policies and allocations from period $(t+1)$ on that satisfies some conditions. By Proposition 1, it follows that the essential feature of the "promise" made in period t is given by the scalar $u'[f(x_{t+1})](m_{t+1} + x_{t+1}) \equiv \theta_{t+1}$ in the Euler equation. Roughly speaking, θ_{t+1} can be seen as the period $(t+1)$ marginal utility of money "promised" by the equilibrium in period t.

Hence we need to study a set Ω defined by:

$$\Omega = \{\theta \in \mathbb{R} : \theta = u'[f(x_0)](m_0 + x_0) \text{ for some } (\mathbf{m}, \mathbf{x}, \mathbf{h}) \in \text{CE}\}.$$

Ω is the set of initial marginal utility of money "promises" consistent with competitive equilibria.

PROPOSITION 2. *Ω is a nonempty and compact subset of $\mathbb{R}_+$.*

Proof. Since *CE* is not empty, Ω is not empty. In any competitive equilibrium, $(m_t + x_t) = h_t m_t \in [0, \bar{\pi}\bar{m}]$. Since $u'[f(x_t)]$ is a positive, continuous function on X, its range is a bounded subset of $\mathbb{R}_+$. Hence Ω is included in some compact interval $[0, \bar{\theta}]$, for some $\bar{\theta}$.

To see that Ω is compact, it is enough to show that Ω is closed. Let $\{\theta^n\}$ be a sequence in Ω converging to $\theta \in [0, \bar{\theta}]$. By definition, there is a

[9] For example, [6].

sequence $(\mathbf{m}^n, \mathbf{x}^n, \mathbf{h}^n)$ in CE such that $\theta^n = u'[f(x_0^n)](m_0^n + x_0^n)$ for each n. Since CE is compact, $(\mathbf{m}^n, \mathbf{x}^n, \mathbf{h}^n)$ can be assumed without loss of generality to converge to some $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ in CE. Finally, continuity of u' and f implies that $\theta = u'[f(x_0)](m_0 + x_0)$. Hence Ω is closed and compact. ∎

Now we can follow [16] and formulate the Ramsey problem in two stages. First, suppose momentarily that the government were constrained not only be the requirement of equilibria, but also by an initial "promise" $\theta \in \Omega$. Then its problem would be:

$$w^*(\theta) = \text{Max} \sum_{t=0}^{\infty} \beta^t [u(f(x_t)) + v(m_t)] \quad \text{s.t.} \quad (\mathbf{m}, \mathbf{x}, \mathbf{h}) \in \Gamma(\theta) \tag{8}$$

where $\Gamma(\theta) = \{(\mathbf{m}, \mathbf{x}, \mathbf{h}) \in CE \mid \theta = u'[f(x_0)](m_0 + x_0)\}$.

Given any initial "promise" θ in Ω, $\Gamma(\theta)$ is a nonempty, compact subset of CE. Since the objective in (8) is clearly continuous, the function w^* is well defined on Ω.

Finally, if the function $w^*(\cdot)$ can be obtained, the value of the Ramsey problem is simply be given by the max of $w^*(\theta)$ on Ω.

The usefulness of recasting the Ramsey problem in this way is that now we obtain a "dynamic programming" formulation:

PROPOSITION 3. *$w^*(\theta)$ satisfies the functional equation*:

$$w(\theta) = \text{Max}\, u[f(x)] + v(m) + \beta w(\theta') \tag{9}$$

$$s.t. \quad (m, x, h, \theta') \in E \times \Omega$$

$$\theta = u'(f(x))(m + x) \tag{10}$$

$$-x = m(1 - h) \tag{11}$$

$$m\{u'[f(x)] - v'(m)\} \leqslant \beta\theta', \text{ with equality if } m < \bar{m}. \tag{12}$$

Conversely, if a bounded function $w : \Omega \to \mathbb{R}$ satisfies the above functional equation, then $w = w^*$.

The proof is closely related to the proof of Bellman's optimality principle and given in the Appendix.

Proposition 3 provides a way in which the Ramsey problem can be solved—provided that we can compute the set Ω, which is not a trivial matter. Now we will show that Ω can be computed taking advantage of the fact that Ω must be the fixed point of a particular operator.[10] It turns out that the analysis of that operator is crucial not only to solve the Ramsey

[10] This approach follows a suggestion of [16]. Our proof of Proposition 5 below is new, although its line of argument is similar to that in [2] and [23].

problem but also, and more importantly, to find the solution of the policy problem *without* commitment.

Let Q be a nonempty and bounded subset of $\mathbb{R}_+$. Define a new set $\mathbb{B}(Q)$ as follows:

$$\mathbb{B}(Q) = \{\theta \in \mathbb{R} : \text{there is } (m, x, h, \theta') \in E \times Q \text{ such that (10)–(12) hold}\}.$$

Then one can show that:

PROPOSITION 4. *(i) $Q \subseteq \mathbb{B}(Q)$ implies that $\mathbb{B}(Q) \subseteq \Omega$. (ii) $\Omega = \mathbb{B}(\Omega)$.*

Proof. Left to the reader (a simple extension of Abreu, Pearce, and Stachetti's arguments).

In other words, Ω is the largest fixed point of the operator $\mathbb{B}$. Following [2], we will refer to property (i) in Proposition 4 as *self generation*, and to property (ii) as *factorization.*

One advantage of this formulation is that it delivers a way to compute Ω as follows. Let $Q_0 = [0, \bar{\theta}]$, where $\bar{\theta}$ is as in Proposition 2. For $n = 1, 2, ...,$ define $Q_n = \mathbb{B}(Q_{n-1})$. Now, the definition of $\mathbb{B}$ clearly implies that $\mathbb{B}$ is monotone in the sense that $Q \subseteq Q'$ implies $\mathbb{B}(Q) \subseteq \mathbb{B}(Q')$. This implies that the sequence $\{Q_n\}_{n=0}^{\infty}$ is decreasing. Also, one can show that $\mathbb{B}$ preserves compactness. Hence each Q_n is a compact set. Define $Q_\infty = \bigcap_{n=0}^{\infty} Q_n$, i.e., Q_∞ is obtained in the limit by repeated application of the operator $\mathbb{B}$, starting with $[0, \bar{\theta}]$. Now it turns out that:

PROPOSITION 5. $\Omega = Q_\infty$.

Proof. Obviously it suffices to show that $Q_\infty \subseteq \Omega$. We shall prove this by showing that $Q_\infty \subseteq \mathbb{B}(Q_\infty)$; the desired result will then follow by self generation.

Suppose that $\theta \in Q_\infty$. By definition of the sequence Q_n, it follows that $\theta \in Q_n = \mathbb{B}(Q_{n-1})$, all $n = 1, 2, ...$ By definition of $\mathbb{B}$, there is for each $n = 0, 1, 2...$ a vector $(m^n, x^n, h^n, \theta'^n)$ in $E \times Q_n$ that satisfies (10)–(12). The sequence $(m^n, x^n, h^n, \theta'^n)$, when seen as a sequence in $E \times [0, \bar{\theta}]$, can be assumed without loss of generality to converge to some $(m, x, h, \theta') \in E \times [0, \bar{\theta}]$. By the continuity of u', v', and f, (m, x, h, θ') satisfies (10)–(12). Finally, θ' can be shown in fact to belong to Q_∞ by the following argument: Fix any $n = 0, 1, ...$ Then, given any $k > n$, $\theta'^k \in Q_k \subseteq Q_n$. Hence θ', which is the limit of the sequence $\{\theta'^k\}$, must also belong to Q_n. Since this is true for any n, $\theta' \in Q_\infty$.

