Topic 5: International Real Business Cycles

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Contents

The general strategy is to build dynamic general equilibrium models and confront them to the data. The focus of International Real Business Cycles (IRBC) is on the international dimension of the data, economic connections across countries, transmission of aggregate fluctuations.

5.1 Empirical Evidence: Backus Kehoe and Kydland (JPE 1992) or Backus Kehoe and Kydland in Cooley et al (1995)

This focus is expressed in terms of the statistical movements and comovements in aggregate variables. Variables are first "detrended". The most commonly used method is the Hodrick-Prescott filter (1997 JMCB, published very late even though it had a huge impact on the literature...)

Take a series y_t and write it as the sum of a trend and a cycle: $y_t = \tau_t + c_t$. The trend is in stochastic in general, but slow moving. We want to recover $c_t = y_t - \tau_t$. The idea is to minimize the variance of c_t but to attach a penalty to making the trend too volatile (otherwise the obvious solution is $y_t = \tau_t$). This is expressed by 'choosing' the degree of smoothness of the trend, *the smoothness coefficient*. So the trend is obtained by minimizing:

$$\sum_{t=1}^{T} (y_t - \tau_t)^2 + \lambda \sum_{t=1}^{T-1} \left(\Delta^2 \tau_{t+1}\right)^2$$

The rationale for the penalty term is that it forces slow moving first derivative of the trend (by penalizing the second derivative). For quarterly data, it is typical to use $\lambda = 1600$.

The first order condition takes the following form:

$$\tau_t = \left[1 + \lambda (1-L)^2 \left(1 - L^{-1}\right)^2\right]^{-1} y_t$$

In practice, the H-P filter is like a bandpass filter that cuts the frequencies lower than 0.4 (i.e. periods longer than 2.5)

How do we implement the HP filter? Consider the FOC for τ_t (noting that $\Delta^2\tau_{t+1} = \tau_{t+1} - 2\tau_t + \tau_{t-1}$), with $2 \leq t \leq T-2$

$$-y_t + \tau_t - 2\lambda(\tau_{t+1} - 2\tau_t + \tau_{t-1}) + \lambda(\tau_t - 2\tau_{t-1} + \tau_{t-2}) + \lambda(\tau_{t+2} - 2\tau_{t+1} + \tau_t) = 0$$

which we can rewrite

$$c_t = y_t - \tau_t = \lambda(\tau_{t+2} - 4\tau_{t+1} + 6\tau_t - 4\tau_{t-1} + \tau_{t-2})$$

so we can write

$$\mathbf{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_T \end{pmatrix} = [\mathbf{I} + \lambda\mathbf{M}]\,\boldsymbol{\tau}$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & -2 & 1 & & & & & & & \\ -2 & 5 & -4 & 1 & & & & & & \\ 1 & -4 & 6 & -4 & 1 & & & & & \\ & 1 & -4 & 6 & -4 & 1 & & & & \\ & & & & & & & & & \\ & & & & 1 & -4 & 6 & -4 & 1 & \\ & & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & & 1 & -4 & 5 & -2 \\ & & & & & & & 1 & -2 & 1 \end{pmatrix}$$

is a TxT matrix. Given $\mathbf{M}$, we can construct HP filtered data as

$$\mathbf{C} = \mathbf{Y} - \boldsymbol{\tau} = \left[\mathbf{I} - [\mathbf{I} + \lambda\mathbf{M}]^{-1}\right]\mathbf{Y}$$

After all variables are H-P filtered, BKK (1992) show that the following pattern hold (see figure 1 and 2):

1. output is more variable than consumption

2. output is highly autocorrelated

3. productivity is strongly procyclical (defined as $\log z = \log(y/n) - \theta \log(k/n)$ where $\theta = 0.36$.

4. trade balance is strongly countercyclical

5. there are positive comovements in output across countries

6. smaller comovements in consumption

Table 1

Properties of Business Cycles in OECD Economies

	Std. Dev. (%)		Ratio of Standard Deviation to that of y					Autocorr. y	Correlation with Output					
Country	y	nx	c	x	g	n	z	y	c	x	g	nx	n	z
Australia	1.45	1.23	.66	2.78	1.28	.34	1.00	.60	.46	.68	.15	−.01	.12	.98
Austria	1.28	1.15	1.14	2.92	.36	1.23	.84	.57	.65	.75	−.24	−.46	.58	.65
Canada	1.50	.78	.85	2.80	.77	.86	.74	.79	.83	.52	−.23	−.26	.69	.84
France	.90	.82	.99	2.96	.71	.55	.76	.78	.61	.79	.25	−.30	.77	.96
Germany	1.51	.79	.90	2.93	.81	.61	.83	.65	.66	.84	.26	−.11	.59	.93
Italy	1.69	1.33	.78	1.95	.42	.44	.92	.85	.82	.86	.01	−.68	.42	.96
Japan	1.35	.93	1.09	2.41	.79	.36	.88	.80	.80	.90	−.02	−.22	.60	.98
Switzerland	1.92	1.32	.74	2.30	.53	.71	.67	.90	.81	.82	.27	−.68	.84	.93
U.K.	1.61	1.19	1.15	2.29	.69	.68	.88	.63	.74	.59	.05	−.19	.47	.90
U.S.	1.92	.52	.75	3.27	.75	.61	.68	.86	.82	.94	.12	−.37	.88	.96
Europe	1.01	.50	.83	2.09	.47	.85	.98	.75	.81	.89	.10	−.25	.32	.85

Notes: Statistics are based on Hodrick-Prescott filtered data. Variables are: y, real output; c, real consumption; x, real fixed investment; g, real government purchases; nx, ratio of net exports to output, both at current prices; n, civilian employment; z, Solow residual, defined in text. Except for the ratio of net exports to output, statistics refer to logarithms of variables. Data are quarterly from the OECD's *Quarterly National Accounts*, except employment, which is from the OECD's *Main Economic Indicators*. The sample period is 1970:1 to 1990:2.

Figure 1

Table 2

International Comovements in OECD Economies

	Correlation with Same U.S. Variable					
Country	y	c	x	g	n	z
Australia	.51	−.19	.16	.23	−.18	.52
Austria	.38	.23	.46	.29	.47	.17
Canada	.76	.49	−.01	−.01	.53	.75
France	.41	.39	.22	−.20	.26	.39
Germany	.69	.49	.55	.28	.52	.65
Italy	.41	.02	.31	.09	−.01	.35
Japan	.60	.44	.56	.11	.32	.58
Switzerland	.42	.40	.38	.01	.36	.43
United Kingdom	.55	.42	.40	−.04	.69	.35
Europe	.66	.51	.53	.18	.33	.56

Notes: See Table 1.

Figure 2

5.2 Introduction to linearization methods: The stochastic growth model in closed economy

5.2.1 The problem

Equations describing the stochastic growth model:

$$\max_{\{c_t, i_t\}} E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

$$\begin{aligned} y_t &= e^{z_t} k_t^{\alpha} \\ c_t + i_t &\leq y_t \\ k_{t+1} &= (1-\delta) k_t + i_t \\ z_{t+1} &= \rho\, z_t + \epsilon_{t+1} \\ \epsilon &\sim N\left(0, \sigma^2\right) \\ & \quad z_0, k_0 \text{ given} \end{aligned}$$

Remark 1 Here, no trend growth. In models with trend growth, it is necessary to rewrite the model in terms of stationary variables (deflate by the appropriate quantities)

Remark 2 The resource constraint takes the form: $c_t + k_{t+1} \leq y_t + (1-\delta) k_t$. It is equivalent to maximize over the sequence $\{c_t, k_{t+1}\}$ since k_0 is given.

Now we can write the Lagrangian for this problem:

$$\mathfrak{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \lambda_t \left(y_t + (1-\delta) k_t - k_{t+1} - c_t \right) \right]$$

The first order conditions are:

$$\begin{aligned} u'(c_t) &= \lambda_t \\ \lambda_t &= \beta E_t \left[\lambda_{t+1} \left(\alpha \frac{y_{t+1}}{k_{t+1}} + 1 - \delta \right) \right] \end{aligned}$$

Combined, we obtain the standard Euler equation:

$$u'(c_t) = \beta E_t \left[u'(c_{t+1}) (1 + r_{t+1}) \right]$$

where $r_{t+1} = \alpha y_{t+1}/k_{t+1} - \delta$ is the rental rate of capital. The equilibrium is characterized by three conditions:

$$\begin{aligned} \beta E_t \left[u'(c_{t+1}) \left(1 + \alpha \frac{y_{t+1}}{k_{t+1}} - \delta \right) \right] &= u'(c_t) \qquad (5.1) \\ k_{t+1} &= y_t + (1-\delta) k_t - c_t \\ z_{t+1} &= \rho\, z_t + \epsilon_{t+1} \end{aligned}$$

The standard solution method involves linearizing the model around the steady state. This requires that we first characterize the steady state.

5.2.2 The steady state

The steady state corresponds to the case where $\sigma^2 = 0$ and $z_t = z^*$. The equilibrium conditions are then:

$$\begin{aligned} u'(c^*) &= \beta u'(c^*) \left(1 + \alpha e^{z^*} k^{*\alpha - 1} - \delta \right) \\ k^* &= e^{z^*} k^{*\alpha} + (1-\delta) k^* - c^* \\ z^* &= \rho\, z^* \end{aligned}$$

this is a system with three equations and three unknowns (z^*, k^*, c^*). In this case, the system is recursive.and we can solve for:

$$\begin{aligned} z^* &= 0 \\ k^* &= \left(\frac{\beta^{-1}+\delta-1}{\alpha}\right)^{1/(\alpha-1)} \\ c^* &= k^{*\alpha} - \delta k^* \end{aligned}$$

5.2.3 The linearization

The set of equations (5.1) forms a nonlinear dynamic system that characterizes the exact solution to the problem. In general, this exact solution cannot be characterized analytically. Instead, we resort to simulation methods. Some simulation methods provide exact solutions (i.e. the solution to (5.1)). These exact solutions are usually more computationally intensive. Other methods linearize the dynamic system around the steady state and solve the linearized model. This method is usually much faster. However, it is only valid in a neighborhood of the steady state, so only when the shocks are not too large. It also requires that the steady state be well defined (more on this for small open economies later).

