SIMON FRASER UNIVERSITY
Department of Economics

Econ 808 | Prof. Kasa
Macroeconomic Theory | Fall 2003

PROBLEM SET 1 - Dynamic Optimization
(Due September 23)

1. **Solving the Continuous-Time LQR Three Different Ways**

   Consider the following dynamic optimization problem:

   $$\min_u \int_0^\infty \frac{1}{2}\left(qx^2 + ru^2\right) e^{-\rho t} dt$$

   subject to:

   $$\begin{aligned} \dot{x} &= ax + bu \\ x(0) &= \bar{x}_0 \\ \lim_{t\to\infty} x(t)e^{-\rho t} &< \infty \end{aligned}$$

   where x is the state variable, u is the control variable, and q and r are parameters reflecting the costs of state and control variability.

   (a) Solve this problem using the calculus of variations. That is, derive the Euler equation, which is a second-order ordinary differential equation in x. Use the boundary conditions to determine the unique solution. How do we know we have a minimum rather than a maximum?

   (b) Solve this problem using Pontryagin's Maximum Principle. That is, define the (current-value) Hamiltonian, and derive the necessary conditions for an optimum (ie, derive a pair of first-order differential equations in x and the co-state variable for the constraint). Show that the eigenvalues of this system are the same as the roots of the Euler equation derived in part (a). Use the boundary conditions to determine a unique solution.

   (c) Solve this problem using dynamic programming. That is, define the Bellman equation, and derive the necessary conditions for an optimum. Posit that the value function is of the form: $V(x) = Px^2$ and use the method of undetermined coefficients to determine a first-order differential equation characterizing P. (This equation is called the 'Riccati equation'). Determine the steady state value of P, and use it to calculate the steady-state optimal feedback policy for u.

   (d) Verify that the optimal paths for x and u are the same in parts a, b, and c.

2. **Solving the Discrete-Time LQR Three Different Ways**

   Consider the following dynamic optimization problem:

$$\min_{\{u_t\}} \frac{1}{2} \sum_{t=0}^{\infty} \beta^t (q x_t^2 + r u_t^2) \qquad \beta < 1$$

   subject to:

$$\begin{aligned} x_{t+1} &= a x_t + b u_t \\ x_0 &= \bar{x}_0 \\ \lim_{t \to \infty} \beta^t x_t &< \infty \end{aligned}$$

   where x_t is a discrete sequence of state variables, u_t is a discrete sequence of control variables, and q and r are parameters reflecting the costs of state and control variability.

   (a) Solve this problem using the calculus of variations. That is, derive the Euler equation, which is now a second-order (linear) difference equation in x_t. Use the boundary conditions to determine the unique solution.

   (b) Solve this problem using the maximum principle, i.e., attach a sequence of Lagrange Multipliers to the constraints. Derive a pair of (linear) first-order difference equations in x_t and the Lagrange Multiplier. Show that the eigenvalues of the first-order system are the same as the roots of the characteristic equation in part (a). Use the boundary conditions to determine a unique solution.

   (c) Solve this problem using dynamic programming. That is, define the Bellman equation, and use it to characterize an optimal feedback policy for u_t. Posit that the value function is of the form $V(x_t) = \frac{1}{2} P_t x_t^2$, and use the method of undetermined coefficients to calculate a first-order (nonlinear) difference equation for P_t. (This equation is called the Riccati equation). Prove that there is a unique positive steady state solution of the Riccati equation.

   (d) Verify that the optimal paths for x_t and u_t are the same in parts a, b, and c.

3. **Who Says You Don't Learn Anything in Graduate School?**

   Use the calculus of variations to prove that the shortest distance between two points in a plane is a straight line. (Hint: Plot t on the horizontal axis and x on the vertical axis. Use the Pythagorean Theorem to show that the (infinitesimal) distance, ds, between two points is: $ds = \sqrt{1 + \dot{x}^2} dt$, where $\dot{x}$ is the derivative of x with respect to t).