The preceding argument shows that $Q_\infty \subseteq \mathbb{B}(Q_\infty)$. By self generation, $Q_\infty \subseteq \Omega$ and the proof is complete. ∎

Together, Propositions 3 and 5 provide a procedure that one can use to solve for $w^*(\theta)$, and hence to compute the Ramsey outcome. Ω can be computed by iterating on the operator $\mathbb{B}$. Once Ω is known, the functional equation in Proposition 4 can be solved by standard methods to obtain w^*. In this sense, the Ramsey problem can be solved recursively.

It should be reiterated that, for our purposes, the computation of Ω is the most important part of the above procedure: The solution of the no commitment case will be found by appropriately modifying the operator $\mathbb{B}$.

Before ending this section, note that our results imply that the Ramsey problem has a Markovian structure. Along an optimal path, the "state" can be defined to be θ_t; the optimal "action" (m_t, x_t, h_t) and next period state θ_{t+1} can be chosen to be time invariant functions of θ_t. The introduction of the state variable θ_t takes care of the requirement that a Ramsey plan be consistent with a perfect foresight competitive equilibrium.[11]

5. SUSTAINABLE PLANS: DEFINITION

Henceforth we will assume that the government does not have the ability to commit to an infinite sequence of money growth rates. Instead, we shall assume that the government sets period t's money growth at the beginning of the period. Under such assumption, it is known from [4] and [15] that the government faces a "credibility" or "time consistency" problem. Characterizing the "credible" outcomes of the model is difficult and requires a well-defined equilibrium concept. In this section we define such a concept, a direct extension of that developed by Chari and Kehoe [6] and Stokey [24] for related environments. Following Chari and Kehoe, we will refer to equilibria as "sustainable plans" (SPs).

A *history* in period t, denoted by $\mathbf{h}^t = (h_0, h_1, ..., h_t)$ describes the actual sequence of money growth rates in every period up to t. Recalling that h_t is assumed to belong to a compact interval Π, a *strategy* for the government is a sequence of functions $\{\sigma_t\}_{t=0}^{\infty}$ such that $\sigma_0 \in \Pi$ and $\sigma_t : \Pi^{t-1} \to \Pi$.

In order to have well-defined decision problems we will impose an additional restriction on the strategy space available to the government. This is, roughly speaking, because some strategies may imply that, after some history, the continuation of the strategy be inconsistent with the existence of a competitive equilibrium. This will be ruled out as follows: Let $CE_\pi = \{\mathbf{h} \in \Pi^\infty : \text{there is some } (\mathbf{m}, \mathbf{x}) \text{ such that } (\mathbf{m}, \mathbf{x}, \mathbf{h}) \in CE\}$. CE_π is the set of infinite horizon money growth sequences that are consistent with competitive equilibria; it is clearly nonempty and compact. A strategy σ will

[11] On the other hand, θ_t is a "fictional" state variable in the sense that the Ramsey problem allows θ_0 to be picked freely.

be called *admissible* if, after any history $\mathbf{h}^{t-1}$, the continuation history $\mathbf{h}_t$, defined by the continuation of σ in the natural way, belongs to CE_π. In what follows we will restrict the government to choose an admissible policy. Intuitively, this says that, after any history, the government has to "announce" a policy for the infinite future that is consistent with the existence of a competitive equilibrium.

Note in particular that, after any history $\mathbf{h}^{t-1}$, the above considerations restrict the government's choice in period t to the set $CE_\pi^0 = \{h \in \Pi : \text{there is } \mathbf{h} \in CE_\pi \text{ with } h = h_0\}$.[12]

Now we are ready to describe market behavior. An *allocation rule* is a sequence of functions $\alpha = \{\alpha_t\}_{t=0}^{\infty}$ such that, for each t, $\alpha_t : \Pi^t \to [0, \bar{m}] \times X$. Here, $\alpha_t(\mathbf{h}^t) = (m_t(\mathbf{h}^t), x_t(\mathbf{h}^t))$ denotes the real value of money and taxes in period t, after history $\mathbf{h}^t$ has been observed.

Given an admissible government strategy σ, an allocation rule α will be called *competitive* if given any history $\mathbf{h}^{t-1}$ and $h_t \in CE_\pi^0$, the continuations of σ and α after $(\mathbf{h}^{t-1}, h_t)$ induce, in the obvious way, a competitive equilibrium sequence.[13]

Finally, a government strategy σ and an allocation rule α constitute a *sustainable plan* if (i) σ is admissible; (ii) α is competitive given σ; (iii) After any history $\mathbf{h}^{t-1}$, the continuation of σ is optimal for the government, that is, the sequence $\mathbf{h}_t$ induced by σ after $\mathbf{h}^{t-1}$ maximizes (1) over CE_π, given α.

The definition of a sustainable plan has some nice properties. One of them is that the continuation of a sustainable plan is itself a sustainable plan. This in turn will enable us to apply recursive methods in the rest of the paper.

PROPOSITION 6. *Given any history $\mathbf{h}^{t-1}$, the continuation of a sustainable plan is itself a sustainable plan.*

Proof. Left to the reader (just a matter of accounting).

To conclude this section, note that any sustainable plan induces, in the natural way, a competitive equilibrium sequence $(\mathbf{m}, \mathbf{x}, \mathbf{h})$. In view of this fact, a competitive equilibrium sequence will be called a *sustainable outcome* if it is induced by some sustainable plan.

[12] Note that $CE_\pi^0 = \{h \in \Pi : \text{there is } (m, \theta') \in [0, \bar{m}] \times \Omega \text{ such that } m[u'(f(h-1)\, m) - v'(m)] \leqslant \beta\theta', \text{ with equality if } m < \bar{m}\}$.

[13] That is, given $\mathbf{h}^{t-1}$ and $h_t \in CE$, define $\mathbf{h}_{t+1}$ recursively by $h_{t+k} = \sigma_{t+k}(\mathbf{h}^{t-1}, h_t, \mathbf{h}_{t+1}^{t+k-1})$, $k = 1, 2, \ldots$. Then define $(m_{t+k}, x_{t+k}) = \alpha_{t+k}(\mathbf{h}^{t-1}, h_t, \mathbf{h}_{t+1}^{t+k})$, $k = 0, 1, 2, \ldots$. For α to be competitive given σ, the sequence $(\mathbf{m}_t, \mathbf{x}_t, \mathbf{h}_t)$ must be in CE.

6. SUSTAINABLE OUTCOMES: A RECURSIVE APPROACH

This section and the next discuss the key results of this paper. The goal of this section is to fully characterize the set of sustainable outcomes in a recursive manner. To do this, two aspects of the model need to be taken into account. The first is that, because the government has a time consistency problem, any SP must provide incentives for the government not to deviate from equilibrium behavior. It will be shown that these incentive constraints can be handled by introducing as a state variable the continuation value of the equilibrium, as advocated by [1] and [23] in other dynamic incentive problems. In our context this is not enough, though, because one has to ensure, after any history, that the continuation of a SP is consistent with a competitive equilibrium for the infinite future. But we saw that, in the Ramsey problem, this constraint can be handled by introducing the promised marginal utility of money as a state variable. Hence one would guess that a recursive approach to the set of sustainable plans should include at least two state variables, one for the continuation values and another for the promised marginal utility of money. In addition, one would guess that it is possible to characterize the state space in a recursive fashion. We shall see that these guesses are in fact correct.