Under some reasonable conditions, the solution to the linearized system provides a first-order accurate solution to the exact non-linear solution (see the Hartman Grobman theorem p118 in Perko, Differential Equations and Dynamical Systems)

We linearize by taking a first order expansion of the set of equations (5.1) around $x^* = (c^*, k^*, z^*)'$. Recall that for sufficiently smooth functions, we can write:

$$F(x) = F(x^*) + DF(x^*)(x - x^*) + o\left(||x||^2\right)$$

We now apply this method to the system to obtain:

$$\begin{aligned} E_t\left[\psi_{1c}\hat{c}_{t+1} + \psi_{1k}\hat{k}_{t+1} + \psi_{1z}\hat{z}_{t+1}\right] &= \phi_{1c}\hat{c}_t + \phi_{1k}\hat{k}_t + \phi_{1z}\hat{z}_t \\ \psi_{2c}\hat{c}_{t+1} + \psi_{2k}\hat{k}_{t+1} + \psi_{2z}\hat{z}_{t+1} &= \phi_{2c}\hat{c}_t + \phi_{2k}\hat{k}_t + \phi_{2z}\hat{z}_t \\ \psi_{3c}\hat{c}_{t+1} + \psi_{3k}\hat{k}_{t+1} + \psi_{3z}\hat{z}_{t+1} &= \phi_{3c}\hat{c}_t + \phi_{3k}\hat{k}_t + \phi_{3z}\hat{z}_t + \hat{\epsilon}_{t+1} \end{aligned}$$

in this notation ψ_{ij} denotes the partial derivative of the LHS of equation i with respect to variable j and ϕ_{ij} denotes the same thing for the RHS, evaluated at the steady state. $\hat{y}_t$ denotes the deviation from steady state $y_t - y^*$ (note that $\hat{\epsilon} = \epsilon$ since $\epsilon^* = 0$). In this notation, many coefficients are equal to zero. In fact, we can rewrite:

$$\psi\hat{x}_{t+1} + J\hat{w}_{t+1} = \phi\hat{x}_t + \hat{r}_{t+1}$$

where

$$\begin{aligned} \psi &= \begin{bmatrix} \beta u''(c^*)(1+r^*) & \beta u'(c^*)\alpha(\alpha-1)k^{*(\alpha-2)} & \beta u'(c^*)\alpha k^{*(\alpha-1)} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \phi &= \begin{bmatrix} u''(c^*) & 0 & 0 \\ -1 & 1+r^* & k^{*\alpha} \\ 0 & 0 & \rho \end{bmatrix} \\ J &= \begin{bmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \hat{w}_{t+1} &= E_t\hat{x}_{t+1} - \hat{x}_{t+1} \\ \hat{r}_{t+1} &= \begin{pmatrix} 0 \\ 0 \\ \hat{\epsilon}_{t+1} \end{pmatrix} \end{aligned}$$

(note that this notation allows to write the equations with expectations and the other all together).

Alternatively, this can be rewritten as:

$$\psi \hat{x}_{t+1} = \phi \hat{x}_t + \hat{f}_{t+1}$$

where

$$\hat{f}_{t+1} = \begin{pmatrix} -\psi_1 \hat{w}_{t+1} \\ 0 \\ \hat{\epsilon}_{t+1} \end{pmatrix}$$

This is a linear system with a forcing term $\hat{f}_{t+1}$.

We solve it by premultiplying by ψ^{-1}:

$$\hat{x}_{t+1} = A\hat{x}_t + \psi^{-1}\hat{f}_{t+1} \tag{5.2}$$

where $A = \psi^{-1}\phi$. This is a canonical form for linear difference models. Observe that the first equation involves expectations of future variables (we have subsumed that term inside f).

Remark 3 In some cases, ψ is not invertible. In that case, we have to use a slightly different method.

There are a number of ways to solve it. We follow Blanchard and Kahn (1980, AER). The BK approach characterizes the nature of the solution (saddle path, stable, explosive) depending on the number of roots of A that lie outside the unit circle. More specifically, suppose that the system contains n variables with $n-m$ pre-determined ones and m non-predetermined. A is then a nxn matrix , and BK prove the following:[2]

Proposition 5.2.1 (Blanchard and Kahn)

- *if A has $m' > m$ eigenvalues outside the unit circle, there is no bounded solution for the system;*
- *if A has $m' < m$ eigenvalues outside the unit circle, there is an infinity of solutions that satisfy the system*
- *if A has $m' = m$ eigenvalues outside the unit circle, the system has the saddle path property and there is a unique and bounded solution to the system for each initial condition on the pre-determined variables.*

In our case, the proposition implies that a unique bounded solution exists if and only if A has a unique eigenvalue that lies outside the unit circle since consumption is the *only* non-predetermined variable. One can check that this condition holds. Moreover, since the solution is bounded, it approximates the solution to the exact equilibrium conditions. Since the solution is unique it gives an approximate characterization of the unique equilibrium.

In practice, define the matrix Q with the n eigenvectors of A on the columns and a diagonal matrix Λ with the n eigenvalues (here $n = 3$)

$$Q = \begin{bmatrix} v_{1(1)} & ... & v_{n(1)} \\ & & \\ v_{1(n)} & ... & v_{n(n)} \end{bmatrix}; \ \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & . & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

By definition of the eigenvectors and eigenvalues,

$$Q'A = \Lambda Q'$$

Premultiply (5.2) by Q' to obtain:

$$Q'\hat{x}_{t+1} = \Lambda Q'\hat{x}_t + Q'\psi^{-1}\hat{f}_{t+1}$$

Define $\hat{y}_t = Q'\hat{x}_t$. Each equation now has the form:

$$\hat{y}^i_{t+1} = \lambda_i \hat{y}^i_t + \eta_i \hat{f}_{t+1}$$

[2] Recall that μ is an eigenvalue if there exists a vector v (called and eigenvector associated with μ) such that: $v'A = \mu\ v'$. Equivalently, μ solves $\det(A - \mu I) = 0$ where I is the identity matrix.

An eigenvalue μ lies outside the unit circle if $|\mu| > 1$ where $|\mu|$ is the modulus of μ.

where η_i is the i^{th} row of the matrix $\eta = Q'\psi^{-1}$.

First, consider the unstable root λ_u such that $|\lambda_u| > 1$. Then, we can calculate:

$$E_t \hat{y}_{ut+1} = \lambda_u \hat{y}_{ut}$$

since $E_t \hat{f}_{t+1} = 0$. Inverting, we obtain:

$$\hat{y}_{ut} = \lambda_u^{-1} E_t \hat{y}_{ut+1} \to 0$$

since by construction the solution is bounded and $\left|\lambda_u^{-1}\right| < 1$. This implies that $\hat{y}_{ut} = 0$ at all times. Denoting $v_u = (v_{uz}, v_{uk}, v_{uc})'$ the eigenvector associated with the eigenvalue λ_u, this implies that

$$v_u' \hat{x}_t = 0$$

In other words, there is a linear combination of $\hat{c}_t$, $\hat{k}_t$ and $\hat{z}_t$ that must be satisfied at all times. We can write for instance:

$$\hat{c}_t = c_k \hat{k}_t + c_z \hat{z}_t \tag{5.3}$$

where $c_k = -v_{uk}/v_{uc}$ and $c_z = -v_{uz}/v_{uc}$.

Second, rewrite the equation for y_{ut} :

$$\hat{y}_{ut+1} = \lambda_u \hat{y}_{ut} + \eta_u \hat{f}_{t+1}$$

using the fact that $\hat{y}_{ut} = 0$, we conclude that there is no role for expectation errors (i.e. no sunspots here): $\eta_u \hat{f}_{t+1} = 0$ or:

$$\psi_1 \hat{w}_{t+1} = -\frac{\eta_{u3}}{\eta_{u1}} \hat{\epsilon}_{t+1}$$

this implies that the innovations to x_{t+1} are simply a function of the fundamental shock.[3]

Finally, we complete the characterization by substituting the expression for $\hat{c}_t$ into (5.1) to obtain the capital dynamics:

$$\hat{k}_{t+1} = \left(\begin{bmatrix} A_{22} & A_{23} \end{bmatrix} + A_{21} \left(c_k, c_z \right) \right) \begin{pmatrix} \hat{k}_t \\ \hat{z}_t \end{pmatrix} \tag{5.4}$$

To summarize, the solution takes the form:

$$\begin{aligned} \hat{c}_t &= c_k \hat{k}_t + c_z \hat{z}_t \\ \hat{k}_{t+1} &= k_k \hat{k}_t + k_z \hat{z}_t \\ \hat{z}_{t+1} &= \rho \hat{z}_t + \epsilon_{t+1} \end{aligned}$$

The steps of the linearization can be performed numerically once the parameter values of the model are set. Standard computational packages such as MATLAB or GAUSS will calculate the eigenvectors and eigenvalues.

5.3 The SOE model

We now apply this methodology to solve open economy macro problems.

[3] It is this result that does not hold anymore when $m' < m$: the exectational innovation is not anchored by fundamental shocks anymore, so that sunspots can drive the dynamics of the system.

