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Dynamic Optimal Taxation with Private Information

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ABSTRACT

We study dynamic optimal taxation in a class of economies with private information. Constrained optimal allocations in these environments are complicated and history-dependent. Yet, we show that they can be attained as competitive equilibria in market economies supplemented with *simple* tax systems. The market structure in these economies is identical to that in Bewley (1986): agents can trade current consumption and risk-free claims to future consumption, subject to a budget constraint and a debt limit. The tax system describes additional transfers that the agents must make to the government. It conditions them upon only two observable characteristics of an agent: her accumulated stock of claims, or wealth, and her current labour income. It implies optimal tax functions that are not additively separable in these variables. The marginal wealth tax is negatively correlated with income and its expected value is generally positive. The marginal income tax is decreasing in wealth.

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1 Introduction

This paper studies dynamic optimal taxation in a class of economies with private information. Specifically, we consider an environment in which each member of a population of infinitely-lived agents receives a privately observed sequence of idiosyncratic skill shocks. The agents' preferences are defined over consumption and labor supply. Constrained optimal allocations in our environment satisfy incentive compatibility and resource feasibility constraints, and a lower bound on lifetime utility at each date. They are complicated and history-dependent. Yet, we show that they can be attained as competitive equilibria in market economies supplemented with *simple* tax systems. The market structure in these economies is identical to that in Bewley (1986), Huggett (1993) or Aiyagari (1994): agents can trade current consumption and risk-free claims to future consumption, subject to a budget constraint and a debt limit. The tax system specifies the transfers that agents must make to the government. Crucially, it conditions these transfers upon only two observable characteristics of an agent: her accumulated stock of claims, or wealth, and her current labour income. An agent's wealth summarises all aspects of her history that are relevant for implementation. Thus, this model has implications for the optimal taxation of both wealth and labour earnings. It implies optimal tax functions that are not additively separable in these variables. In particular, it implies a marginal wealth tax that is negatively correlated with an agent's labour income and a marginal income tax that is decreasing in an agent's wealth.

Most models of dynamic optimal taxation follow the Ramsey approach, an approach in which the set of fiscal instruments available to the government is *exogenously specified.* Linear labor and capital income taxes are typically included in this set, while lump-sum taxes are typically ruled out[1]. The exclusion of lump sum taxes is justified by appealing to incentive or administrative constraints, but these are not explicitly modelled. Hence, the exogenous restrictions on fiscal instruments implicit in this approach become themselves a source of frictions. The government's optimal taxation problem reduces to one of selecting

[1]Classic contributions include Judd (1985) and Chamley (1985) on capital income taxation and Lucas and Stokey (1983) on labour income taxation and debt policy in stochastic economies. Chari and Kehoe (1999) provide an excellent overview of this literature.

from amongst a limited number of policy instruments so as to ameliorate these frictions.

The approach we adopt in this paper builds on the optimal non-linear income taxation literature initiated by Mirrlees (1971). This literature emphasises that incentive compatibility constraints in private information environments give rise to *endogenous* restrictions on optimal government policies. Optimal non-linear taxes reproduce the patterns of wedges, gaps between individual marginal rates of substitution and marginal rates of transformation, associated with these constraints. This analysis has mostly been conducted in a static framework[2]. Thus, while this literature has provided a rich source of insight into the nature of optimal labor income taxation, its implications for the optimal conduct of tax policy in dynamic economies are largely unexplored. We also build upon the large body of work on dynamic contracting[3]. This literature studies the properties of constrained Pareto optimal allocations in a variety of dynamic, private information economies. However, it limits the analysis to implementation via direct mechanisms. While such mechanisms can be interpreted as tax systems, they seem divorced from the actual combination of markets and taxes that are used in practice to allocate resources, at least within modern industrialized economies.[4] Golosov, Kocherlakota and Tsyvinski (2002) provide a partial characterization of the pattern of wedges implied by these constrained optimal allocations in a very general setting. They make a connection with fiscal policy, but they do not present results on how these wedges might be implemented without relying on direct mechanisms[5]. We derive a tax system that implements the pattern of wedges implied by incentive compatibility in a market economy, albeit in a simpler environment. We implement the lower bound on lifetime utility with a debt limit. While we focus on decentralized tax systems rather than direct mechanisms, we do build on a key insight from the dynamic contracting literature:

[2]Brito et al (1991), Diamond and Mirrlees (1978) and da Costa and Werning (2001) are important exceptions.

[3]A non-exhaustive list includes: Rogerson (1985), Green (1987), Thomas and Worrall (1990), Phelan and Townsend (1991), Atkeson and Lucas (1992, 1995) and Phelan (1994).

[4]They may, however, more closely resemble the trading arrangements used in simple village economies, see Townsend (1995) or Ligon (1998).

[5]In a recent interesting paper, Golosov and Tsyvinski (2002) derive an arrangement for implementing an optimal disability insurance model. The arrangement is part direct mechanism, it relies on reports of an agent's shock, and part tax system.

optimal allocations are naturally recursive in promised utilities. We recast this observation in our market economy with taxes, and establish recursivity of the equilibrium and the tax system in agents' wealth rather than utility promise. Hence, our tax system is simple and conditions transfers only on an agent's current wealth and income.

The tax system that implements a constrained optimal allocation must satisfy two requirements[6]. It must reproduce the pattern of wedges implied by the optimal allocation, so that the agent's first order conditions in the market economy with taxes are satisfied at this allocation. This, however, is not enough. It must also be true that the agent's sufficient conditions for optimality are satisfied at the constrained optimal allocation. This imposes more structure on the optimal tax system than can be obtained by just "matching wedges". Indeed, we provide an example in which a tax system that matches wedges admits two solutions to the agent's first order conditions. One involves the choices implied by the constrained optimal allocation. However, the agent attains higher utility at the other solution. This involves the agent saving too much in an early period, and working too little in a later one relative to the constrained optimal allocation.

The wedges stem from a basic trade-off between insurance and incentives. The public finance literature has focussed on an *insurance wedge*, which implies that agents are exposed to consumption risk, and an *effort wedge,* which implies that agents' labor supply is distorted downwards. Dynamic models introduce a third wedge. As Golosov, Kocherlakota and Tsyvinski (2002) have emphasised, the need to provide incentives in such settings implies that a marginal increase in savings has an additional social cost beyond the usual cost in terms of forgone current consumption. Higher wealth reduces the correlation between consumption and labor supply and thus reduces the incentive for an agent to work in the subsequent period. This gives rise to an *intertemporal wedge*, a gap between the economy's discount factor, Q_t, and the agent's expected intertemporal marginal rate of substitution at the constrained optimal allocation:

$$Q_t u'(c_t) - \beta E_t \left[u'(c_{t+1})\right] < 0,$$

[6] The exact link between tax systems and direct mechanisms is given by the "taxation principle" developed by Hammond (1979) and Rochet (1985) and elaborated by Guesnerie (1995). According to this principle, any incentive compatible and resource feasible direct mechanism can be implemented in a market economy with a tax system that conditions transfers on the observable market trades of agents.

This result was derived initially by Diamond and Mirrlees (1978) and Rogerson (1985). Golosov, Kocherlakota and Tsyvinski (2002) have suggested that a positive marginal asset tax may be required to implement this wedge, thus giving a rationale for the taxation of assets that is often absent from complete information Ramsey models. However, we provide an example in which, despite a positive intertemporal wedge, the expected marginal asset tax is zero.

To understand the implications of the intertemporal wedge for marginal asset taxes, note that the agent's intertemporal Euler equation in the market economy is:

$$Q_t u'(c_t) - \beta E_t \left[\left(1 - T_{t+1,b}\left(b_{t+1}, y_{t+1}\right)\right) u'(c_{t+1}) \right] = 0.$$

Here, Q_t represents the price of a risk-free claim, $T_{t+1}\left(b_{t+1}, y_{t+1}\right)$ denotes the tax payment at period $t+1$ as a function of an agent's savings b_{t+1} and labour income y_{t+1}, and $T_{t+1,b}$, is the marginal asset tax. Notice that the optimal tax system implies a stochastic marginal asset tax, since the realisation of y_{t+1} is unknown to an agent selecting b_{t+1} at time t. The agent's intertemporal Euler equation can be rewritten as:

$$\begin{aligned} & \beta E_t \left[u'(c_{t+1}) \right] - Q_t u'(c_t) \\ = \quad & \beta \left\{ E_t \left[T_{t+1,b} \right] E_t \left[u'(c_{t+1}) \right] + Cov_t \left[T_{t+1,b}, u'(c_{t+1}) \right] \right\} \geq 0. \end{aligned}$$

Consequently, if $Cov_t \left[T_{t+1,b}, u'(c_{t+1}) \right]$ is sufficiently positive, the intertemporal wedge does not imply a positive expected marginal asset tax, $E_t \left[T_{t+1,b} \right]$. The extent to which $E_t \left[T_{t+1,b} \right]$ and $Cov_t \left[T_{t+1,b}, u'(c_{t+1}) \right]$ contribute to the intertemporal wedge is determined by the requirement that the constrained optimal allocation satisfies sufficient conditions for optimality for the agent. In a two period example, we establish that $Cov_t \left[T_{t+1,b}, u'(c_{t+1}) \right]$ must be positive to rule out deviations in which the agent saves more in the first period and works less in the second period. In addition, we show that if the labor income to be implemented in the second period, y_{t+1}, does not depend on the savings level, b_{t+1}, then $E_t \left[T_{t+1,b} \right]$ is zero. Otherwise, $E_t \left[T_{t+1,b} \right]$ is positive. It follows that under a recursive tax system, the expected marginal asset tax will in general be positive if the preference specification admits wealth effects on labor supply.