The analysis takes for granted that there exists at least one sustainable plan. For the model at hand this can in fact be proven. However, the details are somewhat peripheral to my main discussion and hence left to the Appendix.

Let $\Theta = \{(\mathbf{m}, \mathbf{x}, \mathbf{h}) \in CE \mid$ there is a SP whose outcome is $(\mathbf{m}, \mathbf{x}, \mathbf{h})\}$ be the set of all sustainable outcomes. Then define:

$$S = \{(w, \theta) \mid \text{there is a sustainable outcome } (\mathbf{m}, \mathbf{x}, \mathbf{h}) \in \Theta \text{ with value } w, \text{ and such that } u'[f(x_0)](m_0 + x_0) = \theta\}.$$

By [A1]–[A2] and Proposition 1, the value of any competitive equilibrium must belong to some compact interval, say $\mathscr{W} = [\underline{w}, \bar{w}]$. Hence S is a subset of the compact set $\mathscr{W} \times \Omega$. It is nonempty since there is at least one SP.

The set S is the set of all pairs of continuation values and promised marginal utilities of money that may emerge in the first period of a SP. Our main objective is to characterize S in a recursive fashion. To this end, the following remarks may be useful.

Consider what a SP must describe in the first period: It must describe an initial, "recommended" action, say $\hat{h}$ and, for each possible deviation h that the government may consider (i.e., for each h in CE_π^0), the SP must specify the real quantity of money $m(h)$ and seigniorage $x(h)$. Moreover, the SP must specify the whole future path of the economy; however, our previous discussion suggests that the future may be summarized by a continuation value, $w'(h)$, and a continuation "promise" $\theta'(h)$, which *must belong to S*. Finally, all of these objects must be such that $\hat{h}$ is optimal for the government, and that the government budget constraint and the Euler equation are satisfied after any deviation. These objects imply a first period (w, θ) which, by assumption, belongs to S.

These considerations motivate the following approach. Let Z denote any nonempty subset of $\mathscr{W} \times \Omega$; the reader can think of Z as a set from which tomorrow's pairs (w', θ') can be chosen. Define a new set $\mathbb{D}(Z) \subseteq \mathscr{W} \times \Omega$, the set of pairs (w, θ) that can be "enforced" today, by:

$$\mathbb{D}(Z) = \{(w, \theta) \mid \text{there is } \hat{h} \in CE^0_\pi \text{ and, for each } h \in CE^0_\pi \text{, a four-tuple } (m(h), x(h), w'(h), \theta'(h)) \text{ in } [0, \bar{m}] \times X \times Z \text{ such that:}$$

$$w = u[f(x(\hat{h}))] + v[(m(\hat{h})] + \beta w'(\hat{h}) \tag{13}$$

$$\theta = u'[f(x(\hat{h}))][m(\hat{h}) + x(\hat{h})] \tag{14}$$

$$\text{and for all } h \in CE^0_\pi:$$

$$w \geqslant u[f(x(h))] + v[m(h)] + \beta w'(h) \tag{15}$$

$$x(h) = m(h)(h-1), \text{ and} \tag{16}$$

$$m(h)[u'(f(h)) - v'(h)] \leqslant \beta\theta'(h), \text{ with equality if } m(h) < \bar{m}\}. \tag{17}$$

The constraints (13)–(14) are usually called "regeneration constraints," while (15) is an "incentive" constraint. (16) and (17) are novel, and are necessary to ensure that the continuation of a sustainable plan after any deviation is consistent with a competitive equilibrium.

As in [2], the operator $\mathbb{D}$ has the following properties, which imply that S is the largest fixed point of $\mathbb{D}$:

PROPOSITION 7. (i) *Self Generation*: *If* $Z \subseteq \mathbb{D}(Z)$, *then* $\mathbb{D}(Z) \subseteq S$; (ii) *Factorization*: $S = \mathbb{D}(S)$.

Proof. (i) Suppose $Z \subseteq \mathbb{D}(Z)$ and let (w, θ) be in $\mathbb{D}(Z)$. Set $(w_0, \theta_0) = (w, \theta)$, and construct a SP (α, σ) recursively as follows. For any $\mathbf{h}^{t-1}$, suppose that we can define $(w_t(\mathbf{h}^{t-1}), \theta_t(\mathbf{h}^{t-1}))$ in Z. Since $Z \subseteq D(Z)$, there is $\hat{h}_t \in CE^0_\pi$ and, for each $h_t \in CE^0_\pi$, a four-tuple $(m(h_t), x(h_t), w'(h_t), \theta'(h_t))$ in $[0, \bar{m}] \times X \times Z$ such that (13)–(17) are satisfied. Define, then, $\sigma_t(\mathbf{h}^{t-1}) = \hat{h}_t$, and $\alpha_t(\mathbf{h}^t) = (m(h_t), x(h_t))$ if $h_t \in CE^0_\pi$, and $= (0, 0)$ if not. Finally, define $(w_{t+1}(\mathbf{h}^t), \theta_{t+1}(\mathbf{h}^t)) = (w'(h_t), \theta'(h_t))$ if $h_t \in CE^0_\pi$, and $= (w, \theta)$ if not. By definition, then, $(w_{t+1}(\mathbf{h}^t), \theta_{t+1}(\mathbf{h}^t)) \in Z$ for all $\mathbf{h}^t$.

Checking that (α, σ) is a SP with value w, initial promise θ is straightforward and left to the reader.

(ii) By self generation, it is enough to show that $S \subseteq \mathbb{D}(S)$. This is easy from the definitions and also left to the reader. ▮

It should be noted that the proof of part (i) of the Proposition shows how one can construct a sustainable plan given any element of S. In addition, the construction reveals that any sustainable outcome has essentially a Markovian structure in the sense that, as is easily seen, $\sigma_t(\mathbf{h}^{t-1})$ and the functions $\alpha_t(\mathbf{h}^{t-1}; h_t)$, $w_{t+1}(\mathbf{h}^{t-1}; h_t)$, and $\theta_{t+1}(\mathbf{h}^{t-1}; h_t)$ depend on $\boldsymbol{h}^{t-1}$ only through the "state" $(w_t(\mathbf{h}^{t-1}), \theta_t(\mathbf{h}^{t-1}))$. Hence our approach to finding S gives a lot of information about the set of SPs.

Next we can derive properties of S by studying the operator $\mathbb{D}$. It is easy to show that $\mathbb{D}$ is nicely behaved, in the sense that it has a monotonicity property and it preserves compactness:

PROPOSITION 8. (i) *Monotonicity*: $Z \subseteq Z'$ *implies* $\mathbb{D}(Z) \subseteq \mathbb{D}(Z')$; (ii) *If* Z *is compact*, $\mathbb{D}(Z)$ *is compact.*

Proof. (i) is obvious from the definition of $\mathbb{D}$. To prove (ii), it is sufficient to show that if Z is compact, $\mathbb{D}(Z)$ is closed.

Let $(w^{(n)}, \theta^{(n)})$ be a sequence in $\mathbb{D}(Z)$ converging to $(w, \theta) \in \mathscr{W} \times \Omega$. By definition, for each n there is a recommended action $\hat{h}^{(n)}$ in CE_π^0 and, for each h in CE_π^0, a 4-tuple $(m(h)^{(n)}, x(h)^{(n)}, w'(h)^{(n)}, \theta'(h)^{(n)})$ in $[0, \bar{m}] \times X \times Z$ that satisfies (13)–(17).