5.3.1 Without uncertainty

We start with a non stochastic small open endowment economy.

$$\begin{aligned} & \max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ c_t + a_{t+1} & \leq y_t + (1+r) a_t \\ \lim_{T\to\infty} \frac{a_{T+1}}{(1+r)^T} & \geq 0 \\ & a_0 \text{ given} \end{aligned}$$

the second condition is a no-ponzi condition.

The trade balance in this economy is $TB_t = y_t - c_t$ while the current account is $CA_t = TB_t + ra_t$.

The Lagrangian for this problem is

$$\mathfrak{L} = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \lambda_t \left(y_t + (1+r) a_t - a_{t+1} - c_t \right) \right]$$

The first order condition for this problem takes the following form:

$$\begin{aligned} u'(c_t) &= \lambda_t \\ \lambda_t &= \beta(1+r)\lambda_{t+1} \end{aligned}$$

When $\beta(1+r) = 1$, we can solve for the consumption level as:

$$c_0 = \frac{r}{1+r} \left\{ (1+r) a_0 + \sum_{t=0}^{\infty} \frac{y_t}{(1+r)^t} \right\}$$

There are two observations that are important at this stage:

1. this is the intertemporal approach to the current account. It implies that the trade balance is positive in response to temporary positive shocks. In other words, this model predicts a procyclical trade balance. Yet we observe instead countercyclical trade balances (see BKK tables). Need a more complicated story.
2. Consider now an increase in assets at time 0, a_0, by Δ The model implies that consumption increases by $r\Delta$. In other words, the household consumes the interest income on the higher level of assets, but saves the principal (consumes the annuity). So a_t is non stationary: if a increases, it stays permanently higher. Any level of foreign assets is consistent with the steady state. This is a problem if we want to linearize to solve the dynamics.

5.3.2 Extensions

5.3.2.1 Adjustment costs to capital

Needed to get the volatility of investment right. Consider the model with production, without adjustment costs. The resource constraint takes the following form:

$$\begin{aligned} a_{t+1} + c_t + i_t &= (1+r) a_t + z_t k_t^{\alpha} \\ k_{t+1} &= (1-\delta) k_t + i_t \end{aligned}$$

The FOC is:

$$\begin{aligned} \lambda_t &= \beta\lambda_{t+1} \left[1 - \delta + z_{t+1} f'(k_{t+1}) \right] \\ \lambda_t &= \beta\lambda_{t+1} (1+r) \end{aligned}$$

so that

$$z_{t+1}\alpha k_{t+1}^{\alpha-1}=r+\delta$$

and investment is decoupled from saving. This pins down the stock of capital as soon as the capital account is open. This makes the investment dynamics completely trivial in this model:

$$\begin{aligned} k_{t+1} &= \left(\frac{r+\delta}{\alpha z_{t+1}}\right)^{1/(\alpha-1)} \\ i_t &= \left(\frac{r+\delta}{\alpha z_{t+1}}\right)^{1/(\alpha-1)} - (1-\delta)\left(\frac{r+\delta}{\alpha z_t}\right)^{1/(\alpha-1)} \end{aligned}$$

To get interesting investment dynamics, we can add adjustment costs so that

$$\begin{aligned} a_{t+1}+c_t+i_t+\frac{\omega}{2}\frac{(i_t-\delta k^*)^2}{k_t} &= (1+r)\,a_t+zf(k_t) \\ k_{t+1} &= (1-\delta)\,k_t+i_t \end{aligned}$$

The Lagrangean is:

$$\mathfrak{L}=\sum_{t=0}^{\infty}\beta^t\left\{u(c_t)+\lambda_t\left(z_tf(k_t)+(1+r)\,a_t-a_{t+1}-c_t-i_t-\frac{\omega}{2}\frac{(i_t-\delta k^*)^2}{k_t}-q_t\left(k_{t+1}-(1-\delta)\,k_t-i_t\right)\right)\right\}$$

where q_t is the relative price of an installed unit of capital in terms of investment goods (Tobin's q).

The FOC are (setting $\omega=1$ and assuming that $\beta(1+r)=1$):

1. investment (i_t)

$$\lambda_t\left(1+\frac{i_t-\delta k^*}{k_t}-q_t\right)=0$$

$$i_t-\delta k^*=k_t(q_t-1)$$

2. Capital (k_{t+1}):

$$\lambda_tq_t=\beta\frac{1}{2}\lambda_{t+1}\frac{(i_t-\delta k^*)}{k_{t+1}^2}+\beta q_{t+1}\lambda_{t+1}(1-\delta)+\beta\lambda_{t+1}z_{t+1}f'(k_{t+1})$$

3. External asset holdings (a_{t+1}):

$$\lambda_t=\beta(1+r)\,\lambda_{t+1}=\lambda_{t+1}$$

this implies:

$$\begin{aligned} q_t &= \beta\frac{1}{2}\frac{(i_t-\delta k^*)}{k_{t+1}^2}+\beta q_{t+1}(1-\delta)+\beta z_{t+1}f'(k_{t+1}) \\ &= \frac{1}{1-\delta}\sum_{s=t+1}^{\infty}\left(\frac{1-\delta}{1+r}\right)^{s-t}\left[z_sf'(k_s)+\frac{1}{2}\left(\frac{i_s-\delta k^*}{k_s}\right)^2\right]+\lim_{T\to\infty}\left(\frac{1-\delta}{1+r}\right)^Tq_{t+T} \end{aligned}$$

the right hand side is the PDV of future MPK plus the impact on the future adjustment costs.

Assume $\delta=0$. In steady state $q=1$ and $r=zf'(k^*)$. The dynamic system is characterized by the following two equations:

$$\begin{aligned} k_{t+1}-k_t &= k_t(q_t-1) \\ q_{t+1}-q_t &= q_tr-z_{t+1}f'(q_tk_t)-\frac{1}{2}(q_{t+1}-1)^2 \end{aligned}$$

These two equations define a saddle-path equilibrium (to convince yourself, draw the phase diagram, or check Blanchard-Fisher).

5.3.2.2 Adding non tradables.

Same model but now consumption is an index of tradable and non tradable:

$$\begin{aligned} u(c) &= \frac{c^{1-\sigma}}{1-\sigma} \\ c &= \left[\gamma \left(c^T\right)^\rho + (1-\gamma)\left(c^N\right)^\rho\right]^{\frac{1}{\rho}} \end{aligned}$$

with $\rho < 1$. Equilibrium on the nontradable goods market requires $c^N = y^N$. Note that $\epsilon = 1/\sigma$ is the IES while $\eta = 1/(1-\rho)$ is the elasticity of substitution between tradable and non tradable. Assume that $\beta(1+r) = 1$ and that the conditions for a convex problem are satisfied (second order sufficient conditions satisfied). This requires that $u_{11} < 0$ where $u = u\left(c^T, c^N\right)$.

With perfect smoothing, the equilibrium conditions satisfy

$$\lambda = u_{1t}$$

where $\lambda > 0$ is constant because $\beta(1+r) = 1$. Recall from previous discussions that the sign of dc^T/dy^N depends upon the sign of $\epsilon - \eta$.

- When $\eta > \epsilon$, the elasticity of substitution between T and N is high so what matters is intertemporal consumption smoothing. If y^N increases, c^T decreases so as to keep aggregate consumption smooth over time.
- When $\eta < \epsilon$, T and N are complements and an increase in y^N requires an increase in c^T. This can generate a countercyclical trade balance (if increase in output coming from increase in y^N)

5.3.2.3 Stationary asset distribution

There are a number of ways to make a stationary so that we can solve the model by linearization (see Schmitt Grohe and Uribe JIE (2004) for a comparison of these different methods)

1. The interest rate is a decreasing function of net foreign assets:

$$1 + r = 1 + r^* + p(A)$$

   where A denotes the aggregate asset level, taken as given by the agents. In equilibrium $A = a$. The function p satisfies $p'(A) < 0$. For instance, we can use:

$$p(A) = \chi\left(e^{A^* - A_t} - 1\right)$$

   The premium is such that the country faces the world interest rate r^* when $A = A^*$.

2. The discount rate is a decreasing function of consumption and leisure. This makes countries with higher levels of assets more impatient, so that they run down these assets. (Mendoza)

$$\beta = \left(1 + C - \frac{\psi_0}{\psi_1} N^\psi\right)^{-\chi}$$

3. Costs of adjusting the international portfolio: $(a_{t+1} - a_t)^2$

4. Overlapping generations with perpetual youth. This also generates a stationary level of assets since young agents are born without initial assets.

5.3.3 Quantitative evaluation of the SOE model (bond only)

Model includes production, as well as consumption-leisure trade-off and adjustment costs of capital.

$$
\begin{aligned}
& \max_{\{c_t,n_t,k_{t+1},b_{t+1}\}} E_0\left[\sum_{t=0}^{\infty}\beta^t u\left(c_t,n_t\right)\right] \\
c_t+k_{t+1}+a_{t+1} &= y_t+g_t-\frac{\omega}{2}\left(k_{t+1}-k_t\right)^2+\left(1-\delta\right)k_t+\left(1+r^*+p\left(A\right)\right)a_t \\
\lim_{T\to\infty}\frac{a_{T+1}}{\left(1+r\right)^T} &\geq 0 \\
& a_0,\ k_0 \text{ and } z_0 \text{ given} \\
y_t &= f\left(k_t,n_t,z_t\right) \\
z_{t+1} &= \rho z_t+\epsilon_{t+1}
\end{aligned}
$$

$\omega>0$ is needed to calibrate the relative volatility of investment (otherwise in the SOE model, capital would jump to its steady state value instantly). Assume that $\beta\left(1+r\right)=1$.