We explore the steady state properties of the optimal tax system in a series of numerical examples. Our parameterisation is consistent with recent calibrations of Bewley economies with endogenous labor supply. The optimal tax systems that we compute have several striking characteristics. These include the high curvature of optimal tax functions in the neighbourhood of the debt limit and a strong dependence of marginal asset and marginal income taxes on wealth. Recent contributions to the public finance literature such as Diamond (1998) and Saez (2001), emphasise that, with appropriate utility function and shock distribution specifications, optimal marginal income taxes are high and decreasing in income at low income levels. They interpret this finding as being consistent with the high empirical marginal income tax rates associated with the phasing out of social benefits at low incomes. In contrast, we find that marginal labour income taxes are strongly decreasing in *wealth*, rather than income. This result is robust to alternative parameterisations, and stems from the government's desire to provide additional insurance to those agents who are close to the debt limit and are thus restricted in their ability to obtain insurance via asset markets. On the other hand, consistent with earlier findings in numerical public finance, we find that the dependence of marginal income taxes on labour income is sensitive to the choice of utility function and shock distribution.

With respect to wealth taxation, we find that the marginal asset tax is negatively correlated with labour income and consumption. The intertemporal wedge is small away from the debt limit, less than 1% over most of the wealth range, but close to this limit it rises steeply to a peak of 16% in our benchmark parameterisation. The expected marginal asset tax is small over most of the wealth range. It peaks at approximately 2% at the debt limit, and then falls steadily with wealth. The covariance component is also decreasing in wealth, but it is much larger close to the limit and falls off much more quickly as wealth increases. The covariance component plays the major role in generating the intertemporal wedge only when the agent's wealth is small and the intertemporal wedge large.

The paper is organized as follows. Sections 2 describes the environment and the primal planning problem. Section 3 develops a recursive dual formulation of this problem that is easier to analyse. We show in section 4 that constrained optimal allocations can be implemented in Bewley economies with simple tax systems. We elaborate on these ideas in section 5 by describing the pattern of wedges that the optimal tax system needs to reproduce. Section 6 provides some simple analytical examples which explore the role of asset taxation, while section 7 presents numerical results.

2 Environment

The economy is populated by a continuum of agents of unit measure. Each agent receives a sequence of preference shocks $\theta = \{\theta_t\}$ with $\theta_t \in \Theta = [\underline{\theta}, \overline{\theta}]$, $\forall t$. Let $\mathcal{B}(\Theta)$ denote the Borel sigma-algebra on Θ. The θ_t shocks are i.i.d. over time and across agents with continuous density π. It is assumed that π also describes the cross sectional distribution of θ_t at each t[7]. Let $\theta^t = \{\theta_0, \theta_1, ...\theta_t\} \in \Theta^{t+1}$ denote a t-period history of preference shocks with probability density $\Pi(\theta^t)$. An agent is assumed to privately observe its history of shocks.

Let $c_t : \Theta^{t+1} \to \mathbb{R}_+$ and $y_t : \Theta^{t+1} \to \mathcal{Y}$ $(\equiv [0, \overline{y}])$ be random variables describing agent consumption and output at date t. Agents produce output by exerting effort, so that y_t may also be interpreted as labour income. Assume that each c_t and y_t is integrable and call the process $\alpha = \{c_t, y_t\}_{t=0}^{\infty}$ an (agent) allocation. Let A denote the set of such allocations. Additionally, call the pair of random variables (c_t, y_t) a current allocation for the agent. The agent's preferences over α are:

$$W(\alpha; \theta) \equiv \sum_{t=0}^{\infty} \beta^t \int_{\Theta^t} [u(c_t) + \theta_t v(y_t)] \, \Pi(\theta^t) d\theta^t.$$

Assume that u and v are bounded, continuously differentiable and, respectively, strictly increasing and decreasing on their domains, and strictly concave. $\theta v(y)$ is to be interpreted as the effort cost of producing output y. Note that the role of preference shocks is to alter this cost and to alter the marginal rate of substitution between consumption and effort. Clearly if v is homogenous of degree k, this formulation is equivalent to one with a productivity shock equal to $\theta^{-1/k}$ and an effort input of $y\theta^{-1/k}$. Let $\widetilde{\mathcal{W}}$ denote the set of possible life time utilities available to an agent from an allocation and define $\mathcal{W} = \widetilde{\mathcal{W}} \cap [\underline{U}, \infty)$, $\underline{U} \in \mathbb{R}$.

Agents are indexed by Pareto-Negishi weights $\gamma \in \Gamma \equiv [0, 1]$. The allocation obtained by γ-indexed agents is denoted $\alpha^\gamma = \{c_t^\gamma, y_t^\gamma\}_t$. A "primal" allocation is a collection $\alpha^\Gamma =$

[7] This amounts to assuming that the law of large numbers holds across agents. As is well known, the law of large numbers may not apply if the underlying index space is a Borel measure. We implicitly rely on the constructions of Judd (1985) and Sun (1996) to resolve this issue.

$\{\alpha^\gamma\}_\gamma$, and A^Γ denotes the set of primal allocations. A planner has preferences over primal allocations of the form:

$$W_0(\alpha^\Gamma; \Psi_0) = \int_\Gamma \gamma \sum_{t=0}^{\infty} \beta^t \int_{\Theta^t} \left[u(c_t^\gamma(\theta^t)) + \theta_t v(y_t^\gamma(\theta^t)) \right] \Pi(\theta^t) d\theta^t d\Psi_0, \tag{1}$$

where Ψ_0 denotes the initial distribution over Pareto-Negishi weights. The planner's continuation payoff at time $s > 0$ is defined as:

$$W_s(\alpha^\Gamma; \Psi_0) = \int_\Gamma \gamma \sum_{t=s}^{\infty} \int_{\Theta^t} \beta^{t-s} \left[u(c_t^\gamma(\theta^t)) + \theta_t v(y_t^\gamma(\theta^t)) \right] \Pi(\theta^t) d\theta^t d\Psi_0.$$

Given that the realisations of θ^t are the private information of agents, the planner cannot condition primal allocations directly upon them. Instead, the planner and the agents play a game in which the planner first selects a direct mechanism that conditions agents' current allocations on their Pareto-Negishi weight and the history of reports that they have made concerning the values of past and current shocks. Denote such a mechanism by $f = \{c_t^\gamma, y_t^\gamma\}_{t,\gamma} \in A^\Gamma$. Agents then select a reporting strategy, $\delta = \{\delta_t\}$, with each $\delta_t : \Theta^{t+1} \to \Theta$ and $\mathcal{B}(\Theta^{t+1})$-measurable, that gives the history contingent reports that they will make. A collection of reporting strategies,$\{\delta^\gamma\}_\gamma$, and a direct mechanism, f, induce a primal allocation $\alpha(\{\delta^\gamma\}_\gamma, f)$.

Let $U^\gamma(\delta, f)$ denote the payoff to a γ-weighted agent from reporting strategy δ under mechanism f. Given f, an agent selects its reporting strategy to solve $\sup_{\delta'} U^\gamma(\delta', f)$. Let $\delta^* = \{\delta_t^*(\theta^t)\}$ be the truthful reporting strategy: $\delta_t^*(\theta^{t-1}, \theta) = \theta \ \forall t, \theta^{t-1}, \theta$. By the revelation principle, there is no loss of generality in restricting the planner's choice of mechanism so as to satisfy the (ex ante) incentive compatibility condition: $U^\gamma(\delta^*, f) \geq U^\gamma(\delta, f), \forall \delta, \gamma$, i.e. truth-telling weakly dominates any other reporting strategy. Evidently, $f = \alpha(\{\delta^*\}_\gamma, f)$ and the planner can be thought of as selecting an induced allocation subject to the incentive compatibility constraints (and other restrictions described below). Let $U^\gamma(\delta, f|\widetilde{\theta}^t)$ denote the continuation payoff to a γ-weighted agent under (δ, f) after making reports $\widetilde{\theta}^t$. Attention will be restricted to mechanisms that satisfy the boundedness condition:

$$\lim_{t\to\infty} \sup_{\delta, \hat{\theta}^t} \beta^t U^\gamma(\delta, f|\widetilde{\theta}^t) = 0. \tag{2}$$

Define $\omega_t^\gamma(\widetilde{\theta}^t, \theta_t; f) = u(c_t^\gamma(\widetilde{\theta}^t)) + \theta_t v^\gamma(y_t(\widetilde{\theta}^t))$ to be the γ-weighted agent's period t payoff from f after making series of reports $\widetilde{\theta}^t$ and receiving a preference shock θ_t. The $(\gamma, t, \widetilde{\theta}^{t-1}, \widetilde{\theta}, \theta_t)$-temporary incentive compatibility constraints (t.i.c.s) are given by:

$$\omega_t^\gamma(\widetilde{\theta}^{t-1}, \theta_t, \theta_t; f) + \beta U^\gamma(\delta^*, f|\widetilde{\theta}^{t-1}, \theta_t) \geq \omega_t^\gamma(\widetilde{\theta}^{t-1}, \widetilde{\theta}, \theta_t; f) + \beta U^\gamma(\delta^*, f|\widetilde{\theta}^{t-1}, \widetilde{\theta}), \qquad (3)$$

$\forall \widetilde{\theta} \in \Theta$. Under these constraints, one period deviations from truth-telling are ruled out after every history. It is more convenient to work with a collection of t.i.c.'s than with the ex ante incentive compatibility constraint. Any f satisfying the ex ante incentive compatibility constraints obviously satisfies the collection of t.i.c.'s for each $(\gamma, t, \widetilde{\theta}^t, \theta_t)$. Moreover, (2) and (3) imply the ex ante incentive compatibility conditions. Thus, the planner will be constrained to select mechanisms satisfying (2) and (3) for all $(\gamma, t, \widetilde{\theta}^t, \theta_t)$.