Since CE_π^0 is compact, there is no loss of generality in assuming that the sequence $\hat{h}^{(n)}$ converges to some $\hat{h}$ in CE_π^0. Likewise, for each h in CE_π^0, $(m(h)^{(n)}, x(h)^{(n)}, w'(h)^{(n)}, \theta'(h)^{(n)})$ is a sequence in the compact set $[0, \bar{m}] \times X \times Z$ and can be assumed to converge to an element $(m(h), x(h), w'(h), \theta'(h))$ of $[0, \bar{m}] \times X \times Z$.

It is easily checked that the continuity of u, f, v, u' and v' ensure that the recommended action $\hat{h}$ and the function $(m(h), x(h), w'(h), \theta'(h))$ thus defined satisfy (13)–(17). Hence (w, θ) belongs to $\mathbb{D}(Z)$. Since $\mathbb{D}(Z)$ contains all its limit points, it is closed, and hence compact. ▮

In particular, the above two properties now imply that S is compact:

PROPOSITION 9. *S is compact.*

Proof. As discussed above, S is bounded. Hence it is enough to show that S is closed. Let $\mathrm{cl}(Z)$ denote the closure of a set Z. Factorization and monotonicity imply $S = \mathbb{D}(S) \subseteq \mathbb{D}(\mathrm{cl}(S))$. Since S is bounded, $\mathrm{cl}(S)$ is compact; hence $\mathbb{D}(\mathrm{cl}(S))$ is compact. It follows that $\mathrm{cl}(\mathbb{D}(\mathrm{cl}\, S)) = \mathbb{D}(\mathrm{cl}(S))$. But then $\mathrm{cl}(S) = \mathrm{cl}(\mathbb{D}(S)) \subseteq \mathrm{cl}(\mathbb{D}(\mathrm{cl}(S))) = \mathbb{D}(\mathrm{cl}(S))$. Self generation implies now that $\mathrm{cl}(S) \subseteq S$, that is, S is closed. ▮

An immediate implication is:

COROLLARY 4. *There are a best and a worst SP.*

This Corollary can be usefully compared against other results available in the literature. [6] and [24], extending arguments of [1], showed that one can fully characterize every sustainable outcome if one can calculate the "worst" sustainable outcome. However, finding the worst sustainable outcome is very difficult in many dynamic models. In contrast, our recursive methods do not rely on finding the worst sustainable outcome; instead, its existence is derived as one implication of the analysis.

Finally, by modifying the proof of Proposition 5 we obtain an algorithm that computes S. Recall that S must be included in $\mathscr{W} \times \Omega$. If we define $S_0 = \mathscr{W} \times \Omega$, the monotonicity of $\mathbb{D}$ implies $S = \mathbb{D}(S) \subseteq \mathbb{D}(S_0) \equiv S_1$. Defining now, for each $n \geqslant 1$, $S_n = \mathbb{D}(S_{n-1})$, we obtain a decreasing sequence of sets $S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$. Moreover, each S_n contains S and is compact (because S_0 is compact and $\mathbb{D}$ preserves compactness). One can then conjecture that the sequence $\{S_n\}$ must converge to S, in the sense that $S_\infty \equiv \bigcap_{n=0}^{\infty} S_n = S$. The following Proposition confirms the validity of such conjecture:

PROPOSITION 10. *Let* $S_0 = \mathscr{W} \times \Omega$ *and* $S_n = \mathbb{D}(S_{n-1}), n = 1, 2, \ldots.$ *Then* $S_\infty \equiv \bigcap_{n=0}^{\infty} S_n = S.$

Proof. The proof is essentially the same as that of Proposition 5 and left to the reader. ∎

The preceding results amount to a complete characterization of the set of sustainable plans. We have learned that the set of sustainable plans is compact. This means, in particular, that a best and a worst sustainable plan exist. Proposition 10 provides a way to compute the set S, and the proof of Proposition 7 provides a way to compute a sustainable plan corresponding to any (w, θ) in S.

The remaining issues are "only" computational. In particular, the difficulty of computing any sustainable plan arises solely from the difficulty of computing the mapping $\mathbb{D}$. For the problem at hand, computing $\mathbb{D}(Z)$ given Z seems fairly complicated in particular by the presence of the constraints (15)–(17). These constraints can be simplified somewhat; this is the subject of the next section.

7. ALTERNATIVE RECURSIVE METHOD

In this section we study a second operator whose largest fixed point is the set S and whose repeated application also yields a decreasing sequence of sets that converges to S. The analysis is related to that developed by Cronshaw and Luenberger [10] for repeated games.

The intuition for a simpler approach is that the government, when considering whether or not to obey an equilibrium "recommendation," need not consider the consequences of all alternative deviations, but only the payoff associated with the "best" deviation. On the other hand, in order to provide incentives for following equilibrium recommendations, one can restrict attention to sustainable plans that prescribe the harshest available punishment in response to a government deviation. In this section I show how these considerations motivate an alternative operator whose largest fixed point is the set S which characterizes sustainable plans.

To start let h be any element of CE_{π}^{0}; the reader can interpret h as a "deviation." Let Z be a compact set such that $S \subseteq Z \subseteq \mathscr{W} \times \Omega$. Now define:

$$P(h; Z) = \text{Min } u[f(x)] + v(m) + \beta w' \text{ subject to} \tag{18}$$

$$-x = m(1-h) \tag{19}$$

$$m[u'[f(x)] - v'(m)] \leqslant \beta\theta', \text{ with equality if } m < \bar{m} \tag{20}$$

$$(m, x, w', \theta') \in [0, \bar{m}] \times X \times Z \tag{21}$$

If Z were equal to S, $P(h; Z)$ would be the worst possible SP continuation after a deviation h in CE_{π}^{0}. This notion is extended to allow for punishments that can be supported by pairs of future (w, θ) in sets Z possibly larger than S.

In the above definition, the condition that Z be a subset of S is required mainly to ensure that the set defined by (19)–(21) be nonempty.

Now let:

$$BR(Z) = \text{Max } P(h; Z) \qquad \text{s.t.} \quad h \in CE_{\pi}^{0}.$$

If Z were equal to S, $BR(Z)$ would be the government's "best deviation."

Finally, define:

$\mathbb{E}(Z) = \{(w, \theta) \in \mathscr{W} \times \Omega \mid \text{there is } (m, x, h, w', \theta') \in E \times Z$ s.t. (19)–(20) hold and:

$$w = u[f(x)] + v(m) + \beta w' \tag{22}$$

$$\theta = u'[f(x)](m + x) \qquad \text{and} \tag{23}$$

$$w \geqslant BR(Z)\}. \tag{24}$$

The intuition behind the operator $\mathbb{E}$ should be clear from the observation that, if Z were equal to S, $\mathbb{E}(Z)$ would include all pairs (w, θ) that could be "enforced" by a threat of reverting to the continuation that is least favorable to the government. Hence the operator $\mathbb{E}$ can be seen as a recursive extension of the methods developed in [1].

From the preceding discussion the following properties of $\mathbb{E}$ should be intuitive:

PROPOSITION 11. *Let Z be a compact set such that $S \subseteq Z \subseteq \mathcal{W} \times \Omega$. Then:* (i) *Self generation*: *$Z \subseteq \mathbb{E}(Z)$ implies $Z = S$*; (ii) *factorization*: *$S = \mathbb{E}(S)$.*

Proof. In Appendix.