The equilibrium conditions are:

$$
\begin{aligned}
a_t &= A_t \\
f_n\left(k_t,n_t,z_t\right) &= -\frac{u_{nt}}{u_{ct}} \\
u_{ct} &= \beta E_t u_{ct+1}\left(1+r+p\left(a_{t+1}\right)\right) \\
u_{ct}\left[1+\omega\left(k_{t+1}-k_t\right)\right] &= \beta E_t u_{ct+1}\left[1-\delta+f_{k,t+1}+\omega\left(k_{t+2}-k_{t+1}\right)\right] \\
c_t+k_{t+1}+a_{t+1} &= f_t-\frac{\omega}{2}\left(k_{t+1}-k_t\right)^2+\left(1-\delta\right)k_t+\left[1+r+p\left(A_t\right)\right]a_t
\end{aligned}
$$

Remark 4 Before we proceed, note two differences with the previous setup:

1. the system involves k_{t+2} as well as k_{t+1} and k_t : it is a second order system. This is due to the adjustment cost term.

2. some equations (intratemporal) do not involve $t+1$ variables at all (e.g. first order condition for n_t)

We follow the usual procedure. First, we characterize the steady state, then we linearize around that steady state, and finally we solve the linear system of difference equations.

We assume the following functional forms:

$$
\begin{aligned}
u\left(c_t,n_t\right) &= \frac{1}{1-\sigma}\left[c-\frac{n^{\psi}}{\psi}\right]^{1-\sigma} \\
f\left(k,n,z\right) &= e^z k^{\alpha} n^{1-\alpha} \\
p\left(A\right) &= \chi\left(e^{A^*-A_t}-1\right)
\end{aligned}
$$

This representation of preferences (due to Greenwood, Hercowitz and Huffman (1988), or GHH preferences) guaranties that there is no wealth effect on the labor supply (i.e. that the labor supply remains constant as consumption increases). An often used alternative is to assume KPR preferences (after King Prescott and Rebelo (1983)):

$$
u\left(c_t,n_t\right)=\frac{1}{1-\sigma}\left(c^{\psi}\left(1-n\right)^{1-\psi}\right)^{1-\sigma}
$$

5.3.3.1 Steady State

In steady state, $\epsilon_t = 0$ and we obtain the following equations:

$$\begin{aligned}
a^* &= A^* \\
z^* &= 0 \\
(1-\alpha)\left(\frac{k^*}{n^*}\right)^{\alpha} &= n^{*\psi-1} \\
1 &= \beta\left(1-\delta+\alpha\left(\frac{k^*}{n^*}\right)^{\alpha-1}\right) \\
c^* &= k^*\left(\frac{r^*+(1-\alpha)\delta}{\alpha}\right)+r^*A^*
\end{aligned}$$

The first two equations determine the steady state capital stock and choice of hours:

$$\begin{aligned}
(1-\alpha)\left(\frac{r^*+\delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} &= n^{*\psi-1} \\
\frac{r^*+\delta}{\alpha} &= \left(\frac{k^*}{n^*}\right)^{\alpha-1}
\end{aligned}$$

while the last equation determines consumption.

Note that the trade deficit is equal to

$$\begin{aligned}
tb^* &= f^* - c^* - i^* \\
&= -r^*A^*
\end{aligned}$$

so that the ratio of the equilibrium trade balance to GDP is given by:

$$tby^* = \frac{tb^*}{f^*} = -\frac{rA^*}{f^*}$$

while the current account is:

$$ca^* = tb^* + r^*A^* = 0$$

5.3.3.2 Log-linearization

We log-linearize the system around the steady state. The log linearization proceeds as follows. First, denote $\hat{x}$ the log-deviation of variable x, i.e.: $x_t = x^* e^{\hat{x}_t}$ (equivalently, $\hat{x}_t = \ln(x_t/x^*)$ is the percent deviation from steady state). Now consider a function of two variables x and y, denoted $g(x,y)$. We write a first order approximation as follows:

$$\begin{aligned}
g(x,y) &= g\left(x^* e^{\hat{x}_t}, y^* e^{\hat{y}_t}\right) \\
&= g(x^*,y^*) + \frac{\partial g}{\partial x}(x^*,y^*)\,x^*\hat{x}_t + \frac{\partial g}{\partial y}(x^*,y^*)\,y^*\hat{y}_t
\end{aligned}$$

Taking logs, we obtain:

$$\begin{aligned}
\ln g(x_t,y_t) &= \ln\left(g^* + \frac{\partial g}{\partial x}(x^*,y^*)\,x^*\hat{x}_t + \frac{\partial g}{\partial y}(x^*,y^*)\,y^*\hat{y}_t + O\left(\|\epsilon_t\|^2\right)\right) \\
&= \ln g^* + \eta_{gx}\hat{x}_t + \eta_{gy}\hat{y}_t + O\left(\|\epsilon_t\|^2\right) \\
&\approx \ln g^* + \eta_{gx}\hat{x}_t + \eta_{gy}\hat{y}_t
\end{aligned}$$

where $\eta_{gx} = \frac{\partial g}{\partial x}(x^*,y^*)\,x^*/g^*$ is the elasticity of g wrt x, evaluated at (x^*,y^*).

We deal with k_{t+2} by defining the variable $k'_t = k_{t+1}$. Observe that k'_t is non-predetermined at time t. We also substitute $A_t = a_t$ directly.

The state variables are (k_t, a_t, z_t) while the control variables (non-predetermined) are c_t, n_t, k'_t. We can write the system as:

$$\begin{aligned}
z_{t+1} &= \rho z_t + \epsilon_{t+1} \\
c_t + k_{t+1} + a_{t+1} &= f_t - \frac{\omega}{2}\left(k_{t+1} - k_t\right)^2 + (1-\delta) k_t + \left[1 + r + p\left(a_t\right)\right] a_t \\
u_{ct} &= \beta E_t u_{ct+1}\left(1 + r + p\left(a_{t+1}\right)\right) \\
u_{ct}\left[1 + \omega\left(k_{t+1} - k_t\right)\right] &= \beta E_t u_{ct+1}\left[1 - \delta + f_{k,t+1} + \omega\left(k'_{t+1} - k'_t\right)\right] \\
e^{z_t} k_t^{\alpha} n_t^{1-\alpha} &= n_t^{\psi - 1} \\
k_{t+1} &= k'_t
\end{aligned}$$

where we note that the last equation is an auxiliary equation so that we can include k_{t+2} in the equations. This is a system of 6 equations and 6 unknowns. We linearize it as follows (note that we are linearizing, not log-linearizing, in z, since $z^* = 0$) :

$$\begin{aligned}
z_{t+1} &= \rho z_t + \epsilon_{t+1} \\
\psi_{yk}\hat{k}_{t+1} + \psi_{ya}\hat{a}_{t+1} &= \phi_{yc}\hat{c}_t + \phi_{yn}\hat{n}_t + \phi_{yk}\hat{k}_t + \phi_{ya}\hat{a}_t + \phi_{yz} z_t \\
\psi_{cc}\hat{c}_{t+1} + \psi_{cn}\hat{n}_{t+1} + \psi_{ca}\hat{a}_{t+1} &= \phi_{cc}\hat{c}_t + \phi_{cn}\hat{n}_t \\
\psi_{kk}\hat{k}_{t+1} + \psi_{kc}\hat{c}_{t+1} + \psi_{kn}\hat{n}_{t+1} + \psi_{kk'}\hat{k}'_{t+1} + \psi_{kz} z_{t+1} &= \phi_{kc}\hat{c}_t + \phi_{kk}\hat{k}_t + \phi_{kk'}\hat{k}'_t \\
z_t + \alpha\hat{k}_t + (1-\alpha)\hat{n}_t &= (\psi - 1)\hat{n}_t \\
\hat{k}_{t+1} &= \hat{k}'_t
\end{aligned}$$

where the various coefficients correspond to the elasticities of the log-linearization
[For instance, $\psi_{kc} = -\sigma c^* / \left(c^* - n^{*\psi}/\psi\right), ...$]

If we define $\hat{x}_t = \left(\hat{k}_t, \hat{a}_t, \hat{z}_t, \hat{c}_t, \hat{n}_t, \hat{k}'_t\right)'$, we can write the system as:

$$\psi \hat{x}_{t+1} = \phi \hat{x}_t + f_{t+1}$$

where $\hat{f}_{t+1}$ takes the following form:

$$f_{t+1} = \begin{pmatrix} \epsilon_{t+1} \\ -\psi_j \hat{w}_{t+1} \\ -\psi_{j'} \hat{w}_{t+1} \\ 0 \end{pmatrix}$$

contains expectational error terms of the form $\psi_j \hat{w}_{t+1} = \psi_j\left(E_t \hat{x}_{t+1} - \hat{x}_{t+1}\right)$ in the equations j that contain expectational terms (the two euler equations)

As was mentioned before, it is often the case that the matrix ψ is not invertible, so that the method proposed in the simple stochastic growth model does not apply any longer. In that case, we have to use slightly more powerful linear algebra results. This is the case when there is an *intratemporal* equilibrium condition (e.g. a leisure-consumption trade-off) since the corresponding row in the linearized system does not involve time $t+1$ variables.