In addition to the incentive compatibility constraint the planner will be required to provide each agent with a continuation utility in excess of some minimal amount:

$$U^\gamma(\delta^*, f|\theta^t) \geq \underline{U} \qquad (4)$$

$\forall t, \theta^t$. Such constraints have been rationalised by Atkeson and Lucas (1995) as representing a planner's ex post concern with equality, and by Phelan (1995) as capturing the agent's inability to commit to a mechanism (and the planner's inability to compel commitment). Collecting these constraints together define:

$$\Omega^\Gamma = \{\alpha^\Gamma \in A^\Gamma : \alpha^\Gamma \text{ satisfies (2), (3) and (4)}\}.$$

The final requirement on the planner is that any induced allocation must satisfy a sequence of aggregate resource constraints, $\forall t$:

$$G_t + \int_\Gamma \int_{\Theta^t} [c_t^\gamma(\theta^t) - y_t^\gamma(\theta^t)] \Pi(\theta^t) d\theta^t d\Psi_0 \leq 0, \qquad (5)$$

where G_t is an exogenously given level of planner consumption. The primal problem at an initial Ψ_0 can now be stated. Solutions to such primal problems will be called "constrained efficient allocations".

Definition 1 *A constrained-efficient allocation at Ψ_0 is a solution to the primal problem:*

$$\sup_{\alpha^\Gamma \in A^\Gamma} \int_\Gamma \gamma \sum_{t=0}^\infty \int_{\Theta^t} \beta^t \left[u(c_t^\gamma(\theta^t) + \theta_t v(y_t^\gamma(\theta^t))\right] \Pi(\theta^t) d\theta^t d\Psi_0 \qquad \text{(Primal problem}(\Psi_0))$$

subject to:

$$\begin{aligned}\alpha^{\Gamma} &\in \Omega^{\Gamma},\\ \forall t &: G_t + \int_{\gamma}\int_{\Theta^t}[c_t^{\gamma}(\theta^t) - y_t^{\gamma}(\theta^t)]\Pi(\theta^t)d\theta^t d\Psi_0 \leq 0.\end{aligned}$$

3 A dual formulation

Following Atkeson and Lucas (1992, 1995) it is more convenient to analyse constrained-efficient allocations by solving a related dual problem. To do this first associate each agent with an initial expected utility promise $w \in \mathcal{W}$ and reindex them and their allocations, α^w, accordingly. A "dual" allocation collects together the utility promise-indexed allocations of agents, $\alpha^{\mathcal{W}} = \{\alpha^w\}_w$. Let Φ_0 denote an initial cross sectional distribution for life time utility promises, and define the dual problem at Φ_0 as:

$$F(\Phi_0) = \inf_{\alpha^{\mathcal{W}} \in A^{\mathcal{W}}} \sup_t \left\{ G_t + \int_{\mathcal{W}}\int_{\Theta^t}[c_t^w(\theta^t) - y_t^w(\theta^t)]\Pi(\theta^t)d\theta^t d\Phi_0 \right\}_{t=0}^{\infty} \qquad \text{(Dual Problem)}$$

subject to

$$\begin{aligned}\alpha^{\mathcal{W}} &\in \Omega^{\mathcal{W}}\\ \forall w &\in \mathcal{W} : w = \sum_{t=0}^{\infty}\int_{\Theta^t}\beta^t\left[u(c_t^w(\theta^t)) + \theta_t v(y_t^w(\theta^t))\right]\Pi(\theta^t)d\theta^t.\end{aligned}$$

Evidently, by reindexing the individual agent allocations according to the utility they induce, any primal allocation α^{Γ} implies a dual allocation, and coupled with an initial distribution over Pareto-Negishi weights, Ψ_0, an initial distribution over initial lifetime utilities, $\Phi(\alpha^{\Gamma}, \Psi_0)$, as well. The most straightforward connection between primal and dual problems links solutions of the former to those of the latter. In particular, if $\alpha^{\Gamma *}$ solves the primal problem at Ψ_0, then it is straightforward to verify that its reindexed dual counterpart solves the dual problem at $\Phi(\alpha^{\Gamma *}, \Psi_0)$. The reverse connection is discussed below.

In the dual problem above a single planner allocates resources across a population of agents. Following Atkeson and Lucas (1992), this problem can be decentralised into a collection of component planner problems. In these problems a "component planner" is responsible for allocating resources only to those agents with a specific initial lifetime utility promise of w_0. The component planner delivers this amount of utility, respecting incentive compatibility and the lower bound on continuation utilities, at minimal cost, where cost is computed using a sequence of prices $q^\infty = \{q_t\}_{t=0}^\infty \in \ell^1$ with q_t denoting the cost of time t consumption in terms of time 0 consumption. In the sequel let $q_r^\infty = \{q_t\}_{t=r}^\infty$. The arrangement can be interpreted as one in which a population of component planners trades claims to future consumption at the prices q^∞.

Definition 2 *A component mechanism, $\mathcal{D} = \left(\underline{U}, \{G_t\}, \Phi_0, q^\infty, \alpha^{\mathcal{W}}\right)$, consists of a lower utility bound $\underline{U}$, a set of utility promises $\mathcal{W} = [\underline{U}, \infty) \cap \widetilde{\mathcal{W}}$, a sequence of planner spending shocks $\{G_t\}$, an initial utility distribution Φ_0 with support$(\Phi_0) \subseteq \mathcal{W}$, a pricing system $q^\infty \in \ell_1$, and a dual allocation $\alpha^{\mathcal{W}} = \{c_t^{w_0}, y_t^{w_0}\}_{t,w_0}$, $w_0 \in \mathcal{W}$, that satisfy:*

1. *Resource feasibility:*

$$\forall t : G_t + \int_{\mathcal{W}} \int_{\Theta^t} \left[c_t^{w_0}(\theta^t) - y_t^{w_0}(\theta^t)\right] \Pi(\theta^t) d\theta^t d\Phi_0 \leq 0. \tag{6}$$

2. *Optimality: For all w_0, $\{c_t^{w_0}, y_t^{w_0}\}_t$ solves*

$$J_0(w_0; q^\infty) = \inf \sum_{t=0}^{\infty} q_t \int_{\Theta^t} [c_t^{w_0}(\theta^t) - y_t^{w_0}(\theta^t)]\Pi(\theta^t) d\theta^t \quad \text{(Component dual problem)}$$

subject to

$$w_0 = \sum_{t=0}^{\infty} \beta^t \int_{\Theta^t} [u(c_t^{w_0}(\theta^t)) + \theta_t v(y_t^{w_0}(\theta^t))]\Pi(\theta^t) d\theta^t, \tag{7}$$

and $c_t^{w_0} \geq \mathbb{R}_+$, $y_t^{w_0} \in \mathcal{Y}$ for each (w_0, t), (3) $\forall (t, \theta^t)$, and

$$\forall \theta^r : \sum_{t=0}^{\infty} \beta^t \int_{\Theta^t} [u\left(c_{r+t}^{w_0}(\theta^{r+t})\right) + \theta_{t+r} v(y_{r+t}^{w_0}(\theta^{r+t}))]\Pi(\theta^{r+t}|\theta^r) d\theta^t \geq \underline{U}. \tag{8}$$

The following lemma links component mechanisms to the dual and primal problems and provides some initial characterisation of the former. It draws on a key theorem in Atkeson and Lucas (1992). The proof of this and most subsequent results are contained in the appendix.

