Asset Pricing with Higher-Order Beliefs

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Objectives and Results

Basic Idea

Incorporate Higher-Order Beliefs into a Standard Linear Present-Value Asset Pricing Model.

Higher-Order Beliefs = Beliefs About Other People's Beliefs

Results

Higher-Order Belief Dynamics Can Explain Apparent Violations of Variance Bound Inequalities and Rejections of Cross-Equation Restrictions.

Some Asset Pricing History

- 1970s-1980s: Linear Present Value Models
  - Constant Discount Rates
  - Allowed for information discrepancies between agents and econometrician, but not among the agents themselves.
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- 1980s-1990s: Time Varying Risk Premia. Stochastic Discount Factors.
  - Consumption-based CAPM. Lucas (1978).
  - Still effectively assumed rep. agent/complete markets.
  - Still empirical failures. (Equity premium puzzle, Hansen-Jagannathan Bounds).

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- Present-Future(?): *Informational* Heterogeneity.
  - Higher-order belief dynamics.

Related Work

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- Morris & Shin. Imperfect Common Knowledge. (Static/No prices)

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4. Background “liquidity traders”. (No-Trade Theorem).
5. Use a reverse engineering, frequency domain approach to solve a complex (infinite-dimensional) signal extraction problem. Futia (1981)
   $\Rightarrow$ Posit Blaschke factors instead of solve Riccati equations.

THE MODEL

- Asset Demands
  - $q_t^i = E_t^i p_{t+1} - \beta^{-1} p_t \qquad \beta^{-1} = (1+r) > 1$

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- Asset Market Equilibrium
  - $p_t = \beta \int_0^1 E_t^i p_{t+1} di - \beta f_t$
- Fundamentals Process
  - $f_t = a_1(L)\varepsilon_{1t} + a_2(L)\varepsilon_{2t}$

THE MODEL (CONT)

- Equilibrium with Two Trader Types
  - Trader 1 observes (p_t, ε_{1t})
  - Trader 2 observes (p_t, ε_{2t})

$$p_t = \beta \left\{ \frac{1}{2} E^1[p_{t+1} | H_p(t), H_{\varepsilon_1}(t)] + \frac{1}{2} E^2[p_{t+1} | H_p(t), H_{\varepsilon_2}] \right\} - \beta f_t$$

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- Moving Average Representation

$$\begin{bmatrix} f_t \\ p_t \end{bmatrix} = \begin{bmatrix} a_1(L) & a_2(L) \\ \pi_1(L) & \pi_2(L) \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

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- Conjectured Equilibrium

$$\pi_i(L) = (L - \lambda)[\rho_i + L g_i(L)] \qquad |\lambda| < 1$$

Signal Extraction

- Full Information Expectations

$$E[p_{t+1}|H_{\varepsilon_1}(t)\bigvee H_{\varepsilon_2}(t)] = [L^{-1}\pi_1(L)]_+\varepsilon_{1t}+[L^{-1}\pi_2(L)]_+\varepsilon_{2t}$$

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- Lemma 1: *The projection* $E[p_{t+1}|H_{\varepsilon_2}(t)] = [\rho_2+(L-\lambda)g_2(L)]\varepsilon_{2t}$ *has the following orthogonal decomposition*:

$$[\rho_2+(L-\lambda)g_2(L)]\varepsilon_{2t} = \left[h(L)\frac{L-\lambda}{1-\lambda L}\right]\varepsilon_{2t}+\frac{constant}{1-\lambda L}\varepsilon_{2t}$$

where $h(L)$ *is an analytic function in D with zeroes outside D*.

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where $h(L)$ *is an analytic function in D with zeroes outside D*.

- Lemma 2: *Given the hypothesized pricing functions, the conditional expectations of Type 1 traders are given by*:

$$E[p_{t+1}|H_p(t) \bigvee H_{\varepsilon_1}(t)] = [\rho_1 + (L-\lambda)g_1(L)]\varepsilon_{1t} + [\rho_2 + (L-\lambda)g_2(L) - \frac{\rho_2(1-\lambda^2)}{1-\lambda L}]\varepsilon_{2t}$$

with an analogous expression for Type 2 traders.