Here is how we proceed in that case:

1. order variables and equations so that:
   (a) $\hat{x}_t$ lists first the predetermined variables (already done here)
   (b) *Intertemporal* conditions come first, *intratemporal* conditions below (already done, see above)
   (c) Euler equations are placed last in the set of intratemporal equations (already done, see above)

2. Perform a *Generalized Complex Schur Decomposition* (or QZ decomposition). Given ψ and ϕ, this returns Q, Z, S and T such that:

$$\begin{aligned}\psi &= QSZ^H \\ \phi &= QTZ^H\end{aligned}$$

where Q and Z are unitary matrices (i.e. matrices such that $Q^H Q = I$, if Q is real, it is an orthogonal matrix) and S and T are upper triangular. Q^H denotes the complex transpose of Q.[4] The dynamic system can then be rewritten as:

$$QSZ^H \hat{x}_{t+1} = QTZ^H \hat{x}_t + \hat{f}_{t+1}$$

The generalized eigenvalues of ψ and ϕ are equal to the diagonal elements of T (t_{ii}) divided by the diagonal elements of S (s_{ii}).[5] The generalized complex Schur decomposition can be obtained in a number of ways. In Matlab, the qz M-file computes the generalized Shur decomposition. In GAUSS, one can use either Paul Soderlind routines (available at http://www.hhs.se/personal/PSoderlind/). One can also use the lapgschur instruction (GAUSS 4.0 and beyond) although Lapgschur will not provide the necessary sorting (see below).

3. Order the decomposition so that the stable generalized eigenvalues come first (again, this is performed automatically in P. Soderlind routine; in MATLAB, you can use Chris Sims' qzdiv and qzswitch M-files). In general these programs require a cut-off value (to decide what generalized eigenvalues are stable or unstable). In most cases the value for the cut-off should be 1. The final result is a set of matrices Q, Z, S and T so that the generalized eigenvalues t_{ii}/s_{ii} are increasing in norm.

4. Count the number of *stable* eigenvalues, i.e. such that $\|t_{ii}/s_{ii}\| < 1$ (call it n_s) and define $n_u = n_{var} - n_s$ where n_{var} is the number of variables in the system (in our example $n_{var} = 6$). The system is saddle point stable, iff n_s equals the number of pre-determined variables. [This is the equivalent of the Blanchard and Kahn result]. In our case, you can check that $n_s = 3$.

How do we characterize the dynamics? This is easy when $n_s \geq n_u$.
Start by premultiplying the system by $Q^H = Q^{-1}$:

$$SZ^H \hat{x}_{t+1} = TZ^H \hat{x}_t + Q^H \hat{f}_{t+1}$$

Now define $\hat{s}_t$ and $\hat{u}_t$ such that:

$$\begin{pmatrix} \hat{s}_t \\ \hat{u}_t \end{pmatrix} = Z^H \hat{x}_t$$

where $\hat{s}_t$ has dimension n_s. This means that we partition the $n_{var}x1$ vector $Z^H \hat{x}_t$ into n_s and n_u elements. This implies:

$$S \begin{pmatrix} \hat{s}_{t+1} \\ \hat{u}_{t+1} \end{pmatrix} = T \begin{pmatrix} \hat{s}_t \\ \hat{u}_t \end{pmatrix} + Q^H \hat{f}_{t+1}$$

Now partition conformably S and T to $\hat{s}_t$ and $\hat{u}_t$ to obtain (recall that $S_{21} = T_{21} = 0$ since S and T are upper triangular and $n_s \geq n_u$)

$$\begin{aligned}S_{11}\hat{s}_{t+1} + S_{12}\hat{u}_{t+1} &= T_{11}\hat{s}_t + T_{12}\hat{u}_t + Q_{11}^H \hat{f}_{st+1} + Q_{12}^H \hat{f}_{ut+1} \\ S_{22}\hat{u}_{t+1} &= T_{22}\hat{u}_t + Q_{21}^H \hat{f}_{st+1} + Q_{22}^H \hat{f}_{ut+1}\end{aligned}$$

where I have partitioned $\hat{f}_{t+1}$ and Q^H conformably as well.

Now since the last n_u general eigenvalues are *unstable*, the only possible solution is $\hat{u}_t = 0$ (the argument is the same as before: iterate and take expectations).

[4] This implies that Q and Z are invertible and that $Q^{-1} = Q^H$, $Z^{-1} = Z^H$.

[5] The generalized eigenvalue problem for a pair of matrices (A, B) is the problem of finding the pairs (α_k, β_k) and the vectors x_k such that :

$$\beta_k A x_k = \alpha_k B x_k$$

The scalars $\lambda_k = \alpha_k/\beta_k$ are called the *generalized eigenvalues.* If $\beta_k = 0$, then λ_k is set to ∞.

Parameters	σ	ψ	α	ω	r	δ	ρ	σ_ϵ^2	A^*	χ
Values	2	1.45	0.32	0.028	0.04	0.1	0.42	0.0129	-0.7442	0.000742

Table 5.1 Parameters of the Schmitt-Grohe and Uribe paper

Substituting $\hat{u}_t = 0$, the first equation becomes:

$$S_{11}\hat{s}_{t+1} = T_{11}\hat{s}_t + Q_{11}^H \hat{f}_{st+1} + Q_{12}^H \hat{f}_{ut+1}$$

while the second equation imposes that (assuming that Q_{22}^H is invertible):

$$\hat{f}_{ut+1} = -\left(Q_{22}^H\right)^{-1} Q_{21}^H \hat{f}_{st+1}$$

Finally, $\hat{u}_t = 0$ implies:

$$\begin{pmatrix} Z_{11} \\ Z_{21} \end{pmatrix} \hat{s}_t = \begin{pmatrix} \hat{x}_{st} \\ \hat{x}_{ut} \end{pmatrix}$$

Suppose that Z_{11} is invertible. Then,

$$\hat{s}_t = Z_{11}^{-1}\hat{x}_{st}$$

and we can substitute to obtain the **law of motion of the state variables** (premultiplying by S_{11}^{-1} and by Z_{11}):

$$\begin{aligned} \hat{x}_{st+1} &= Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}\hat{x}_{st} + Z_{11}S_{11}^{-1}\left(Q_{11}^H\hat{f}_{st+1} + Q_{12}^H\hat{f}_{ut+1}\right) \\ \hat{x}_{st+1} &= Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}\hat{x}_{st} + Z_{11}S_{11}^{-1}\left(Q_{11}^H - Q_{12}^H\left(Q_{22}^H\right)^{-1}Q_{21}^H\right)\hat{f}_{st+1} \end{aligned}$$

The **law of motion for the controls** (non predetermined variables) satisfies:

$$\begin{aligned} \hat{x}_{ut} &= Z_{21}\hat{s}_t \\ &= Z_{21}Z_{11}^{-1}\hat{x}_{st} \end{aligned}$$

This completes the analysis.[6]

See the Schmitt-Grohe and Uribe simulations for impulse response functions.

SGU's parameters are reported in Table 5.1. χ is selected to match the variability of the current account from the data.

The model's statistics are reported in figure 3 and the impulse responses in figure 4. We observe that the model is able to reproduce a number of stylized facts (the data is for Canada, a SOE)

1. ranking of volatilities (ascending order): consumption, output and investment
2. component of aggregate demand and hours are procyclical
3. trade balance slightly negative or close to zero (note the role of investment here).

But the model overestimate correlation output and hours (1 in the model, only 0.8 in the data).

As for the impulse reponse functions to a productivity shock, the model predicts that consumption and investment increase, so that the country runs initially a trade deficit. Hours increase after the productivity shock. This is in part because there is no wealth effect on labor supply. With KPR preferences, the response of hours would be smaller as the increase in wealth and consumption would lead to an increase in leisure too.[7]

[6] Note that this is not the only method for solving this problem. Other common methods are: (a) a linear quadratic approach (similar to Kydland and Prescott), (b) an undetermined coefficients methods (see Christiano: http://pubweb.nwu.edu/~rdi611/larry/main.htm for details).

[7] KPR preferences take the following form:

$$u(c,n) = \left(c^\psi (1-n)^{1-\psi}\right)^{1-\sigma} / (1-\sigma)$$

You can check that

$$-\frac{u_n}{u_c} = \frac{1-\psi}{\psi}\frac{c}{1-n}$$

so that the labor supply changes when consumption changes.

Observed and implied second moments

	Data	Model 1	Model 1a	Model 2	Model 3	Model 4
Volatilities:						
std(y_t)	2.8	3.1	3.1	3.1	3.1	3.1
std(c_t)	2.5	2.3	2.3	2.7	2.7	1.9
std(i_t)	9.8	9.1	9.1	9	9	9.1
std(h_t)	2	2.1	2.1	2.1	2.1	2.1
std$\left(\frac{tb_t}{y_t}\right)$	1.9	1.5	1.5	1.8	1.8	1.6
std$\left(\frac{ca_t}{y_t}\right)$		1.5	1.5	1.5	1.5	
Serial correlations:						
corr(y_t, y_{t-1})	0.61	0.61	0.61	0.62	0.62	0.61
corr(c_t, c_{t-1})	0.7	0.7	0.7	0.78	0.78	0.61
corr(i_t, i_{t-1})	0.31	0.07	0.07	0.069	0.069	0.07
corr(h_t, h_{t-1})	0.54	0.61	0.61	0.62	0.62	0.61
corr$\left(\frac{tb_t}{y_t}, \frac{tb_{t-1}}{y_{t-1}}\right)$	0.66	0.33	0.32	0.51	0.5	0.39
corr$\left(\frac{ca_t}{y_t}, \frac{ca_{t-1}}{y_{t-1}}\right)$		0.3	0.3	0.32	0.32	
Correlations with output:						
corr(c_t, y_t)	0.59	0.94	0.94	0.84	0.85	1
corr(i_t, y_t)	0.64	0.66	0.66	0.67	0.67	0.66
corr(h_t, y_t)	0.8	1	1	1	1	1
corr$\left(\frac{tb_t}{y_t}, y_t\right)$	−0.13	−0.012	−0.013	−0.044	−0.043	0.13
corr$\left(\frac{ca_t}{y_t}, y_t\right)$		0.026	0.025	0.05	0.051	

Note. The first column was taken from Mendoza (1991). Standard deviations are measured in percent per year.