Lemma 1 *Let $\mathcal{D} = \left(\underline{U}, \{G_t\}, \Phi_0, q^{\infty}, \alpha^{\mathcal{W}}\right)$ be a component mechanism satisfying:*

$$\forall t : G_t + \int_{\mathcal{W}} \int_{\Theta^t} \left[c_t^{w_0}(\theta^t) - y_t^{w_0}(\theta^t)\right] \Pi(\theta^t) d\theta^t d\Phi_0 = 0.$$

Then:

1. *$\alpha^{\mathcal{W}}$ solves the dual problem at Φ_0.*
2. *The component dual problem is a convex programming problem and $J_0(.; q^{\infty})$ is strictly convex.*
3. *$J_0(.; q^{\infty})$ is strictly increasing. It is continuous and has range equal to an interval $[\underline{b}_0, \infty)$.*
4. *Suppose $\widetilde{\gamma} : \mathcal{W} \to \Gamma$ is Borel measurable and Φ_0 is a probability measure on $(\mathcal{W}, \mathcal{B}(\mathcal{W}))$. Define $\Psi_0(\Phi_0, \widetilde{\gamma})$ setwise as $\Psi_0(S) = \Phi_0(\gamma^{-1}(S))$, $S \in \mathcal{B}(\Gamma)$. Let $\partial J_0(w; q^{\infty})$ denote the set of superdifferentials of $J_0(.; q^{\infty})$ at w. There exists a measurable selection, $\gamma(.)$, from $\partial J_0(., q^{\infty})$ such that if every agent is assigned a Pareto-Negishi weight equal to $\gamma(w_0)$, then the solution to the primal problem at $\Psi_0(\Phi_0, \gamma)$ coincides with the component mechanism.*

The component dual problem is naturally recursive in the agent's utility promise. In particular, if the continuation of a solution to that problem after some history of shocks θ^t delivers a continuation utility to the agent of w_t, then this continuation allocation solves:

$$J_t(w_t; q_t^{\infty}) = \inf \sum_{s=0}^{\infty} \frac{q_{t+s}}{q_t} \int_{\Theta^s} [c_s^{w_t}(\theta^s) - y_s^{w_t}(\theta^s)] \Pi(\theta^t) d\theta^t \tag{9}$$

subject to the incentive compatibility constraints (3) $\forall$ (s, θ^s), the participation constraint

$$w_t = \sum_{s=0}^{\infty} \beta^s \int_{\Theta^s} [u(c_s^{w_t}(\theta^s)) + \theta_s v(y_s^{w_t}(\theta^s))] \Pi(\theta^t) d\theta^t, \tag{10}$$

the boundary constraints $c_s^{w_t} \geq \Re_+$, $y_s^{w_t} \in \mathcal{Y}$ $\forall s$, and the utility bounds

$$\sum_{s=0}^{\infty} \beta^s \int_{\Theta^s} [u\left(c_{r+s}^{w_t}(\theta^{r+s})\right) + \theta_{r+s} v(y_{r+s}^{w_t}(\theta^{r+s}))] \Pi(\theta^{r+s}|\theta^r) d\theta^s \geq \underline{U}, \tag{11}$$

for each θ^r, $r \geq t$. It follows from this observation, the recursive formulation of the constraints in Atkeson and Lucas (1992) and the strict concavity of the component dual problem, that if an optimal solution to the component dual problem exists, then it can be recovered from the collection of policy functions $\left\{c_t^*, y_t^*, w_{t+1}^*\right\}$ that solve the recursive problems:[8]

$$J_t(w_t; q_t^{\infty}) = \inf \int_{\Theta} \left[c(\theta) - y(\theta) + Q_t J_{t+1}(w'(\theta); q_{t+1}^{\infty})\right] \pi(\theta) d\theta \qquad \text{(recursive component dual problem)}$$

subject to

$$w_t = \int_{\Theta} [u(c(\theta)) + \theta v(y(\theta)) + \beta w'(\theta)] \pi(\theta) d\theta, \tag{12}$$

$$\forall \theta, \widetilde{\theta} : u(c(\theta)) + \theta v(y(\theta)) + \beta w'(\theta) \geq u(c(\widetilde{\theta})) + \theta v(y(\widetilde{\theta})) + \beta w'(\widetilde{\theta}), \tag{13}$$

$$\forall \theta : w'(\theta) \in \mathcal{W},$$

and the boundary constraints $c \geq \mathbb{R}_+$, $y \in \mathcal{Y}$. Here $Q_t = q_{t+1}/q_t$ is the price of a non-contingent claim to period $t+1$ consumption in terms of period t consumption. These policy functions are of the form $c_t^* : \mathcal{W} \times \Theta \to \mathbb{R}_+$, $y_t^* : \mathcal{W} \times \Theta \to \mathcal{Y}$, $w_{t+1}^* : \mathcal{W} \times \Theta \to \mathcal{W}$. Moreover, the sequence of functions $\left\{w_{t+1}^*\right\}$, together with Φ_0, induce a corresponding sequence of cross sectional utility distributions $\{\Phi_t\}$ that evolve according to:

$$\forall t, S \in \mathcal{B}(\mathcal{W}) : \quad \Phi_{t+1}(S) = \int 1_{\left\{w_{t+1}^*(w,\theta) \in S\right\}} \pi(\theta) d\theta d\Phi_t.$$

Finally, these distributions and policy functions satisfy the resource constraints:

$$\forall t : G_t + \int \left[c_t^*(w, \theta) - y_t^*(w, \theta)\right] \pi(\theta) d\theta d\Phi_t \leq 0.$$

In the remainder of the paper, such a collection $\left\{\underline{U}, \{G_t\}, \{\Phi_t\}, q^{\infty}, \{J_t\}, \left\{c_t^*, y_t^*, w_{t+1}^*\right\}\right\}$ will be referred to as a recursive component mechanism. Note that each J_t, like J_0, is strictly increasing and convex in its first argument.

[8] The recursive representation of the constraint set relies on the assumed boundedness of plans at infinity. See Atkeson and Lucas (1992).

4 Implementation

We now show that a constrained-efficient allocation can be obtained as part of a competitive equilibrium in a market economy with taxes. The market structure in the economy corresponds to that in Bewley (1986), Huggett (1993) and Aiyagari (1994): agents can trade current consumption and non-contingent claims to next period consumption. A government administers the tax system that supplements this market arrangement. This tax system conditions transfers to agents on their current output (labour income) and their accumulated stock of claims. Thus, it is simple.

We define a market economy with asset limits and taxes, denoted $\mathcal{E}(\{\underline{b}_t\}, \Lambda_0, \{T_t\}, \{G_t\})$, a market for non-contingent claims to one period ahead consumption in each period, a sequence of debt limits $\{\underline{b}_t\}$, a continuum of agents with initial cross sectional distribution of claim holdings Λ_0, a government with a sequence of spending shocks $\{G_t\}$, and a sequence of tax functions $\{T_t\}$ with each $T_t : B_t \times \mathcal{Y} \to \mathbb{R}$, $B_t \equiv [\underline{b}_t, \infty)$. A competitive equilibrium of the market economy $\mathcal{E}(\{\underline{b}_t\}, \Lambda_0, \{T_t\}, \{G_t\})$ is defined as follows:

Definition 3 *A sequence of prices q^∞, policy functions $\left\{\widehat{c}_t, \widehat{y}_t, \widehat{b}_{t+1}\right\}$, value functions $\{V_t\}$ and cross sectional distributions of claim holdings $\{\Lambda_t\}$ is a **competitive equilibrium** of the market economy $\mathcal{E}(\{\underline{b}_t\}, \Lambda_0, \{T_t\}, \{G_t\})$ if :*

1. *$\forall t$, $\widehat{c}_t : B_t \times \Theta \to \Re_+$, $\widehat{y}_t : B_t \times \Theta \to \mathcal{Y}$, $\widehat{b}_{t+1} : B_t \times \Theta \to B_{t+1}$ and $V_t : B_t \to \mathbb{R}$.*

2. *V_t is the value function for and $\left(\widehat{c}_t, \widehat{y}_t, \widehat{b}_{t+1}\right)$ are the optimal policy functions for the recursive problem[9]:*

$$V_t(b) = \sup \int [\{u(c(\theta)) + \theta v(y(\theta))\} + \beta V_{t+1}(b'(\theta))]\pi(\theta)d\theta \qquad (14)$$

subject to

$$b = c(\theta) - y(\theta) + T(b, y(\theta)) + Q_t b'(\theta)$$

$b'(\theta) \in B_{t+1}$, $y(\theta) \in \mathcal{Y}$, $c(\theta) \geq 0$ for each θ. And the policy functions attain the optima in this collection of optimisations.

[9] We implicitly invoke the boundedness of the agent's utility functions to justify focussing on this recursive problem.

3. $\forall t$, *support*$(\Lambda_t) = B_t$, *and* $\forall S \in \mathcal{B}(B_{t+1})$, $\Lambda_{t+1}(S) = \int 1_{\left\{\widehat{b}_{t+1}(b,\theta) \in S\right\}} \pi(\theta) d\theta d\Lambda_t$.

4. $\forall t$, $G_t + \int \left[\widehat{c}_t(b,\theta) - \widehat{y}_t(b,\theta)\right] \pi(\theta) d\theta d\Lambda_t = 0$.

The formal definition of implementation now follows.

Definition 4 *We say that a recursive component mechanism* $\{\underline{U},\ \{G_t\},\ \{\Phi_t\},\ q^\infty,\ \{J_t\},\ \left\{c_t^*, y_t^*, w_{t+1}^*\right\}\ \}$, *can be* ***implemented*** *in a market economy* $\mathcal{E}(\ \{\underline{b}_t\}, \Lambda_0, \{T_t\}, \{G_t\})$ *if*

1. $\forall t$, $J_t(\mathcal{W}) = B_t$,

2. *for each* $S \in \mathcal{B}(\mathcal{W})$, $\Lambda_0(S) = \Phi_0(J_0^{-1}(S))$,

3. *and the economy has a competitive equilibrium* $\left\{\{\Lambda_t\}, q^\infty, \{V_t\}, \left\{\widehat{c}_t, \widehat{y}_t, \widehat{b}_{t+1}\right\}\right\}$ *such that for each* $w_0 \in \mathcal{W}$, *the policy functions* $\left\{\widehat{c}_t, \widehat{y}_t, \widehat{b}_{t+1}\right\}$ *starting at* $J_0(w_0)$ *induce the same allocation as the functions* $\left\{c_t^*, y_t^*, w_{t+1}^*\right\}$ *beginning at* w_0.