EQUILIBRIUM

- Heterogeneous Beliefs:

$$E_t^1 p_{t+1} - E_t^2 p_{t+1} = \frac{1-\lambda^2}{1-\lambda L}(\rho_1 \varepsilon_{1t} - \rho_2 \varepsilon_{2t})$$

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- Equilibrium Pricing Functions:

Proposition 1: *Under Assumption 1 (given below), there exists a unique heterogeneous beliefs Rational Expectations pricing function with z-transforms given by*:

$$\begin{aligned}
\pi_1(z) &= (z-\lambda)\left[2a_1(\lambda) + \frac{z}{z-\beta}\left\{-2a_1(\lambda) + \frac{\beta}{z-\lambda}\left[2a_1(\lambda) - a_1(z) - \frac{a_1(\lambda)(1-\lambda^2)}{1-\lambda z}\right]\right\}\right] \\
\pi_2(z) &= (z-\lambda)\left[2a_2(\lambda) + \frac{z}{z-\beta}\left\{-2a_2(\lambda) + \frac{\beta}{z-\lambda}\left[2a_2(\lambda) - a_2(z) - \frac{a_2(\lambda)(1-\lambda^2)}{1-\lambda z}\right]\right\}\right]
\end{aligned}$$

and $|\lambda| < 1$ given implicitly by the equation

$$2\lambda a_1(\lambda) = \beta[a_1(\beta) + a_1(\lambda)(1-\lambda^2)/(1-\lambda\beta)]$$

Existence Condition

Assumption 1: *There exists a unique* $|\lambda| < 1$ *with* $\lambda \neq \beta$, *that solves the two equations*:

$$2\lambda = \beta \left[\frac{a_i(\beta)}{a_i(\lambda)} + \frac{1-\lambda^2}{1-\lambda\beta} \right] \qquad i = 1, 2$$

Comment: AR(1) doesn't work. ARMA(1,1) does.

- Assumption 2: *There exists a* $|\lambda| > 1$ *that solves the two equations*:

$$2\lambda = \beta\left[\frac{a_i(\beta)}{a_i(\lambda)} + \frac{1-\lambda^2}{1-\lambda\beta}\right] \qquad i = 1, 2$$

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- Proposition 2: *Given Assumption 2, there exists a homogeneous beliefs Rational Expectations equilibrium given by*:

$$\pi_i^s(z) = -\beta \left\{ a_i(\beta) + \frac{z}{z-\beta} [a_i(z) - a_i(\beta)] \right\} \qquad i = 1, 2$$

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- Assumption 3: *There exist* $a_1(L)$ *and* $a_2(L)$ *polynomials that simultaneously satisfy Assumptions 1 and 2*.

Higher-Order Beliefs

- Proposition 3: *Given Assumption 3, there exists both a heterogeneous expectations equilibrium and a homogeneous expectations equilibrium, with z-transforms related as follows*:

$$\pi_i(z) = \pi_i^s(z) + a_i(\lambda)(1-\lambda^2) \cdot \frac{\beta}{z-\beta}\left(\frac{\beta}{1-\lambda\beta} - \frac{z}{1-\lambda z}\right)$$

where $|\lambda| < 1$ is given in Proposition 1, and the $\pi_i^s(z)$ are given by Proposition 2.

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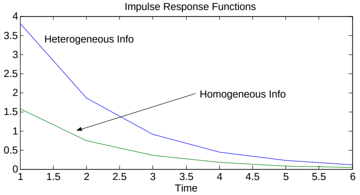
- Corollary 1: *Higher-order beliefs amplify the initial response of asset prices to innovations in fundamentals*

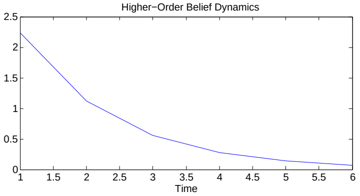
$$\pi_i(0) = \pi_i^s(0) - a_i(\lambda)\cdot\frac{1-\lambda^2}{1-\lambda\beta} < \pi_i^s(0)$$

EQUILIBRIUM COMPARISON

Calibration:

$$a_1(L) = \frac{L-\phi}{1-\gamma L} \qquad \begin{array}{lcl} \lambda & = & 0.5 \\ \gamma & = & 0.476 \end{array} \qquad \begin{array}{lcl} \phi & = & 0.83 \\ \beta & = & 0.90 \end{array}$$





Observed Fundamentals

Proposition 4: *With just two trader types, there does not exist a heterogeneous beliefs Rational Expectations equilibrium if aggregate fundamentals are observable. Prices must be revealing*.

Proof: Suppose Type 1 observes $(p_t, f_t, \varepsilon_{1t})$ and Type 2 observes $(p_t, f_t, \varepsilon_{2t})$. Then Type 1 effectively observes $a_2(L)\varepsilon_{2t}$. p_t also depends on the history of ε_{2t}. A classic theorem from the theory of analytic functions implies that $\{p_t, a_2(L)\}$ spans $H_{\varepsilon_2}(t)$ unless $a_2(L)$ and $\pi_2(L)$ have identical non-invertible roots. However, $a_2(\lambda) \neq 0$, by the existence condition given in Proposition 1. QED.