Figure 3 Schmitt-Grohe and Uribe

Finally, the predictions in terms of second moments are very similar regardless of the method adopted to stationarize the economy (this is the main point of the SGU paper). The only exception is when markets are complete. Note that the complete market case is also stationary since the claims are traded at time 0 and after that relative wealth remains unchanged. In that case, consumption is smoother and the trade balance is more procyclical. Still, observe how stationary incomplete markets and complete markets are similar.

See http://www.econ.duke.edu/~uribe/closing.htm for some MATLAB code.

5.4 The LOE model (incomplete)

The large economy model typically assumes that there are two countries, and that the bonds are in zero net supply.

5.4.1 The setup

We borrow the set-up from Kehoe and Perri (2002). There are two countries, $i = 1, 2$. Each country has an infinitely lived representative agent. The countries produce the same good and they have identical preferences. The production process is similar in both countries, but labor input and productivity shocks are local.

In each period, the world experiences one of many finite events s_t. Define $s^t = (s_0, s_1, ..., s_t)$ for the world history up to and including time t. We endow s^t with the probability $\pi(s^t)$, as of time 0 (by convention,

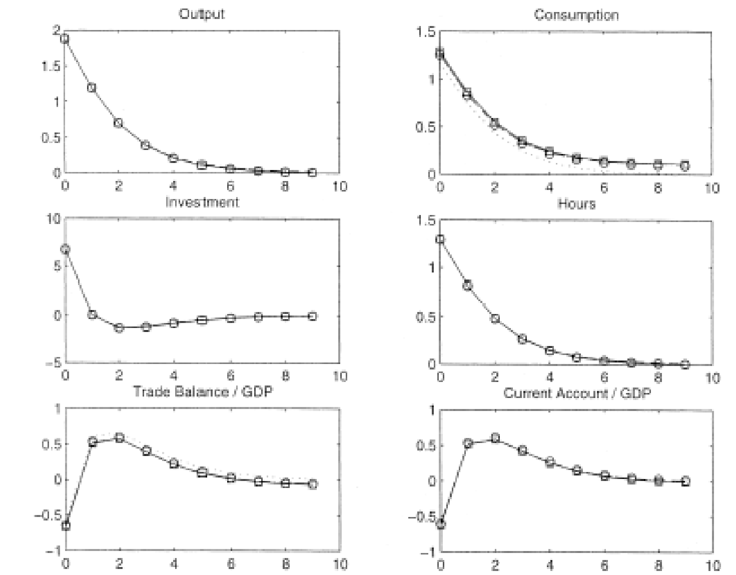

Fig. 1. Impulse response to a unit technology shock in Models 1–5. Note. Solid line: Endogenous discount factor model; Squares: Endogenous discount factor model without internalization; Dashed line: Debt-elastic interest rate model; Dash–dotted line: Portfolio adjustment cost model; Dotted line: complete asset markets model; Circles: Model without stationarity inducing elements.

Figure 4 Schmitt-Grohe and Uribe

$\pi\left(s^0\right)=1$). We can also define $\pi\left(s^T/s^t\right)$ as the probability of history $(s_{t+1}, s_{t+2}, ..., s_T)$ as of time t, given history s^t. This is given by $\pi\left(s^T/s^t\right)=\pi\left(s^T\right)/\pi\left(s^t\right)$.

In each period, a single good is produced in each country. The initial amount of capital in country i is denoted $k_i\left(s^{t-1}\right)$ and the amount of labor engaged in production is $l_i\left(s^t\right)$. The notation makes clear that the capital stock in period t is chosen in period $t-1$ given history s^{t-1}. Output is given by $y_i\left(s^t\right)=F\left(k_i\left(s^{t-1}\right), A_i\left(s^t\right) l_i\left(s^t\right)\right)$ where $A_i\left(s^t\right)$ denotes the labor-neutral country specific productivity shock.

Preferences take the following form:

$$\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi\left(s^t\right)U\left(c_i\left(s^t\right), l_i\left(s^t\right)\right) \tag{5.5}$$

where the summation is over periods and states of the world. Finally, the resource constraint of the world is:

$$\sum_{i=1,2}\left[c_i\left(s^t\right)+k_i\left(s^t\right)\right]=\sum_{i=1,2}\left[y_i\left(s^t\right)+(1-\delta)k_i\left(s^{t-1}\right)-\phi k_i\left(s^{t-1}\right)\left(\frac{x_i\left(s^t\right)}{k_i\left(s^{t-1}\right)}-\delta\right)^2\right] \tag{5.6}$$

The last term represents the effect of the adjustment costs on capital.

The bond economy assumes that there is a one-period risk free bond in zero net supply. Define $q\left(s^t\right)$ as the time t price of the risk free bond (that pays 1 next period) and denote $b_i\left(s^t\right)$ the bond holdings of country i. The budget constraint becomes:

$$c_i\left(s^t\right)+k_i\left(s^t\right)+q\left(s^t\right)b_i\left(s^t\right)=w_i\left(s^t\right)l_i\left(s^t\right)+r_i\left(s^t\right)k_i\left(s^{t-1}\right)+b_i\left(s^{t-1}\right)-\phi k_i\left(s^{t-1}\right)\left(\frac{x_i\left(s^t\right)}{k_i\left(s^{t-1}\right)}-\delta\right)^2 \tag{5.7}$$

where w_i and r_i are respectively the wage and the return on domestic capital. The representative agent of country i solves the following problem.

($\mathcal{P}_i$): Maximize (5.5) subject to (5.7) taking the wage and the price processes as given, and imposing the following no-ponzi condition:

$$\lim_{T\to\infty} \beta^T q\left(s^T\right) b_i\left(s^T\right) = 0$$

In addition, market must clear:

1. market for goods (5.6)
2. market for labor:
$$A_i\left(s^t\right) F_l\left(k_i\left(s^{t-1}\right), A_i\left(s^t\right) l_i\left(s^t\right)\right) = w_i\left(s^t\right)$$
3. market for domestic capital:
$$1 + F_k\left(k_i\left(s^{t-1}\right), A_i\left(s^t\right) l_i\left(s^t\right)\right) - \delta = r_i\left(s^t\right)$$
4. the market for the international bond:
$$\sum_{i=1,2} b_i\left(s^t\right) = 0$$

Remark 5 The model with two countries and incomplete markets *also suffers from a unit root*. Although the world interest rate will now increase when one country's willingness to borrow increases (limiting the extent to which borrowing takes place in equilibrium), it is still the case that transitory shocks have permanent effects on the wealth distribution. Think about a positive productivity shock at home. This increases domestic wealth permanently. So the linearization methods will also require that the problem be 'stationarized' using one of the standard methods. One possibility is to add very small costs of bond holdings. Another option is to 'assume the problem away' and linearize around the initial debt/output ratio and hope that the shocks will not be too big. But one should keep in mind that this amounts to having one too many unstable roots.

Remark 6 The economy exhibits wealth effects since shocks are smoothed by adjusting wealth levels permanently. Whether this will deliver very different results from the complete market model depends upon the nature of the shocks. Baxter and Crucini (1995) show that when productivity shocks are either transitory or correlated across countries (so that the shocks are symmetric) then a bond only economy is sufficient to quasi replicate the complete market allocation.

Writing the Lagrangian for the home problem and maximizing over the choices of $c_i\left(s^t\right), l_i\left(s^t\right), k_i\left(s^t\right)$ and $b_i\left(s^t\right)$, the following equilibrium conditions obtain (in the case without adjustment costs):

$$\begin{aligned} L &= \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi\left(s^t\right)\left[U\left(c_i\left(s^t\right), l_i\left(s^t\right)\right) + \right. \\ &\quad \left. + \lambda_i\left(s^t\right)\left(w_i\left(s^t\right) l_i\left(s^t\right) + r_i\left(s^t\right) k_i\left(s^{t-1}\right) + b_i\left(s^{t-1}\right) - c_i\left(s^t\right) - k_i\left(s^t\right) - q\left(s^t\right) b_i\left(s^t\right)\right)\right] \end{aligned}$$

We obtain:

$$\begin{aligned} U_c\left(s^t\right) &= \lambda_i\left(s^t\right) \\ U_n\left(s^t\right) &= -\lambda_i\left(s^t\right) w_i\left(s^t\right) \\ \lambda_i\left(s^t\right) q\left(s^t\right) &= \beta \sum_{s^{t+1}|s^t} \pi\left(s^{t+1}|s^t\right)\lambda_i\left(s^{t+1}\right) \\ \lambda_i\left(s^t\right) &= \sum_{s^{t+1}|s^t} \beta\pi\left(s^{t+1}/s^t\right)\lambda_i\left(s^{t+1}\right) r_i\left(s^{t+1}\right) \end{aligned}$$

These are simply the usual equilibrium conditions. The model's state variables are: k_i, A_i and b_i while the control variables are c_i, l_i and q, for a total of 11 variables. There are 11 equilibrium conditions, rewritten as follows:

$$\begin{aligned}
U_n\left(s^t\right) &= -U_c\left(s^t\right) A_i\left(s^t\right) F_l\left(k_i\left(s^{t-1}\right), A_i\left(s^t\right) l_i\left(s^t\right)\right) \\
U_c\left(s^t\right) q\left(s^t\right) &= \beta \sum_{s^{t+1}|s^t} \pi\left(s^{t+1}|s^t\right) U_c\left(s^{t+1}\right) \\
U_c\left(s^t\right) &= \sum_{s^{t+1}|s^t} \beta\pi\left(s^{t+1}/s^t\right) U_c\left(s^{t+1}\right)\left[1-\delta+F_k\left(k_i\left(s^{t-1}\right), A_i\left(s^t\right) l_i\left(s^t\right)\right)\right] \\
c_i\left(s^t\right)+k_i\left(s^t\right)+q\left(s^t\right) b_i\left(s^t\right) &= w_i\left(s^t\right) l_i\left(s^t\right)+r_i\left(s^t\right) k_i\left(s^{t-1}\right)+b_i\left(s^{t-1}\right) \\
b_1\left(s^t\right)+b_2\left(s^t\right) &= 0 \\
\begin{pmatrix} \log A_1\left(s^t\right) \\ \log A_2\left(s^t\right) \end{pmatrix} &= \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}\begin{pmatrix} \log A_1\left(s^{t-1}\right) \\ \log A_2\left(s^{t-1}\right) \end{pmatrix}+\begin{pmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{pmatrix}
\end{aligned}$$

The standard method can then be applied to linearize the system around the equilibrium.