Asset holdings will be used in the market economy to keep track of the expected discounted value of future transfers available to an agent. The first condition above says that the asset limits are not too tight in the sense that the set of expected discounted future transfer values delivered to agents under the mechanism at each date t, $J_t(\mathcal{W})$, equals the set of asset levels potentially available to agents. The second condition says that the initial distribution over expected future transfer values induced by the mechanism coincides with the initial asset distribution in the market economy.

Definition 5 *If the recursive component mechanism* $m = \{\underline{U},\ \{G_t\},\ \{\Phi_t\},\ q^\infty,\ \{J_t\},\ \left\{c_t^*, y_t^*, w_{t+1}^*\right\}\ \}$ *can be implemented in a market economy* $\mathcal{E}(\ \{\underline{b}_t\}, \Lambda_0, \{T_t\}, \{G_t\})$ *with competitive equilibrium* $\left\{\{\Lambda_t\}, q^\infty, \{V_t\}, \left\{\widehat{c}_t, \widehat{y}_t, \widehat{b}_{t+1}\right\}\right\}$, *then we call* $\{\{\underline{b}_t\},\ \{G_t\},\ \{\Lambda_t\},\ q^\infty,\ \{T_t\},\ \{V_t\},\ \left\{\widehat{c}_t, \widehat{y}_t, \widehat{b}_{t+1}\right\}\ \}$ *a* ***fiscal decentralisation*** *of* m *with taxes* $\{T_t\}$. *If there exists a fiscal decentralisation of* m *with taxes* $\{T_t\}$, *we say that* m *can be implemented with taxes* $\{T_t\}$.

The remainder of this section provides sufficient conditions for a recursive component mechanism to admit a fiscal decentralisation. The approach exploits the recursivity of the component dual problem: subdividing it into a collection of static problems. Thus, as a precursor, consider the following static implementation problem. Imagine that an agent's preferences over resource-output allocations (x, y) are given by:

$$\int \left[d(x(\theta)) + \theta v(y(\theta))\right] \pi(\theta) d\theta,$$

where d is strictly increasing, continuous and strictly concave and consider the family of static dual problems indexed by $w \in \mathcal{W}$:

$$J(w) = \inf \int \left[x(\theta) - y(\theta)\right] \pi(\theta) d\theta, \qquad \text{(Static dual problem)}$$

s.t.

$$w = \int \left[d(x(\theta)) + \theta v(y(\theta))\right] \pi(\theta) d\theta,$$

$$d(x(\theta)) + \theta v(y(\theta)) \geq d(x(\widetilde{\theta})) + \theta v(y(\widetilde{\theta})), \ \forall \theta, \widetilde{\theta} \in \Theta,$$

$x(\theta) \in X \subseteq \mathbb{R}$, X convex, $y(\theta) \in \mathcal{Y}$. The following lemma describes some properties of this problem.

Lemma 2 *1. A solution to the static dual problem exists for each $w \in \mathcal{W}$.*

2. Such a collection of solutions can be described by a pair of functions (x^*, y^*), $x^* : \mathcal{W} \times \Theta \to X$ and $y^* : \mathcal{W} \times \Theta \to \mathcal{Y}$, with each $(x^*(w, \cdot), y^*(w, \cdot))$ monotone (in θ).

3. J is strictly convex and increasing.

Next consider a static economy with taxes in which agents have wealths $b \in B = J(\mathcal{W})$ and solve:

$$D(b) = \sup \int \left[d(x(\theta)) + \theta v(y(\theta))\right] \pi(\theta) d\theta$$

subject to, for each $\theta \in \Theta$,

$$b = x(\theta) - y(\theta) + T(b, y(\theta)),$$

$x(\theta) \in X$ and[10]

$$y(\theta) \in \widehat{\mathcal{Y}}(b) \equiv \left\{ y : y = y^*(J^{-1}(b), \widetilde{\theta}) \text{ some } \widetilde{\theta} \in \Theta \right\} \subseteq \mathcal{Y}. \qquad \text{(OF)}$$

[10]This constraint is rather unnatural from the point of view of a market economy. It may be motivated by assuming that the agent faces a penal tax for output choices that can not be rationalised by the mechanism. Its removal will be discussed later.

If there exists a tax function T such that for all (b, θ), the agent makes the same choices as under the mechanism, $(x^*(J^{-1}(b), \theta), y^*(J^{-1}(b), \theta))$, then we will say that (x^*, y^*) can be implemented in the static economy (with tax function T). It is straightforward to show that such implementation is possible:

Lemma 3 *(Static taxation principle) Consider the static dual problem above. An optimal solution (x^*, y^*) can be implemented in a static economy with a tax function $T : B \times \mathcal{Y} \to \mathbb{R}$.*

This result motivates the subdivision of the recursive component dual problem into two steps. In the first step, the planner solves a "static dual" problem of the sort described in the preceding lemma; in the second step, the planner allocates resources between current and future consumption so as to minimise the cost of attaining an interim utility promise that excludes the cost of producing current output. Thus, in the second step at date t, conditional on the interim utility promise d_t, the planner solves:

$$X_t(d_t; q_t^\infty) = \inf_{c(\theta), w'(\theta)} c(\theta) + Q_t J_{t+1}(w'(\theta); q_{t+1}^\infty), \qquad \text{(Step 2 problem)}$$

subject to

$$d_t = u(c(\theta)) + \beta w'(\theta), \tag{15}$$

$c(\theta) \geq 0$ and $w'(\theta) \in \mathcal{W}$. Constraint (15) is the participation constraint for the step 2 problem. It is straightforward to show that X_t is strictly increasing and convex. Let $D_t(.; q_t^\infty) = X_t^{-1}(.; q_t^\infty)$, so that D_t is strictly increasing and concave. Preceding back to the first step, the planner solves:

$$J_t(w_t; q_t^\infty) = \inf \int_\Theta [x(\theta) - y(\theta)] \, \pi(\theta) d\theta \qquad \text{(Step 1 problem)}$$

subject to

$$w_t = \int_\Theta [D_t(x(\theta); q_t^\infty) + \theta v(y(\theta))] \pi(\theta) d\theta, \tag{16}$$

$$D_t(x(\theta); q_t^\infty) + \theta v(y(\theta)) \geq D_t(x(\widetilde{\theta}), q_t^\infty) + \theta v(y(\widetilde{\theta})), \ \forall \theta, \widetilde{\theta} \in \Theta. \tag{17}$$

$x(\theta) \geq \underline{b}_{t+1}$. This problem is mathematically equivalent to the static dual problem given above, and its solution can be interpreted as a static mechanism that allocates a demand for output and resources to an agent at date t. From the previous lemma, there is a tax function T_t that can be used to implement this static mechanism, conditional on the agent valuing resources according to D_t. As in the proof of Lemma 3, T_t ensures that, for each $b \in B_t$, the set of budget feasible output and resource choices for an agent coincide with those available under the optimal component dual mechanism at the corresponding lifetime utility level $J_t^{-1}(b)$. Applying this logic successively to all dates, a family of tax functions $\{T_t\}$ can be constructed. To verify that these do indeed implement the recursive decentralised mechanism, it remains to check that agents use the sequence of value functions $\{D_t\}$ to value resources. The following simple proposition does this.

Proposition 1 *Assume that u and v are bounded, and that $m = \{\underline{U}, \{G_t\}, \{\Phi_t\}, q^\infty, \{J_t\}, \{c_t^*, y_t^*, w_{t+1}^*\}\}$ is a recursive component mechanism. Then, there exists a sequence of taxes $\{T_t\}$ that can be used to implement the mechanism.*

Proof: As described above, the component dual problems associated with m define a sequence of static dual planner problems. Hence, they define a sequence of tax functions $\{T_t\}$ that can be used to implement the solutions to these problems, provided agents value resources according to D_t at each t, where D_t is constructed from J_{t+1} and q^∞ by inverting the X_t from the step 2 problem above. We now check this.