Hence the operator $\mathbb{E}$ has the key properties of self generation and factorization. As in previous sections, we can derive a lot of mileage from showing that $\mathbb{E}$ has nice properties:

PROPOSITION 12. (i) *$\mathbb{E}$ is monotone in the sense that $S \subseteq Z_1 \subseteq Z_2$ implies $S \subseteq \mathbb{E}(Z_1) \subseteq \mathbb{E}(Z_2)$.* (ii) *If Z is compact and $S \subseteq Z$, then $\mathbb{E}(Z)$ is compact.*

Proof. (i) Let $S \subseteq Z_1 \subseteq Z_2$ be given and suppose that (w, θ) is in $\mathbb{E}(Z_1)$. To show that (w, θ) is in $\mathbb{E}(Z_2)$ it is sufficient to show that $BR(Z_1) \geqslant BR(Z_2)$. This follows from the definition of BR. Hence $\mathbb{E}(Z_1) \subseteq \mathbb{E}(Z_2)$. Now, $S \subseteq \mathbb{E}(Z_1)$ follows by applying the preceding result to $S \subseteq Z_1 \subseteq Z_2$ and noting that $S = \mathbb{E}(S)$. (ii) The proof is easy and left to the reader. ∎

Finally, we can use Propositions 11–12 to obtain an algorithm to compute S as follows. We know that $S \subseteq \mathcal{W} \times \Omega$. Set $Z_0 = \mathcal{W} \times \Omega$ and, for all $n = 1, 2, ..., Z_n = E(Z_{n-1})$. By the preceding results, the sequence $\{Z_n\}$ is a decreasing sequence of compact sets which include S. Hence a plausible conjecture is that $S = Z_\infty \equiv \bigcap_{n=0}^{\infty} Z_n$. This is in fact true, as shown by:

PROPOSITION 13. $S = Z_\infty$.

Proof. See Appendix.

In summary, the approach in this section also provides a useful characterization of the set of sustainable outcomes and yields a successful algorithm for computing it. The operator $\mathbb{E}$ seems somewhat simpler to implement than the operator $\mathbb{D}$ of the previous section.

8. COMPUTATIONAL ISSUES

In order to examine computational issues related to the theory just advanced, this section presents and analyzes a parametric example. Our objective will be to show that implementing the theory is feasible and to illustrate some of the difficulties involved. We have not attempted here to develop efficient and accurate computational algorithms for the theory; we believe that task to be a nontrivial endeavor and better left for future research.

We will focus on the question of computing the set S of SP (w, θ) pairs given functional forms and parameter values for the model of the previous sections. Our choices were motivated by the objective of illustrating the implementation of the theory and are not intended to be necessarily realistic. Our assumptions on functional forms and parameters were:

$$u(c) = 10000 \log c,$$

$$f(x) = 64 - (0.2x)^2,$$

$$v(m) = 40m - m^2/2,$$

$$\beta = 0.9,$$

$$\Pi = [\underline{\pi}, \bar{\pi}] = [0.25, 1.25], \qquad \text{and}$$

$$\bar{m} = 40 = m^f.$$

This parameterization of the model ensures that assumptions [A1]–[A7] are met. As shown above, we assume u to be logarithmic in consumption, and f and v to be quadratic.[14] Maximal feasible output was set at 64 and the satiation level of money was set at 40. The assumption on Π, the range of permitted values of $h_t = M_{t-1}/M_t$, implies that the nominal quantity of money can at most quadruple between periods, and that it can shrink by twenty percent. The assumptions that β is equal to 0.9 and that $\bar{m} = m^f$ were made mostly for simplicity.

With these values, the set of possible values of seigniorage revenue is given by $X = [-30, 10]$; then [A8] is satisfied. Now one can calculate ranges of values of w and θ that are consistent with this parameterization. The representative agent's utility, w, must be bounded above by $\bar{w} = \{u[f(0)] + v(m^f)\}/(1-\beta)$, which is the discounted value of extracting no seigniorage while enjoying the satiation level of real balances. A lower bound for w is in turn given by $\underline{w} = [u[f(\text{Max}\{\bar{x}, \underline{x}\})] + v(0)]/(1-\beta)$, the discounted value of living with the worst possible tax distortions and worthless money. Hence any equilibrium value of w will belong to $\mathscr{W} = [\underline{w}, \bar{w}]$. For our example, $\underline{w} = 144716$ and $\bar{w} = 188618$. Note, for future reference, that the value of the nonmonetary equilibrium is 180618, the value of the constant money supply equilibrium is 187397, and the value of the competitive equilibrium associated with the Friedman rule (i.e., deflation at the rate of time preference) is 188078. Both the constant money

[14] The description of v is almost but not exactly accurate. If v were quadratic on $\mathbb{R}_+$, $v'(0)$ would be finite, contradicting [A3]. To remedy this, in our actual computations we assumed $v(m)$ to be a square root function for m very small; the parameters of this function were adjusted so as to satisfy [A2].

supply rule and the Friedman rule come very close to achieving the maximum feasible utility level $\bar{w}$.

As for θ, recall that $\theta = u'[f(x)](m+x) = u'[f(x)]\, hm$. Since u', h, and m are nonnegative, θ is bounded below by zero. An upper bound for θ is given by $\bar{\theta} \equiv u'[f(\bar{x})]\, \bar{\pi} m^f$; for our example, $\bar{\theta} = 17857$.

The next task is to compute Ω, the set of θ's consistent with a competitive equilibrium. To do this, we can implement Proposition 5, taking $Q_0 = [0, \bar{\theta}]$ and applying $\mathbb{B}$ repeatedly to obtain a decreasing sequence of sets $Q_{n+1} = \mathbb{B}(Q_n)$ which converges to Ω. This procedure presents two main difficulties. The first is that, although Q_0 is a "nice" compact interval, the sets Q_n need not be as well behaved. In particular, those sets may not be convex, which greatly complicates their representation in a computer. To deal with this, we approximated the interval $Q_0 = [0, \bar{w}]$ by 101 equally spaced points. Any subset of Q_0 can be then represented, in the obvious way, by a 101-tuple of ones and zeros, with ones denoting inclusion in the set.

The second difficulty is related to the definition of $\mathbb{B}$. Suppose that an approximation to Q_n, call it $\hat{Q}_n$, is given (by a vector of zeros and ones). The computation of $\hat{Q}_{n+1} = \mathbb{B}(\hat{Q}_n)$ amounts to checking, given any θ in $\hat{Q}_n$, whether there is (m, x, h) in E and θ' in $\hat{Q}_n$ that solve (10)–(12). Given the nonlinearity of (10)–(12), this is a nontrivial task, and we proceeded as follows. We eliminated x by inserting (11) in (10) and (12). Then, the ranges of values of m and h, given by $[0, \bar{m}]$ and Π in the model, were discretized: The interval $[0, \bar{m}]$ was approximated by 121 equally spaced points, and Π by 51 equally spaced points. In other words, the set $[0, \bar{m}] \times \Pi$ was represented by a 121×51 matrix which we will refer to as Egrid for the discussion.