- Given Proposition 4, let's add a third trader type, who observes a third component of fundamentals. That is, we now have:

$$f_t = a_1(L)\varepsilon_{1t} + a_2(L)\varepsilon_{2t} + a_3(L)\varepsilon_{3t}$$

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- Therefore, the observer system for Type 1 becomes:

$$\begin{bmatrix} \varepsilon_{1t} \\ f_t \\ p_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a_1(L) & a_2(L) & a_3(L) \\ (L-\lambda)[\rho_1 + Lg_1(L)] & (L-\lambda)[\rho_2 + Lg_2(L)] & (L-\lambda)[\rho_3 + Lg_3(L)] \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}$$

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- Everything goes through as before, except the existence condition gets modified to: $\exists\lambda < 1$ *such that*

$$2\lambda = \beta[a_i(\beta)/a_i(\lambda) + (1-\lambda^2)/(1-\lambda\beta)] \qquad i = 1, 2, 3$$

VARIANCE BOUNDS

Proposition 5: *If fundamentals are ARMA(1,1) (i.e., $a_i(L) = (1 - \phi_i L)/(1 - \gamma_i L)$), then asset prices violate the standard variance bound whenever λ is sufficiently close to β, and ϕ_i and γ_i are sufficiently small.*

Perfect Foresight Price

$$\pi_1^{pf}(z) = -\beta(1 - \beta z^{-1})^{-1} a_1(z)$$

From Parseval's formula,

$$\begin{aligned} \text{var}(p_1^{pf}) &= \frac{1}{2\pi i} \oint \pi_1^{pf}(z) \pi_1^{pf}(z^{-1}) \frac{dz}{z} \\ &= \beta^2 \frac{(1 - \phi_1\gamma_1)(1 - \phi_1\beta) + (\phi_1 - \beta)(\phi_1 - \gamma_1)}{(1 - \gamma_1\beta)(1 - \beta^2)(1 - \gamma_1^2)} \end{aligned}$$

$$\text{var}(p_1) = \frac{1}{2\pi i} \oint \pi_1^s(z)\pi_1^s(z^{-1}) + \kappa(z)\kappa(z^{-1}) + 2\pi_1^s(z)\kappa(z^{-1})] \frac{dz}{z}$$

$$= \left(\frac{\beta(1 - \phi_1\beta)}{1 - \gamma_1\beta}\right)^2 \frac{(1 - x\gamma_1)(1 - x/\gamma_1)}{1 - \gamma_1^2} + \left(\frac{\beta(1 - \phi_1\lambda)}{(1 - \lambda\beta)(1 - \gamma_1\lambda)}\right)^2 (1-\lambda^2) + 2\frac{\beta^2(1 - \lambda^2)(1 - \phi_1\beta)(1 - \phi_1\lambda)(1 - x\lambda)}{(1 - \gamma_1\beta)(1 - \lambda\beta)(1 - \gamma_1\lambda)^2}$$

where $x = \phi_1(1 - \beta\gamma_1)/(1 - \beta\phi_1)$.

- Proof: Setting $\lambda = \beta$ and $\gamma_1 = \phi_1 = 0$ yields,

$$\text{var}(p_1^{pf}) = \frac{\beta^2}{1-\beta^2}$$

and

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- Interpretation: What happened to the logic of Shiller's variance bound. It's a *theorem* after all!
  - Answer: Higher-order beliefs play the role of *missing fundamentals*. Traders don't just care about their own expectations of future fundamentals, they also care about, and try to forecast, *other traders'* expectations about fundamentals.

Cross-Equation Restrictions

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- RE imposes the *cross-equation restrictions*

$$\pi_i(L) = \Pi^s(a_i(L)) + \Pi^h(a_i(L))$$

  across the rows of the MA representation.

Cross-Equation Restrictions (cont)

- Prior studies find $\pi_i(L) \neq \Pi^s(a_i(L))$. Clearly, this does *not* necessarily imply a rejection of the (heterogeneous info) PV model!

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- Even the clever VAR methodology of Campbell & Shiller (1987) can be tricked into false rejections. Why? Because it tests the wrong restrictions. Campbell & Shiller presume the law of iterated expectations holds, i.e., $E_t(E_{t+1}x_{t+n})) = E_t x_{t+n}$. However, the law of iterated expectations does not (in general) apply to the averaged expectations operator, i.e., $\bar{E}_t(\bar{E}_{t+1}x_{t+n})) \neq \bar{E}_t x_{t+n}$. Conditioning down doesn't work!

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- This paper argues that a different kind of heterogeneity, an *informational heterogeneity*, can help to reconcile observed asset prices with the data.
- The key challenge is to prevent prices from revealing agents' private information. Informational heterogeneity does not automatically translate into belief heterogeneity!
- We argue that a reverse engineering, frequency domain approach, is useful in doing this.

Extensions

- Verify that existence conditions (i.e, Assumption 1) can be satisfied with empirically plausible specifications for fundamentals.
- Investigate other asset pricing puzzles, e.g., Equity Premium, Hansen-Jagannathan Bounds, etc.
- General Equilibrium. Optimizing Asset Demands.

  Nonlinearity + Higher-Order Beliefs = Hard Problem!