Assume that bond prices in the market economy are given by q^∞ and let the tax system be given by $\{T_t\}$. Truncate the economy at some date τ and suppose that for all $b_{\tau+1} \in B_{\tau+1}$, an agent receives the continuation payoff $V_{\tau+1}(b_{\tau+1}) = J_{\tau+1}^{-1}(b_{\tau+1})$. At each date $t \in \{0, ..., \tau\}$, the agent's choice problem in the economy can be divided into two stages. In the second stage, the agent allocates resources between consumption and bond purchases. In the first stage, the agent chooses her effort and pays her taxes conditional on her output and wealth. Beginning at period τ, it is immediate that the second stage of the agent's problem is the dual of the second stage component planner's problem at τ. Thus, it induces the agent value function D_τ and agents value period τ resources in the same way in the truncated economy as under the mechanism. Moving back to the first stage of the agent's problem at τ, since T_τ is constructed as in the proof of Lemma 3, at each b, the set of current resource-output allocations, (x, y) that satisfy $y(\theta) \in \widehat{\mathcal{Y}}(b) \equiv \left\{\widetilde{y} : \widetilde{y} = y_\tau^*(J_\tau^{-1}(b), \widetilde{\theta}) \text{ some } \widetilde{\theta} \in \Theta\right\}$, $\forall \theta$ and are budget feasible coincide with the set of allocations available to the agent under the mechanism at $J_\tau^{-1}(b_\tau)$. Hence, the optimal resource-output allocation for the agent in the truncated economy at date τ with wealth b_τ is the same as that under the mechanism at $J_\tau^{-1}(b_\tau)$, date τ. The stage 1 problem induces a value function over b_τ's, which satisfies $V_\tau = J_\tau^{-1}$. Iterating back to period 0, it is clear that agents make identical choices in the truncated economy as under the mechanism, and receive the same allocations. This is true for arbitrary finite τ. It follows from this and boundedness of the utility functions, that for $\tau = \infty$, the agent obtains a higher payoff from the allocation implied by the mechanism than any other feasible allocation in the (untruncated) economy. ■

5 Characteristics of the optimal mechanism

This section describes some characteristics of the optimal component mechanism. Since most of these results are already known in the literature, it proceeds heuristically. The aim is to give a brief overview of the distortions generated by the incentive compatibility condition in the private information economy. Since most of these results are already known in the literature, it proceeds heuristically. For simplicity, assume that the mechanism and each value function $J_t(., q_t^\infty)$ is piecewise differentiable.[11] Denote partial derivatives with respect to θ by dots. Given a triple of piece-wise differentiable functions $c : \Theta \to \mathbb{R}_+$, $y : \Theta \to \mathcal{Y}$ and $w' : \Theta \to \mathcal{W}$, define the functions $U : \Theta \to \mathbb{R}$ and $W : \Theta \to \mathbb{R}$ as follows:

$$U(\theta) = u(c(\theta)) + \theta v(y(\theta)) + \beta w'(\theta) \quad (18)$$

$$W(\theta) = \int_{\underline{\theta}}^{\theta} U(\widetilde{\theta})\pi(\widetilde{\theta})d\widetilde{\theta}. \quad (19)$$

By standard arguments, e.g. Salanié (1997), any triple of piecewise differentiable functions $\{c, y, w'\}$ is incentive compatible for an agent if and only if:

$$\dot{U}(\theta) = v(y(\theta)) \quad \text{a.e. } \theta, \quad (20)$$

$$y(\theta) \text{ non-decreasing in } \theta. \quad (21)$$

Imposing the constraint that y is non-decreasing would require explicitly introducing $\dot{y}$ as a control variable and requiring that $\dot{y} \geq 0$. The standard approach is to drop this restriction

[11] Kahn (1993) provides conditions for an optimal static mechanism to be absolutely continuous. These conditions require that the planner's value functions be twice continuously differentiable.

and either i) impose conditions on the problem that guarantee it or ii) check that the solution satisfies the condition ex post. For now the constraint is dropped. Also, $\dot{W}(\theta) = U(\theta)\pi(\theta)$. The variables $U(\underline{\theta})$ and $U(\overline{\theta})$ are free, while $W(\underline{\theta}) = 0$ and $W(\overline{\theta}) = w$, for an agent with continuation utility promise w. All other boundary constraints on the control variables are dropped except $w'(\theta) \geq 0$.

The Hamiltonian for the component planner at w can be formulated as:

$$\begin{aligned} H_t^w(\theta, U, y, w'; \chi^w, \lambda^w, \phi^w, Q_t, J_{t+1}) = -\chi^w U(\theta)\pi(\theta) - \lambda^w(\theta)v(y(\theta)) + \\ \left[C(U(\theta) - \theta v(y(\theta)) - \beta w'(\theta)) - y(\theta) + Q_t J_{t+1}(w'(\theta); q_{t+1}^{\infty})\right]\pi(\theta) \\ + \phi^w(\theta)(w'(\theta) - \underline{U})\pi(\theta). \end{aligned}$$

Here χ^w is the multiplier on the constraint $\int_{\underline{\theta}}^{\overline{\theta}} U(\theta)\pi(\theta)d\theta = w$, λ^w is the costate variable associated with the incentive compatibility constraint (20) and $\phi^w(\theta)$ is the multiplier on the constraint $w'(\theta) \geq \underline{U}$. The component planner's problem is:

$$\sup_{\{U,y,w'\}} \int H_t^w(\theta, U, y, w'; \chi^w, \lambda^w, \phi^w, Q_t, J_{t+1})d\theta,$$

with first order conditions:

$$\dot{\lambda}^w(\theta) = \left[\chi^w - \frac{1}{u'(c(\theta))}\right]\pi(\theta), \tag{22}$$

$$0 = -\frac{\beta}{u'(c(\theta))} + Q_t J'_{t+1}(w'(\theta); q_{t+1}^{\infty}) + \phi^w(\theta), \tag{23}$$

$$0 = -\lambda^w(\theta)\, v'(y(\theta)) - \left[\frac{\theta v'(y(\theta))}{u'(c(\theta))} + 1\right]\pi(\theta). \tag{24}$$

Additionally, the transversality conditions imply that $\lambda^w(\overline{\theta}) = \lambda^w(\underline{\theta}) = 0$. The costate variable λ^w captures the distortions created by the incentive compatibility constraints. It is readily shown that the optimal $c(\theta)$ is non-decreasing[12], hence, by (22), $\dot{\lambda}^w(\theta)$ is non-increasing. This coupled with the condition $\lambda^w(\overline{\theta}) = \lambda^w(\underline{\theta}) = 0$ implies that $\lambda^w(\theta) \geq 0$, all θ. As the first order conditions above indicate, λ^w generates a pattern of wedges. These are described below:

[12] The incentive constraints imply that $u(c(\theta)) + \beta w'(\theta)$ is non-decreasing. The strict convexity of J_{t+1} and strict concavity of u then establish that $c(\theta)$ is non-decreasing.

1. **The effort wedge**. Equation (24) implies that in those θ states in which $\lambda^w(\theta) \neq 0$, an agent's marginal rate of substitution between effort and consumption is not equated with the corresponding marginal rate of transformation (which is always 1). In particular, if $\lambda^w > 0$ on the interval $(\underline{\theta}, \overline{\theta})$, then effort is distorted downwards given consumption. Since, $\lambda^w(\overline{\theta}) = \lambda^w(\underline{\theta}) = 0$, there is no distortion at the end points.

2. **The insurance wedge**. Absent the incentive compatibility condition, the optimal arrangement would call for agents to receive the same consumption bundle across θ states. In contrast, equation (22) implies that if $\dot{\lambda}^w(\theta) > 0$ over some interval of θ's, a spread in agent consumptions and marginal utilities occurs.

3. **The intertemporal wedge**. The envelope condition of the component planner, (22), and (23) imply that the solution to the component planner's problem satisfies:

$$\frac{1}{u'(c_t^*(w,\theta))} = \frac{Q_t}{\beta} E_{\widetilde{\theta}} \left\{ \frac{1}{u'(c_{t+1}^*(w_{t+1}^*(w,\theta),\widetilde{\theta}))} \right\}. \tag{25}$$

   When the incentive constraint binds, consumption will vary across θ realisations and, hence, (25) and Jensen's inequality imply:

$$Q_t u'(c_t^*(w,\theta)) < \beta E_{\widetilde{\theta}} \left\{ u'(c_{t+1}^*(w_{t+1}^*(w,\theta),\widetilde{\theta})) \right\}. \tag{26}$$

The insurance and effort wedge are well known from the static public finance literature and have direct implications for income taxation. The next lemma makes the connection explicit and drops the (OF) condition. For a proof see Albanesi and Sleet (2003).

Lemma 4 *Let* $m = \{\underline{U}, \{G_t\}, \{\Phi_t\}, q^\infty, \{J_t\}, \{c_t^*, y_t^*, w_{t+1}^*\}\}$ *be a recursive component mechanism such that 1) at each* t*,* J_{t+1} *is differentiable and 2) at each* t *and* w*,* $\{c_t^*(w,\cdot)$, $y_t^*(w,\cdot)$, $w_{t+1}^*(w,\cdot)\}$ *is differentiable and coincides with a solution to*

$$\sup_{\{U,y,w'\}} \int H_t^w(\theta, U, y, w'; \chi^w, \lambda^w, \phi^w, Q_t, J_{t+1}) d\theta.$$

Assume that m *can be implemented with taxes* $\{T_t\}$*. Then,*

1. *the marginal output tax,* $T_{t,y} \equiv \partial T_t/\partial y$*, satisfies, at each* (w,θ)*,* $T_{t,y}(J_t(w), y_t^*(w,\theta)) = \left[\frac{\theta v'(y_t^*(w,\theta))}{u'(c_t^*(w,\theta))} + 1\right]$.

2. *The set* $\widehat{\mathcal{Y}}_t(b) = \left\{\widetilde{y} : \widetilde{y} = \widehat{y}_t(J_t^{-1}(w),\theta) \text{ some } \theta \in \Theta\right\}$ *is an interval of the form.* $\left[\underline{\widehat{y}}(b), \overline{\widehat{y}}(b)\right]$.