Now, given any θ in $\hat{Q}_n$, a finite search suffices to check whether there is (m, h) in Egrid and θ' in $\hat{Q}_n$ that solve (10)–(12). There is one more detail to deal with. Because of the discretization procedure, it is possible that no (m, h, θ') in Egrid $\times \hat{Q}_n$ solve (10)–(12) exactly, even if there is (m, h, θ') in $E \times Q_n$ that solve (10)–(12). To correct for this, we allowed an element θ of $\hat{Q}_n$ to be an element of $\hat{Q}_{n+1}$ if there was (m, h, θ') in Egrid $\times$ $\hat{Q}_n$, such that (10)–(12) were satisfied approximately. For the example, the maximum margin of (combined) error was set at one tenth of the size of the intervals of the grid for θ.

Summarizing, given an approximation $\hat{Q}_n$ (a vector of zeros and ones) one can compute $\hat{Q}_{n+1}$ by checking, for each nonzero element θ of $\hat{Q}_n$, whether there is an (m, h, θ') in Egrid $\times \hat{Q}_n$ that approximately solves (10)–(12). If there is such (m, h, θ'), θ is kept at one; if not, θ is set to zero. One can iterate on this procedure until convergence to obtain an approximation to Ω; because of the discretization, convergence is guaranteed in a finite number of iterations.

To perform the computations, we wrote a GAUSS program and ran it in my personal computer (a Pentium 200).[15] These are relatively modest resources, in spite of which the computation of Ω was relatively quick, taking less than 10 minutes. However, the amount of computation and the time required to compute Ω seem to increase quite fast as one increases the number of points used to approximate the different sets involved.

The computations converged on a vector whose first 47 elements were ones and the rest zeros. This result implies that Ω can be approximated by the interval $[0, 8399]$. Hence, the procedure revealed that Ω is much smaller than $[0, \bar{\theta}]$. It also suggests that Ω is an interval, although further investigation on this issue seems to be warranted.[16]

With the estimate of Ω in hand (call it $\hat{\Omega}$), we proceeded to compute an approximation to S, the set of all (w, θ)'s consistent with a sustainable plan. To this end one can exploit Proposition 13, taking $Z_0 = \mathscr{W} \times \Omega$ and applying $\mathbb{E}$ recursively to obtain a sequence $Z_{n+1} = \mathbb{E}(Z_n)$ which converges to S. The computation of $\mathbb{E}$ presents essentially the same difficulties as the computation of $\mathbb{B}$, except that the amount of computation required is now much more demanding.

To start the computations one needs a finite approximation to $Z_0 = \mathscr{W} \times \Omega$. Ω was naturally approximated by $\hat{\Omega}$, while $\mathscr{W} = [\underline{w}, \bar{w}]$ was approximated by a grid of 51 equally spaced points. Hence the subsets of Z_0 that emerge in the iterative procedure can be represented by matrices of ones and zeros, with ones denoting inclusion. It will be seen that, for some questions, one would like to study a finer approximation of $\mathscr{W}$. This is where our computational constraints became importantly binding: Even with a grid this coarse, the computation of S took about 24 hours, and we found that the amount of computing time grew very quickly with the fineness of the grid for $\mathscr{W}$. Nevertheless, we were able to take a further step to improve the quality of the computation. My first computations showed conclusively that very low values of w would not be consistent with any sustainable plan. Since the application of Proposition 13 only requires that Z_0 contain the sustainable set S, in a final run we restricted the w values under consideration to the interval $[179837, 188618]$, which was in turn approximated by a grid of 51 points.

The other details related to the computation of S are very similar to those associated with computing Ω and need not be repeated. Figure 1 displays the resulting approximation to S. Only the "relevant" part of the computation is depicted. The algorithm sets entries belonging to the

[15] The GAUSS programs for the calculations of this section are available on request.

[16] There are some theoretical reasons to suspect that Ω may not be an interval in this class of models. For instance, in the model of Obstfeld and Rogoff [19], which is closely related to [4], the variable corresponding to θ assumes a discrete number of values.

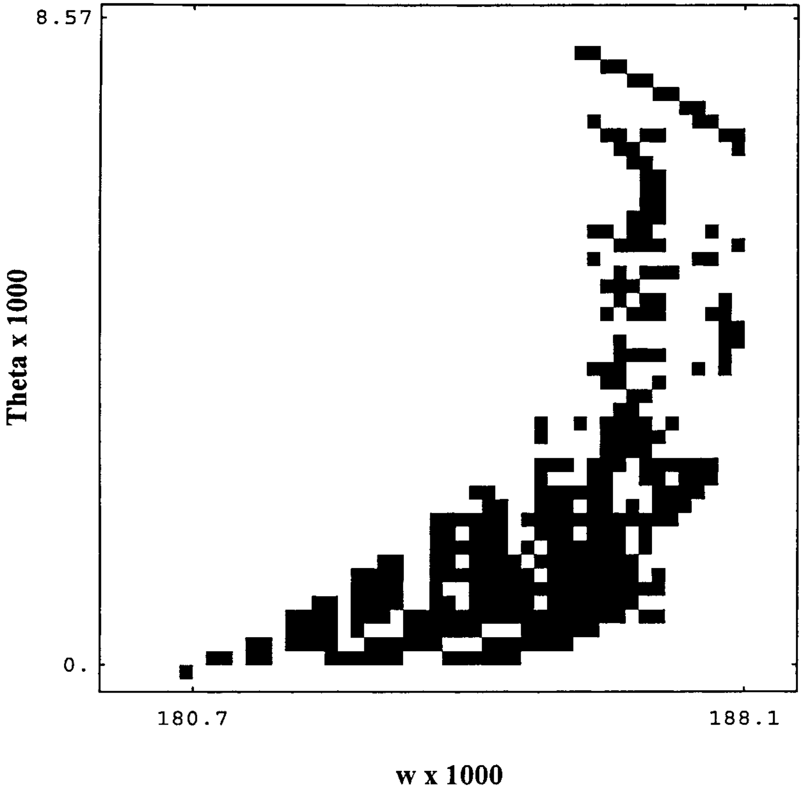

FIG. 1. For the parameterization of the text, the rectangle depicts (w, θ) candidates in the region $[179837, 188618] \times [0, 8399]$. The dark set represents a computed approximation to the sustainable set S.

approximation to S equal to one; they are the black squares in the figure. The remaining, omitted entries were found to be zeros in our calculations and hence need not be displayed. It has to be noted that our procedure starts with $\mathscr{W} \times [0, \bar{\theta}]$ and shrinks to less than one tenth its original size.

Figure 1 shows that the SP values of w are approximated by the interval $[180716, 188091]$. The lower bound is rather close to the value of the nonmonetary equilibrium, and the difference is likely to be due to approximation error. Hence our computation suggests that the nonmonetary equilibrium is the worst sustainable outcome for our example. Perhaps one can arrive at the same conclusion by analytical means;[17] it is nonetheless remarkable that the result emerges directly from the computation.

[17] One can show directly that the nonmonetary equilibrium is a sustainable outcome. It is not obvious, though, that it is the worst one.

On the other hand, the upper bound of the computed S coincides almost exactly with the value of the Friedman rule. Again, even though one may be able to derive this result analytically, it is remarkable that it emerges directly from the computation.

Finally, Fig. 1 suggests that S is not convex. It also suggests that S may not be connected, although it is unclear whether this result is due to our approximation procedure.

Computing sustainable paths for money growth, real balances, output, etc. is now straightforward. We omit the details for brevity and because in this example the result seems to be, simply, that any competitive equilibrium path, whose continuation value is at all times no less than the value of the nonmonetary equilibrium, is sustainable.