The parameters are in figure 5. Quantitative results and impulse responses in table 6–9.

The results indicate that the main puzzles (consumption correlation, cross country correlation of investment and employment, volatility of investment and trade balance) remain largely identical in the bond only economy as in the complete market economy. Looking at the impulse response to a productivity shock, we see:

1. a decline in foreign output, foreign investment and foreign labor
2. an increase in domestic and foreign consumption
3. a trade deficit (a good thing)
4. small deviations from perfect risk sharing.

5.4.2 The endogenous borrowing constraint version (Kehoe and Perri 2002).

Kehoe and Perri propose an alternative model where borrowing is endogenously constrained by enforcement constraints. The key idea is that the foreign country cannot force the home country to repay (this is a central idea in the literature on sovereign borrowing), so any level of borrowing must be sustained in equilibrium by the threat of some punishment. The punishment that is considered here is that the country that fails to repay will not have access to international borrowing any more (financial autarky).

The extent to which a country will be able to borrow will then depend upon how severe a punishment autarky would be. A direct implication is that the more a country receives positive shocks, the harder it will be for that country to borrow. The reason is that better productivity (especially when productivity is persistent) make autarky relatively more attractive.

Consider now the effect of a positive productivity shock. In the bond only economy, this leads to an increase in investment at home, as domestic investment is more productive. In the enforcement economy, a positive productivity shock may lead to a decline in investment and a trade surplus. The reason? It becomes relatively harder for the country to borrow, and that dampens the investment response . One possible interpretation is that the country needs to 'post a bond' by investing abroad, even if domestic productivity is high.

The model is the same as before, except that countries must satisfy the following *enforcement constraints*:

$$\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \pi\left(s^r|s^t\right) U_i\left(s^r\right) \geq V_i\left(k_i\left(s^{t-1}\right), s^t\right) \tag{5.8}$$

TABLE 1. Parameter values

Experiments		Parameters
Baseline Experiments		
	Preferences	$\beta = .99, \quad \sigma = 2, \quad \gamma = .36$
	Technology	$\alpha = .36, \quad \delta = .025$
	Productivity shocks	$a_1 = .95, a_2 = 0$
		$\text{var}(\varepsilon_1) = \text{var}(\varepsilon_2) = .007^2$, $\text{corr}(\varepsilon_1, \varepsilon_2) = .25$
Sensitivity Experiments		
	Adjustment costs*	$\phi = .6$
	High persistence	$a_1 = .99, a_2 = 0$
	High spillover	$a_1 = .85, a_2 = .15$
	BKK	$a_1 = .906, a_2 = .088$

*In the other sensitivity analysis experiments, the adjustment cost parameter in the bond economy is set to match the relative volatility of investment in the data.

Figure 5 Kehoe-Perri

where V_i represents the value of autarky onward and solves the following problem:

$$\begin{aligned} V_i\left(k_i\left(s^{t-1}\right), s^t\right) &= \max \sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \pi\left(s^r | s^t\right) U_i\left(s^r\right) \\ & \quad s.t. \\ c_i\left(s^r\right) + k_i\left(s^r\right) &\leq y_i\left(s^r\right) + (1-\delta) k_i\left(s^{r-1}\right) \end{aligned}$$

Kehoe and Perri's model allows for a full menu of contingent claims, but impose the enforcement constraints. In fact, given that all the state contingent claims are available, the competitive equilibrium will be equivalent to the solution to the constrained planner problem:

$$\max \sum_{i=1,2} \sum_{t=0}^{\infty} \sum_{s^t} \lambda_i \beta^t \pi\left(s^t\right) U_i\left(s^t\right)$$

subject to the ressource constraint (5.6) as well as the enforcement constraints (5.8).

In general, this is a difficult problem since the autarky values V_i at future dates depend on the consumption and investment choices today. So future decision variables enter the current enforcement constraints and direct dynamic programming techniques are not going to work.

Instead, Kehoe and Perri extend the work of Marcet and Marimon (1999) and show that we can introduce a new state variable that captures "promised utility". With this additional state, recursive contracts can be derived.

Write $\beta^t \pi\left(s^t\right) \mu_i\left(s^t\right)$ the Lagrange multiplier on the enforcement constraint at time t for country i. The trick is to realize that the contribution of the enforcement constraints to the Lagrangian

$$\beta^t \pi\left(s^t\right) \mu_i\left(s^t\right) \left[\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \pi\left(s^r | s^t\right) U_i\left(s^r\right) - V_i\left(k_i\left(s^{t-1}\right), s^t\right)\right]$$

TABLE 2. Business cycle statistics: Baseline parameters

Statistic	Data	Economy with: *No Adjustment Costs* — Complete Markets	Bond	Enforcement	*Adjustment Costs* — Complete Markets	Bond
Volatility						
% Standard deviations						
GDP	1.72 (.20)	2.01	1.94	1.33	1.37	1.34
Net Exports/GDP	0.15 (.01)	13.04	12.42	0.06	0.36	0.33
% Standard deviations relative to GDP						
Consumption	0.79 (.05)	0.19	0.21	0.28	0.27	0.29
Investment	3.24 (.17)	25.23	25.06	3.04	3.42	3.24
Employment	0.63 (.04)	0.56	0.54	0.50	0.52	0.49
Domestic Comovement						
Correlations with GDP						
Consumption	0.87 (.03)	0.90	0.93	0.93	0.90	0.94
Investment	0.93 (.02)	0.07	0.08	0.99	0.95	0.95
Employment	0.86 (.03)	0.99	0.99	0.99	0.99	0.99
Net Exports/GDP	−0.36 (.09)	0.06	0.06	0.27	−0.02	−0.05
International Correlations						
Home and Foreign GDP	0.51 (.13)	−0.46	−0.43	0.25	0.09	0.12
Home and Foreign Consumption	0.32 (.17)	0.28	0.13	0.29	0.77	0.62
Home and Foreign Investment	0.29 (.17)	−0.99	−0.99	0.33	−0.17	−0.09
Home and Foreign Employment	0.43 (.11)	−0.58	−0.53	0.23	−0.15	−0.04

Note: The statistics in the first 9 rows of the data column are calculated from U.S. quarterly time series, 1970:1–1998:4. The statistics in the last 4 rows of the data column are calculated from U.S. variables and an aggregate of 15 European countries. The data statistics are GMM estimates of the moments based on logged (except for net exports) and Hodrick-Prescott–filtered data with a smoothing parameter of 1,600. The numbers in parentheses are standard errors. The model statistics are computed from a simulation of 100,000 periods, where the relevant series have been logged and HP-filtered as the data series.

Source: See Appendix.

Figure 6 Kehoe and Perri

TABLE 3. Business cycle statistics: Sensitivity to technology shocks

Statistic	Data	*Enforcement Economy*				*Bond Economy with Adjustment Costs*			
		Baseline	High Persistence	High Spillover	BKK	Baseline	High Persistence	High Spillover	BKK
Volatility									
% Standard deviations									
GDP	1.72 (.20)	1.33	1.26	1.27	1.20	1.34	1.36	1.30	1.23
Net Exports/GDP	0.15 (.01)	0.06	0.07	0.03	0.03	0.33	0.32	0.31	0.34
% Standard deviations relative to GDP									
Consumption	0.79 (.05)	0.28	0.38	0.27	0.37	0.29	0.24	0.27	0.35
Investment	3.24 (.17)	3.04	2.76	3.08	2.78	3.24	3.24	3.24	3.24
Employment	0.63 (.04)	0.50	0.44	0.51	0.46	0.49	0.53	0.51	0.48
Domestic Comovement									
Correlations with GDP									
Consumption	0.87 (.03)	0.93	0.96	0.92	0.91	0.94	0.96	0.91	0.89
Investment	0.93 (.02)	0.99	0.98	0.99	0.99	0.95	0.87	0.95	0.93
Employment	0.86 (.03)	0.99	0.99	0.99	0.98	0.99	0.99	0.99	0.97
Net Exports/GDP	−0.36 (.09)	0.27	0.25	0.42	0.52	−0.05	−0.34	0.07	0.02
International Correlations									
Home and Foreign GDP	0.51 (.13)	0.25	0.25	0.32	0.20	0.12	0.08	0.25	0.14
Home and Foreign Consumption	0.32 (.17)	0.29	0.28	0.62	0.76	0.62	0.52	0.77	0.85
Home and Foreign Investment	0.29 (.17)	0.33	0.33	0.35	0.08	−0.09	−0.51	0.08	−0.30
Home and Foreign Employment	0.43 (.11)	0.23	0.22	0.21	−0.16	−0.04	−0.14	0.07	−0.30

Note: The statistics in the first 9 rows of the data column are calculated from U.S. quarterly time series, 1970:1–1998:4. The statistics in the last 4 rows of the data column are calculated from U.S. variables and an aggregate of 15 European countries. The data statistics are GMM estimates of the moments based on logged (except for net exports) and Hodrick-Prescott–filtered data with a smoothing parameter of 1,600. The numbers in parentheses are standard errors. The model statistics are computed from a simulation of 100,000 periods, where the relevant series have been logged and HP-filtered as the data series.