3. *Let* $\widetilde{T}_t$ *denote the extension of* T_t *off of the graph of* $\widehat{\mathcal{Y}}_t$ *obtained by setting, for each* $(b,y) \in B \times \mathcal{Y}$*,* $\widetilde{y}(b,y) = \sup(\inf(y, \overline{\widehat{y}}(b)), \underline{\widehat{y}}(b))$ *and* $\widetilde{T}_t\ (b,y) = T_t\ (b, \widetilde{y}(b,y)) + T_{t,y}(b, \widetilde{y}(b,y))(y - \widetilde{y}(b,y))$*. The sequence of taxes* $\left\{\widetilde{T}_t\right\}$ *can be used to implement the mechanism. With these taxes, the output feasibility condition (OF) may be dropped.*

Golosov, Kocherlakota and Tsyvinski (GKT) (2002) show that the intertemporal wedge arises in a large class of dynamic models with private information. In these models higher wealth reduces the correlation between an agent's consumption and her labor supply, and thus higher savings reduce the incentive for an agent to work in the subsequent period. The wedge adjusts for this additional marginal social cost of saving. Based on condition (26), GKT suggest that implementation may call for a positive marginal asset tax, thus, providing a rationale for the taxation of capital income that is often absent from Ramsey models. We show below, however, that the presence of the wedge in (26) does *not* in general imply such a rationale.

To understand this issue, first note that the tax function T_t implies a stochastic marginal asset tax. Assuming that T_t is differentiable, $T_{t,b}(b_t, y_t) \equiv \partial T_t/\partial b(b_t, y_t)$ depends upon the realisation of y_t which is unknown to an agent selecting b_t. Thus, the agent's intertemporal Euler equation, absent a binding debt limit, can be rewritten as:

$$\begin{aligned} & \beta E_t\left[u'(c_{t+1})\right] - Q_t u'(c_t) \\ = \quad & \beta\left\{E_t\left[T_{t+1,b}\right] E_t\left[u'(c_{t+1})\right] + Cov_t\left[T_{t+1,b}, u'(c_{t+1})\right]\right\} \geq 0. \end{aligned}$$

Consequently, if $Cov_t\left[T_{t+1,b}, u'(c_{t+1})\right]$ is sufficiently positive, the expected marginal asset tax, $E_t\left[T_{t+1,b}\right]$, may be zero or even negative. A positive covariance between $T_{t+1,b}$ and the marginal utility of consumption "damages the asset" by inducing a positive correlation between after tax returns on the assets and income. In doing so it boosts incentives at $t+1$ by reinforcing the covariance between consumption and labour income.

To explore this issue further, we study a pair of simple two period examples. We show that implementation of the constrained efficient allocation in these examples calls for $Cov_t\left[T_{t+1,b}, u'(c_{t+1})\right] > 0$. This positive covariance is needed to rule out joint deviations in which the agent saves too much in the first period and works too little in the second. In addition, we show that if the allocation to be implemented in the subsequent period $\{c_{t+1}, y_{t+1}\}$ does not depend on the savings level, b_{t+1}, then $E_t\left[T_{t+1,b}\right] = 0$. Otherwise, $E_t\left[T_{t+1,b}\right]$ can be strictly positive.

6 Revealing Examples

The economy lasts for two periods. In the first period agents consume and do not supply labour. In the second period, they receive an ability shock, $\theta \in \left\{\theta^1, \theta^2\right\}$, $\theta^1 < \theta^2$, choose a labor supply from the discrete set: $\left\{\underline{y}, \overline{y}\right\}$, $\underline{y} < \overline{y}$.and consume. We assume that the planner faces an intertemporal price Q and attempts to minimise the expected cost of providing agents with initial utility w_1.[13]. Hence, the planner solves:

$$J_1(w_1) = \inf_{w_2} -c_1 + QJ_2\left(w_2\right) \tag{27}$$

$$\text{s.t.}: w_1 = u(c_1) + \beta w_2,$$

$$J_2\left(w_2\right) = \inf_{\left\{c_2^j, y_2^j\right\}} \sum \left\{y_2^j - c_2^j\right\} \pi^j$$

$$\text{s.t. } w_2 = \sum \left\{u(c_2^j) + \theta^j v(y_2^j)\right\} \pi^j$$

$$u(c_2^1) + \theta^1 v(y_2^1) \geq u(c_2^2) + \theta^1 v(y_2^2).$$

Denote with w_2^* and $\alpha^* = \left\{c_1^*, \left\{c_2^{j*}, y_2^{j*}\right\}_{j=1,2}\right\}$ the continuation utility and the allocation that solve this problem. It is straightforward to show that there exists two values of w_1,

[13] We do not impose a lower bound on continuation utility for this example. We discuss the implications of this bound below.

$\underline{\phi}$ and $\overline{\phi}$ such that for $w_1 \in [\underline{\phi}, \overline{\phi}]$, $w_2^*(w_1)$, $c_1^*(w_1)$, $J_2(w_2)$, $c_2^{1*}(w_2)$, $c_2^{2*}(w_2)$ are monotone and $y_2^{1*}(w_2^*(w_1)) = \overline{y}$, $y_2^{2*}(w_2^*(w_1)) =\underline{y}$. We restrict attention to $w_1 \in [\underline{\phi}, \overline{\phi}]$.

The incentive and continuation promise keeping constraint completely pin down the second period allocation. Defining $\Delta v = v(\overline{y}) - v(\underline{y}) < 0$ and using the incentive compatibility and promise keeping constraints to eliminate the consumption levels, the planner's continuation payoff is:

$$J_2(w_2) = \left[C(w_2 - E\theta v(\underline{y}) - \theta^1 \Delta v) - \underline{y}\right] \pi^1 + \left[C(w_2 - E\theta v(\underline{y})) - \overline{y}\right] \pi^2.$$

The solution to this optimisation problem implies an intertemporal wedge:

$$Qu'(c_1^*) < \beta E u'(c_2^*). \tag{28}$$

6.1 Implementation

We now implement the allocation that solves the planning problem in market arrangement with taxes. At time 1, agents are endowed with b_1 bonds. They choose consumption, c_1, and trade risk-free claims to consumption at time 2 at price Q. At time 2, they choose consumption, c_2, output, y_2, based on their wealth b_2 and preference shock θ. They face a tax $T(b_2, y_2)$ conditional on wealth and output. Assume that $T(\cdot, y)$ is differentiable for each y. The agent's problem is:

$$\sup_{c_1, \left\{c_2^j, y_2^j\right\}_{j=1}^2, b_2} u(c_1) + \beta \left\{ \left[u(c_2^1) + \theta^1 v(y_2^1)\right] \pi + \left[u(c_2^2) + \theta^2 v(y_2^2)\right] (1-\pi) \right\}$$

subject to:

$$\begin{aligned} b_1 &= c_1 + Q b_2 \\ \forall j &: \quad b_2 = c_2^j + T(b_2, y_2^j) - y_2^j. \end{aligned}$$

We denote the solution to the agents' problem in the market economy with: $\{\widehat{c}_1(b_1)$, $\left\{\widehat{c}_2^j(b_2),\ \widehat{y}_2^j(b_2)\right\}_{j=1,2}$, $\widehat{b}_2(b_1)\}$.

The government selects T so as to implement α^*. We set $b_1 = J_1(w_1)$. Then, the absence of taxes in period 1 and the agent's period 1 budget constraint this implies that the tax system must induce agents to select the savings level $b_2^* = J_2(w_2^*)$[14]. Additionally, it must induce the appropriate output levels $\left\{y_2^{j*}\right\}$. Using the budget conditions to substitute the consumption levels out, the tax system must satisfy:

$$\overline{y} \in \arg\max_{y \in \{\underline{y}, \overline{y}\}} u(b_2^* - T(b_2^*, y) + y) + \theta^1 v(y), \tag{29}$$

$$\underline{y} \in \arg\max_{y \in \{\underline{y}, \overline{y}\}} u(b_2^* - T(b_2^*, y) + y) + \theta^2 v(y), \tag{30}$$

and

$$Qu'(c_1^*) = \beta \left\{ \left[(1 - T_b(b_2^*, \overline{y}))u'(c_2^{1*}) \right] \pi^1 + \left[(1 - T_b(b_2^*, \underline{y}))u'(c_2^{2*}) \right] \pi^2 \right\}. \tag{31}$$

We restrict attention to a parsimonious candidate tax function of the form[15]:

$$T(b, y) = T_0(y) + T_1(y)b. \tag{32}$$

Implementation of α^* clearly requires a tax system $T(b, y)$ consistent with the satisfaction of the agent's necessary conditions for optimality at α^*. Thus, the tax system must satisfy (29)-(31). However, these conditions are not sufficient for implementation. There are, in fact, many tax functions of the form (32) that satisfy them One of these is the simple tax function $T_0(y) + T_1 b$, with $T_1 > 0$. Can the planner's optimal allocation be implemented with a marginal asset tax that does not depend on output? Lemma 5 shows that the answer

[14] The planner's optimal mechanism determines the pattern of optimal transfers to an agent, it does not determine the extent to which they are obtained through an initial endowment of bonds or through appropriate initial lump sum transfers. This indeterminacy is common to all the examples and the general decentralisation.

[15] The fiscal decentralisation permits a larger set of agent choices than the mechanism selected by the planner. In the latter, an agent's only choice is her shock announcement in period 2. Since agents have a common initial utility promise in this example, the solution to the planner's problem delivers a unique continuation utility, w_2^*, and corresponding expected transfer, $J_2(w_2^*)$, to truthful agents in period 2. In the market economy, agents make a savings choice in period 1 and an labor choice in period 2. This potentially gives them access to a larger set of allocations.