Although the results for the example just discussed are quite simple, our discussion (we believe) has been fruitful. We know now that computing an approximation to the whole set of sustainable plans is possible. In this example, it has turned out that the nonmonetary equilibrium seems to be the worst sustainable outcome; that result may not hold for other parameterizations, though, while our procedures are still applicable and deliver the whole sustainable set. One can perhaps criticize the quality of my approximations, but improving them is just a matter of using more powerful computer resources. In any case, we have learned that computational constraints may be binding and, therefore, that developing alternative procedures is an important topic for future research. Progress has been made recently by Conklin and Judd [9] and Cronshaw [11],[18] but much more remains to be done.

9. FINAL REMARKS

This paper has provided recursive methods that yield a complete characterization of the set of sustainable outcomes in Calvo's monetary economy. The recursive characterization yields valuable insights about the set of sustainable outcomes, and suggests algorithms for computing it.

It should be clear that the methods of this paper are applicable to a wide variety of models. The essential requisite seems to be that the set of competitive equilibria should have a recursive structure. This will typically be the case if competitive equilibria can be described by the solution of (possibly stochastic) difference equations. Hence, it is clear that the recursive approach will be applicable to models that have physical state

[18] Both [9] and [11] focus on the problem of computing the fixed point of the Abreu, Pearce, and Stachetti operator. They both assume that the fixed point is a convex set, which simplifies the computer representation and implies a much smaller computational burden.

variables, uncertainty, etc. Indeed Phelan and Stachetti [22] have arrived to a similar conclusion in the context of capital–labor taxation.

The results of this paper reduce the time consistency problem to a question of computing two operators between sets. We implemented a brute force way to deal with this computation that clearly leaves room for improvement. It should be noted that this computing problem is still not very well understood, as witnessed by Conklin and Judd's and Cronshaw's recent attempts to create algorithms for applying the methods of [2]. Hence the development of simple and efficient computational procedures should be a priority of future research.

The methods of this paper may be useful to investigate policy problems under alternative assumptions. For example, they may be adapted to cases in which governments' objectives differ from maximizing public welfare, as assumed by the rapidly expanding literature on political economy.

Finally, the approach in this paper may be adapted to some problems in which information may be imperfect, because of imperfect monitoring for example. However, it is unclear whether other time consistency problems with asymmetric information can be handled with the same approach. These questions are also interesting for future research.

APPENDIX

Proof of Proposition 1. In any competitive equilibrium, $m_t \in [0, \bar{m}]$ by our assumptions on the household's problem. That $h_t \in H$ follows from [A5]. It has already been shown that $x_t \in X$.

(6) follows from the government budget constraint. Finally, straightforward analysis implies that the Euler equation for the consumer is

$$m_t\{u'[f(x_t)] - v'(m_t)\} \leqslant \beta u'[f(x_{t+1})]\, h_{t+1} m_{t+1}, \qquad \text{with equality if} \quad m_t < \bar{m}. \tag{25}$$

Using (5), $h_{t+1} m_{t+1} = m_{t+1} + x_{t+1}$. Inserting this in (25) gives (7).

Conversely, suppose $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ satisfy (6), (7), and $(m_t, x_t, h_t) \in E$, all t. Define $M_t = M_{t-1}/h_t$, $q_t = m_t/M_t$, $c_t = f(x_t)$. Then it is easy to check that the policy $(\mathbf{h}, \mathbf{x})$ and the allocation $(\mathbf{c}, \mathbf{m}, \mathbf{y}, \mathbf{q})$ are a competitive equilibrium. To see this, note that (6) and (7) ensure that, respectively, the government budget constraint and the representative agent's Euler conditions are satisfied. It is then sufficient to prove that the transversality condition for the representative agent holds, that is, that $\beta^t u'[f(x_t)]\, m_t h_t \to 0$ as $t \to \infty$. Now, since E is compact, the continuity of u' and f ensures that

$u'[f(x)]\,mh$ must belong to a compact interval for any (x, m, h) in E. Hence $u'[f(x_t)]\,m_t h_t$ is a uniformly bounded sequence, and $\beta^t u'[f(x_t)]\,m_t h_t$ must indeed converge to zero. ▮

Proof of Corollary 2. CE is a nonempty subset of the compact set E^∞. Let $(\mathbf{m}^n, \mathbf{x}^n, \mathbf{h}^n)$ be a sequence in CE converging to some sequence $(\mathbf{m}, \mathbf{x}, \mathbf{h})$. Since E^∞ is compact, it is closed, and hence $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ belongs to E^∞. By [A1]–[A2] and the fact that (6) and (7) are satisfied for each n we can conclude that $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ must satisfy (6)–(7) as well. Hence $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ belongs to CE. This implies that CE is a closed subset of the compact set E^∞, hence it is compact. ▮

Proof of Proposition 3. For any bounded function $w\colon \Omega \to \mathbb{R}$, let Tw be the sup of $u[f(x)] + v(m) + \beta w(\theta')$ over all $(m, x, h, \theta') \in E \times \Omega$ that satisfy (10)–(12). The first claim of the Proposition is that $w^* = Tw^*$, and that the sup is in fact achieved. To prove this, fix $\theta = \theta_0 \in \Omega$, and let $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ attain the max in (8). Define $\theta_1 = u[f(x_1)](m_1 + x_1)$. Then $(m_0, x_0, h_0, \theta_1)$ satisfies (10)–(12). Hence $Tw^*(\theta_0) \geqslant u[f(x_0)] + v(m_0) + \beta w^*(\theta_1) \geqslant w^*(\theta_0)$. Suppose the inequality is strict. Then there is $(m_0', x_0', h_0', \theta_1')$ in $E \times \Omega$ that satisfies (10)–(12) and such that $u[f(x_0')] + v(m_0') + \beta w^*(\theta_1') > w^*(\theta_0)$. But then there must be a plan $(\mathbf{m}', \mathbf{x}', \mathbf{h}') \in \Gamma(\theta_0)$ whose value is more than $w^*(\theta_0)$, which is a contradiction. Hence $w^* = Tw^*$, and the sup is achieved by $(m_0, x_0, h_0, \theta_1)$.

To prove the second claim, let w be bounded and satisfy (9)–(12). Given any $\theta = \theta_0$ in Ω, define a sequence $(\mathbf{m}, \mathbf{x}, \mathbf{h})$ recursively as follows: If θ_t is given in Ω, choose $(m_t, x_t, h_t, \theta_{t+1})$ in $E \times \Omega$ that satisfies (10)–(12); such a choice is possible by assumption. Clearly $(\mathbf{m}, \mathbf{x}, \mathbf{h}) \in \Gamma(\theta_0)$ and has value $w(\theta_0)$ by the boundedness of w. Hence $w^*(\theta_0) \geqslant w(\theta_0)$. The proof that $w(\theta_0) \geqslant w^*(\theta_0)$ is easy and left to the reader. ▮

Proof of the Existence of a Sustainable Plan. It is easy to prove that there is a sustainable plan in which money has no value; the reader can supply the details. There is also a sustainable plan whose outcome is a constant supply of money. The proof below may be of independent interest.

Let $\hat{m}$ be the real quantity of money associated with zero money growth and no taxes; that is, $\hat{m}$ is the only solution to

$$u'[f(0)](1 - \beta) = v'(\hat{m}). \tag{26}$$

Then we claim that the following is a sustainable plan

$$\sigma_t(\mathbf{h}^{t-1}) = 1$$

$m_t(\mathbf{h}^t) = z(h_t)$, where $z(h_t)$ is the (only) value of $z \in [0, m^f]$ that solves:[19]

$$z\{u'[f(z(h_t - 1))] - v'(z)\} = \beta u'[f(0)]\,\hat{m}. \tag{27}$$

Finally, set $x_t(\mathbf{h}^t) = (h_t - 1)\, m_t(\mathbf{h}^t)$.