Source: See Appendix.

Figure 7 Kehoe and Perri

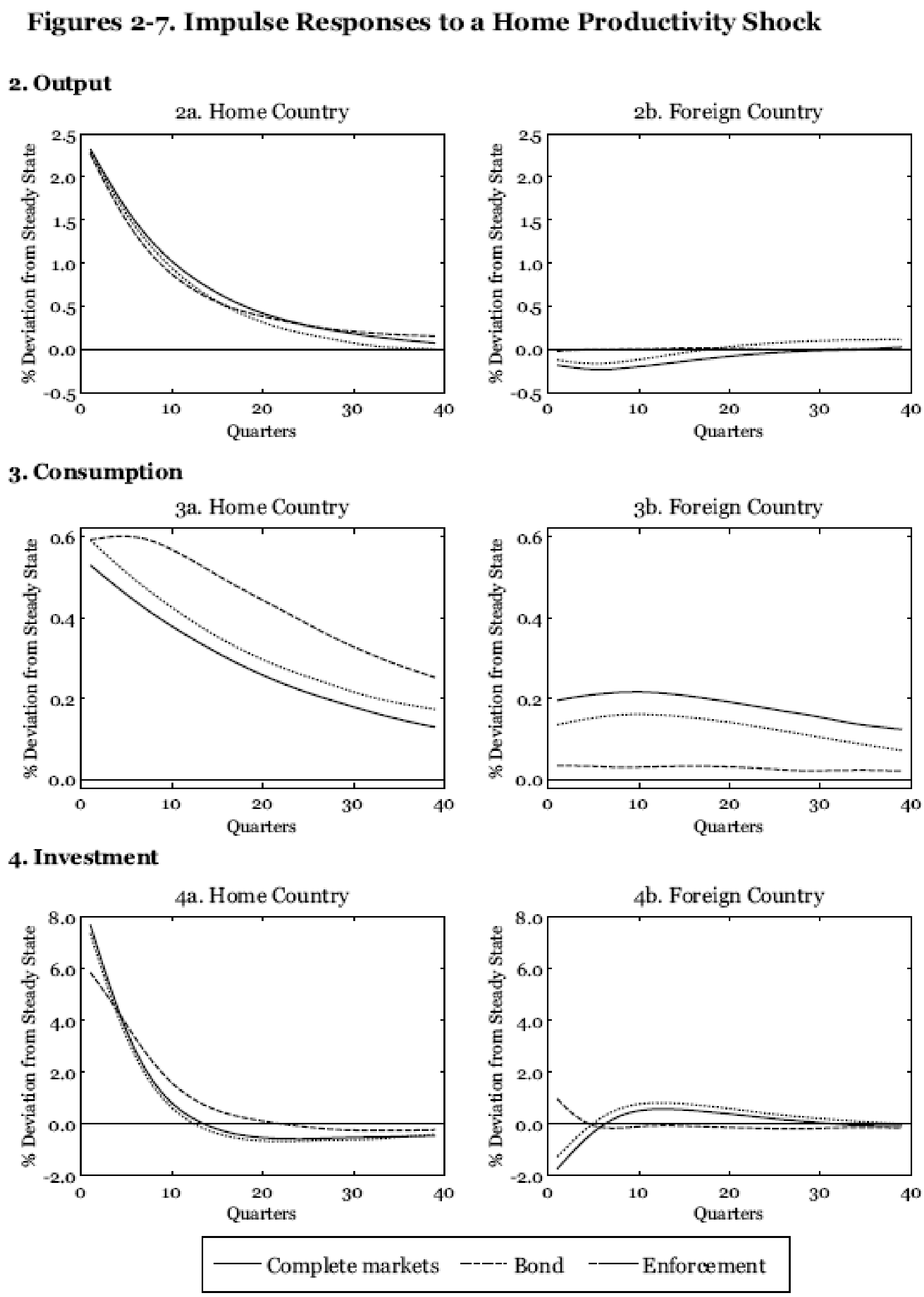

Figure 8 Kehoe and Perri

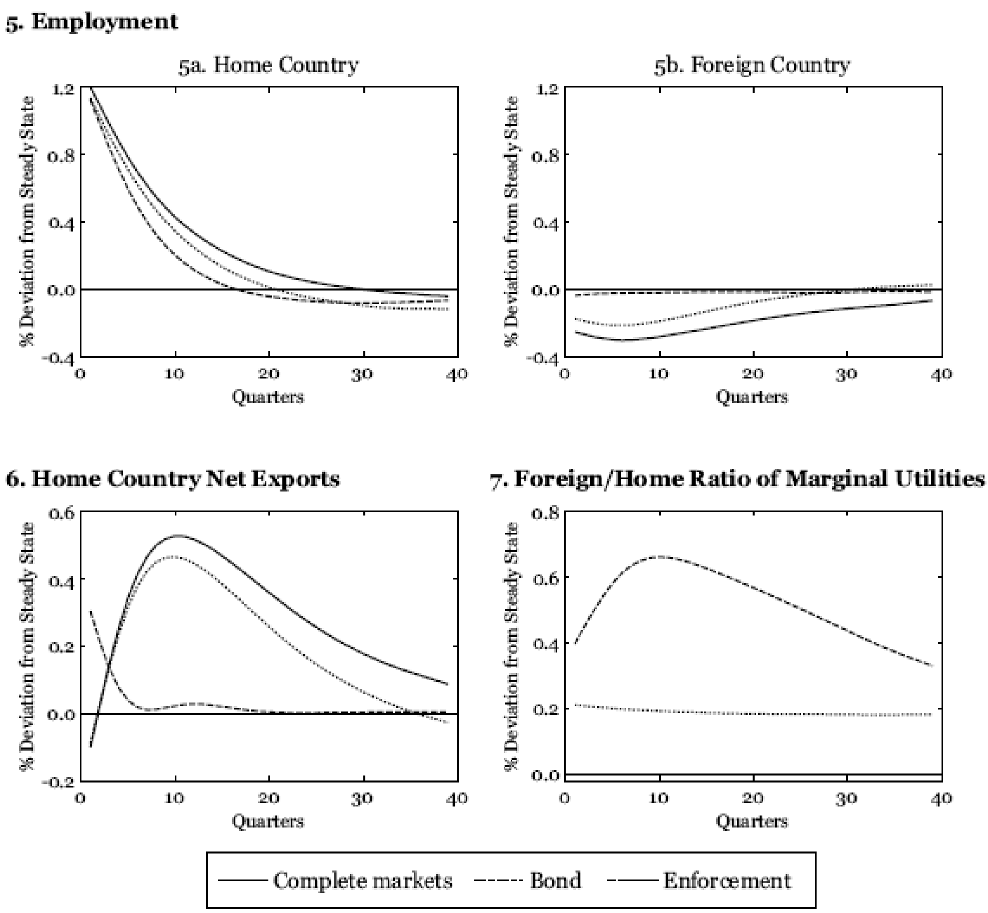

Figure 9 Kehoe and Perri

can be re-arranged using the fact that $\pi\left(s^t\right)\pi\left(s^r|s^t\right)=\pi\left(s^r\right)$ to give:

$$\sum_{t=0}^{\infty}\sum_{s^t}\sum_{i}\beta^t\pi\left(s^t\right)\left[M_i\left(s^{t-1}\right)U_i\left(s^t\right)+\mu_i\left(s^t\right)\left[U_i\left(s^t\right)-V_i\left(k_i\left(s^{t-1}\right),s^t\right)\right]\right]$$

plus usual terms having to do with the resource constraint. In this expression, $M_i\left(s^t\right)$ satisfies:

$$\begin{aligned} M_i\left(s^t\right) &= M_i\left(s^{t-1}\right)+\mu_i\left(s^t\right) \\ M_i\left(s^{-1}\right) &= \lambda_i \end{aligned}$$

So the weights are simply the original planner's weights, *incremented each time the enforcement constraint is binding.*

Why? when the enforcement constraint binds, the planner must allocate more utility to the country whose IC constraint binds, otherwise the country would choose autarky. This is equivalent to increasing the weight that the planner puts on the country. This will give higher current and future consumption to the country.

In fact, the equilibrium conditions are:

$$\begin{aligned} \frac{U_{1c}\left(s^t\right)}{U_{2c}\left(s^t\right)} &= \frac{M_2\left(s^t\right)}{M_1\left(s^t\right)} \\ -\frac{U_{il}\left(s^t\right)}{U_{ic}\left(s^t\right)} &= F_l\left(s^t\right) \\ U_{ic}\left(s^t\right) &= \beta\sum_{s^{t+1}|s^t}\pi\left(s^{t+1}|s^t\right)\left[\frac{M_i\left(s^{t+1}\right)}{M_i\left(s^t\right)}U_{ic}\left(s^{t+1}\right)\left[1-\delta+F_{ik}\left(s^{t+1}\right)\right]-\frac{\mu_i\left(s^{t+1}\right)}{M_i\left(s^t\right)}V_{ik}\left(s^{t+1}\right)\right] \end{aligned}$$

The last condition incorporates the fact that a higher capital stock today changes the promised utility $V_i\left(s^{t+1}\right)$. After a positive productivity shock in country 1, not so much capital is shifted because that would increase V_i.

5.4.3 The 2-good, 2-country model.

TBD.