Before proceeding, note the intuition behind the candidate sustainable plan. The government's strategy σ prescribes keeping the money supply constant after any history. The allocation rule implicitly defined by (27) states that, after any history $\mathbf{h}^t$, the private sector believes that the money supply will be constant from period $(t+1)$ on. In other words, any deviation from the constant money supply rule is taken to be temporary. Hence, the price level in period t adjusts to ensure that the supply of money in period t is willingly held.

Now we check that the strategy σ and the allocation rule $\alpha_t(\mathbf{h}^t) = (m(\mathbf{h}^t), x(\mathbf{h}^t))$ are in fact a SP. σ is clearly admissible. To check that α is competitive given σ, note that the continuation of σ after any $\mathbf{h}^t$ implies that the money supply will be constant from period t on.

To check that σ is optimal given α, we shall show that the government cannot gain from any one shot deviation from the constant money supply rule, given any $\mathbf{h}^{t-1}$. Then the Principle of Optimality applies and implies that no (finite or infinite) deviation from the constant money supply rule can be profitable.

The allocation rule and the government's strategy imply that, after any one shot deviation, the continuation of (α, σ) deliver the same utility no matter the value of the initial deviation. Hence a one-shot deviation is profitable if and only if there is h_t such that $u[f((h_t - 1)\, m_t] + v(m_t) > u[f(0)] + v(\hat{m})$, i.e., if it improves current utility, with $m_t = z(h_t)$. Now, the definition of $z(h_t)$ implies that $z(h_t)$ is maximized at $h_t = 1$. But then the above inequality cannot hold for any h_t. ∎

Proof of Proposition 11. (i) Suppose $Z \subseteq \mathbb{E}(Z)$ and let $(w, \theta) \in Z$. We shall construct a SP that "delivers" (w, θ) as follows. Set $w_0 = w$, $\theta_0 = \theta$. Consider any period $t \geqslant 1$, and an arbitrary history $\mathbf{h}^{t-1}$. To use an inductive step, assume that $(w_t(\mathbf{h}^{t-1}), \theta_t(\mathbf{h}^{t-1})) \in Z$. By hypothesis, $(w_t(\mathbf{h}^{t-1}), \theta_t(h^{t-1})) \in \mathbb{E}(Z)$; hence there is $(\tilde{m}, \tilde{x}, \tilde{h}, \tilde{w}', \tilde{\theta}')$ in $E \times Z$ s.t. (19)–(20) and (22)–(24) are satisfied. Set $\sigma_t(\mathbf{h}^{t-1}) = \tilde{h}$, and define $(m_t(\mathbf{h}^t), x_t(\mathbf{h}^t), w_{t+1}(\mathbf{h}^t), \theta_{t+1}(\mathbf{h}^t))$ to be equal to $(\tilde{m}, \tilde{x}, \tilde{w}', \tilde{\theta}')$ if $h_t = \tilde{h}$, and equal to any solution to the problem (18)–(21) if $h_t \neq \tilde{h}$ but $h_t \in CE^0_\pi$. If h_t is not in CE^0_π, set $(m_t(\mathbf{h}^t), x_t(\mathbf{h}^t)) = (0, 0)$ and $(w_{t+1}(\mathbf{h}^t), \theta_{t+1}(\mathbf{h}^t)) = (w, \theta)$.

[19] To see that (27) has a unique solution for each h_t in Π, let $T(z; h_t)$ be the LHS of (27). For given $h_t \in \Pi$, $T(\cdot\,; h_t)$ is continuous and strictly increasing in $[0, m^f]$, with $T(0) = 0$ and $T(m^f) = m^f u'[f(m^f(h_t - 1))] > m^f u'[f(0)] > \beta u'[f(0)]\,\hat{m}$. The result follows.

By induction, the strategy $\sigma_t(\mathbf{h}^{t-1})$, and the allocation rule $\alpha_t(\mathbf{h}^t) = (m_t(\mathbf{h}^t), x_t(\mathbf{h}^t))$ are well-defined for all $\mathbf{h}^t$. It can be easily checked that (α, σ) are a SP, and that it "delivers" (w, θ).

(ii) It suffices to show that $S \subseteq \mathbb{E}(S)$. This is easy and left to the reader. ∎

Proof of Proposition 13. By Self generation it suffices to show that $Z_\infty \subseteq \mathbb{E}(Z_\infty)$. Let (w, θ) be in Z_∞. By definition, (w, θ) belongs to $\mathbb{E}(Z_n)$ all n. Hence, for each n, there is $(m^n, x^n, h^n, w'^n, \theta'^n)$ in $E \times Z_n$ that satisfies (19)–(20), (22)–(23), and $w \geqslant BR(Z_n)$. Without loss of generality, assume that $(m^n, x^n, h^n, w'^n, \theta'^n)$ converges to some (m, x, h, w', θ') in $E \times \mathscr{W} \times \Omega$. Clearly (m, x, h, w', θ') satisfies (19)–(20) and (22)–(23). Moreover, it is easily shown that $(w', \theta') \in Z_\infty$.[20]

It remains to show that $w \geqslant BR(Z_\infty)$. Fix an arbitrary n. For $k \geqslant n$, $w^k \geqslant BR(Z_k) \geqslant BR(Z_n)$. Hence $w \geqslant BR(Z_n)$ for all n. Suppose that $w < BR(Z_\infty)$. Then there is an $h \in CE^0_\pi$ such that $BR(Z_\infty) = P(h; Z_\infty) > w$. But this implies a contradiction, because $P(h; Z_n)$ must converge to $P(h; Z_\infty)$ for all $h \in CE^0_\pi$, as shown next.

Suppose that $P(h; Z_n)$ does not converge to $P(h; Z_\infty)$ for some $h \in CE^0_\pi$. Then there is an $\varepsilon > 0$ such that, given any N, there is $n \geqslant N$ such that $P(h; Z_\infty) - P(h; Z_n) > \varepsilon$. This means that for any $k = 1, 2, ...$ there is $n(k)$ and a subsequence $(m^{n(k)}, x^{n(k)}, w'^{n(k)}, \theta'^{n(k)})$ in $[0, \bar{m}] \times X \times Z_{n(k)}$ such that $n(k) \geqslant n(k-1)$, (19)–(21) are satisfied, and $P(h; Z_{n(k)}) = u[f(x^{n(k)})] + v(m^{n(k)}) + \beta w'^{n(k)}$. The subsequence $(m^{n(k)}, x^{n(k)}, w'^{n(k)}, \theta'^{n(k)})$ can be assumed without loss of generality to converge to some (m, x, w', θ') in $[0, \bar{m}] \times X \times Z_\infty$. This (m, x, w', θ') satisfies (19)–(21) and is such that $P(h; Z_\infty) \geqslant u[f(x)] + v(m) + \beta w' + \varepsilon$. But this contradicts the optimality of $P(h; Z_\infty)$. The proof is complete. ∎

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[20] Fix any n. For all $k \geqslant n$, $(w'^k, \theta'^k) \in Z_k \subseteq Z_n$. Hence $(w', \theta') \in Z_n$, all n.

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