The solution to the planner's problem only pins down the tax function at the savings level b_2^* in the market economy. For other savings levels, we rely on a linear extension of this tax system in b. In the next example, we establish a mapping between savings in the market economy and continuation utilities in the planner's problem. This allows us to use the solution to the planner's problem to tie down the tax system for a range of savings levels.

is no: an output-contingent marginal asset tax is required to rule out an otherwise profitable deviation for the agent in which she sets $b_2 > b_2^*$ and $y_2 = \underline{y}$ irrespective of the realization of θ.

Lemma 5 *An output-contingent marginal asset tax is necessary for implementation.*

Proof: Assume the converse, i.e. assume that the solution to the planner's problem can be implemented with a tax function of the form $T_0(y) + T_1 b$. Note that because the incentive constraints bind under the mechanism:

$$u(c_2^{1*}) + \theta_1 v(\overline{y}) = u(c_2^{2*}) + \theta_1 v(\underline{y}).$$

Hence, in the decentralisation, the agent can obtain the utility level w_1 in one of two ways. It can save the required amount, b_2^*, and can then select the output and consumption levels prescribed by the mechanism or, alternatively, it can save b_2^*, and select $\underline{y}$ (and consumption c_2^{2*}) in all states. Now, the agent's Euler equation in the decentralisation implies:

$$qu'(c_2^{1*}) = \beta(1 - T_1)\left[u'(c_2^{1*})\pi + u'(c_2^{2*})(1 - \pi)\right] < \beta(1 - T_1)u'(c_2^{2*}).$$

It follows that the second choice described above is dominated by one in which the agent saves $b_2^* + \varepsilon$ (for ε small) and produces $\underline{y}$ regardless of his shock. But then this latter choice gives a payoff of more than w_1, contradicting the optimality of the mechanism-prescribed allocation. ■

To understand the intuition behind the potential for joint deviations, suppose that a tax function, $T(b, y) = T_0(y) + T_1 b$, satisfies the necessary conditions (29)-(30). Let $v_1(b_2)$ denote the agent's continuation utility under this tax function, if she chooses to produce $\overline{y}$ when the shock is θ^1 and $\underline{y}$ when the shock is θ^2, at savings level b_2:

$$v_1(b_2) = \left\{u(b_2 - T(b_2, \overline{y}) + \overline{y}) + \theta^1 v(\overline{y})\right\}\pi + \left\{u(b_2 - T(b_2, \underline{y}) + \underline{y}) + \theta^2 v(\underline{y})\right\}(1 - \pi).$$

Let $v_2(b_2)$ denote the agent's continuation utility under this tax function at b_2, if she chooses to produce $\underline{y}$ regardless of the shock:

$$v_2(b_2) = u(b_2 - T(b_2, \underline{y}) + \underline{y}) + \left\{\theta^1\pi + \theta^2(1 - \pi)\right\}v(\underline{y}).$$

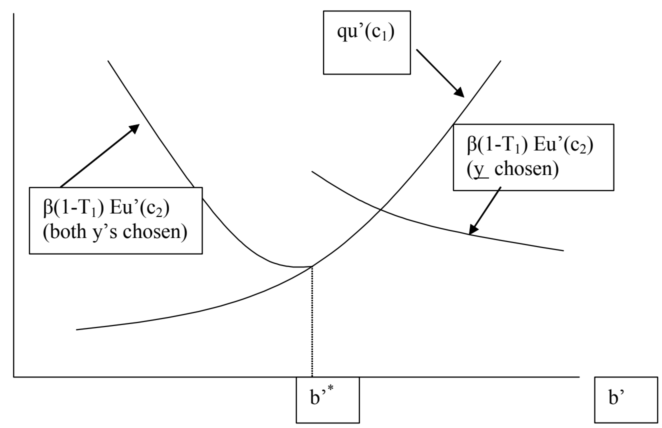


Figure 1: Savings tradeoff with a non-contingent asset tax

Evidently, $v_1(b_2^*) = v_2(b_2^*)$. Moreover, v_2 crosses v_1 from below at b_2^*: for lower levels of b_2, the agent produces the high amount of output in the low cost state and has relatively higher consumption then; for higher levels of b_2, the agent produces the low level of output in all states and has relatively lower consumption in the low cost state. The continuation value function is the upper envelope of these functions[16], and, thus, is non-concave and has a kink at b_2^*. Figure 1 shows an agent's marginal utility of consumption at time 1 and her discounted expected marginal utility of consumption at time 2 as a function of b_2 for a fixed b_1. The agent chooses y_2 optimally conditional on b_2. The kink at b_2^* corresponds to an upward jump in the slope of the continuation value function, $(1 - T_1)Eu'(c_2)$, which equals $v_1'(b_2)$ to the left of b_2^* and $v_2'(b_2)$ to the right of b_2^*. Thus, the agent's intertemporal Euler equation has two solutions, as indicated in the Figure. Lemma 6 completes the analysis by showing that a marginal asset tax that has a negative covariance with consumption and a zero average does implement the planner's optimal allocation in the market economy.

Lemma 6 *Any tax function $T(b, y) = T_0(y) + T_1(y)b$ implements the planner's optimal allocation only if:*

[16]Strictly, speaking over a range of b_2's large enough that agent does not wish to produce high output in all shock states.

- $T_1(\underline{y}) > T_1(\overline{y})$.
- $T_1(\underline{y})\pi + T_1(\overline{y})(1-\pi) = 0$

Proof: Define $\underline{T}_1$ and $\overline{T}_1$ as follows:

$$qu'(c_1^*) = \beta(1-\underline{T}_1)u'(c_2^{2*})$$

$$qu'(c_1^*) = \beta(1-\overline{T}_1)u'(c_2^{1*}).$$

These marginal tax values are consistent with the agent's intertemporal Euler equation at the planner's optimal allocation:

$$qu'(c_1^*) = \beta\left[(1-\overline{T}_1)u'(c_2^{1*})\pi + (1-\underline{T}_1)u'(c_2^{2*})(1-\pi)\right]. \tag{33}$$

We now assert that $T_1(\underline{y}) = \underline{T}_1 > \overline{T}_1 = T_1(\overline{y})$. To see this, firstly suppose $T_1(\underline{y}) < \underline{T}_1$. Then, as in the proof of the previous lemma, the agent can save the amount implied by the solution to the planner's problem (b_2^*) and select an output $\underline{y}$ regardless of her shock. This is feasible and delivers the same payoff, w_1, to the agent as the planner's solution. However, if

$$qu'(c_1^*) < \beta(1-T_1(\underline{y}))u'(c_2^{2*}),$$

the agent can obtain a payoff above w_1 by saving slightly more than b_2^*, and selecting an output of $\underline{y}$ regardless of her shock. Hence, $T_1(\underline{y}) \geq \underline{T}_1$. Similarly, if $T_1(\underline{y}) > \underline{T}_1$, the agent can improve on the planner's solution by saving slightly less than b_2^* and selecting an output $\underline{y}$ regardless of her shock. Hence, $T_1(\underline{y}) = \underline{T}_1$. (33) then ensures $T_1(\overline{y}) = \overline{T}_1$. Next note that

$$\begin{aligned}\overline{T}_1 &= 1 - \frac{q}{\beta}\frac{u'(c_1^*)}{u'(c_2^{1*})},\\ \underline{T}_1 &= 1 - \frac{q}{\beta}\frac{u'(c_1^*)}{u'(c_2^{2*})}.\end{aligned}$$

so

$$\overline{T}_1\pi + \underline{T}_1(1-\pi) = 1 - \frac{q}{\beta}u'(c_1^*)\left\{\frac{\pi}{u'(c_2^{1*})} + \frac{(1-\pi)}{u'(c_2^{2*})}\right\} = 0,$$

which completes the argument. ■

The terms $T_0(y)$ can then be chosen to guarantee that the desired consumption levels are obtained, i.e.

$$\begin{aligned} T_0(\overline{y}) &= (1-\overline{T}_1)b_2^* + \overline{y} - c_2^{1*} \\ T_0(\underline{y}) &= (1-\underline{T}_1)b_2^* + \underline{y} - c_2^{2*}. \end{aligned}$$

6.2 Recursive implementation

We now extend the previous example. The planner faces a heterogeneous population of agents, distinguished by their different initial utility promises. If all agents have a common initial utility promise w_1, given the absence of shocks in the first period, all agents must be induced to undertake a common specific savings level to implement the optimal allocation. Instead, heterogeneity in utility promises translates into a dispersed wealth distribution in the market economy. Implementation then necessitates a tax function that recreates the appropriate pattern of wedges at all admissible wealth levels. Proposition 1 provides a strategy for achieving this with a recursive tax system. Under the optimal tax system, the government implements the allocation $\left\{c_t^*\left(J_t^{-1}\left(b_t\right),\theta\right), y_t^*\left(J_t^{-1}\left(b_t\right),\theta\right), w_t^*\left(J_t^{-1}\left(b_t\right),\theta\right)\right\}$ for an agent with wealth b_t with a tax system $T_t\left(b_t, y_t\right)$. We explore the implications of the recursive nature of the tax system for the properties of the marginal asset tax in the two period economy.

Assume the economy is populated by a continuum of agents indexed by $w_1 \in \mathcal{W}$, and that the planner's problem is analogous to the one in the previous section for each w_1. In addition, the planner seeks to attain a distribution over initial utilities, Ψ. Let $\left\{c_1^*(w_1), \left\{c_2^{j*}(w_2^*(w_1)), y_2^{j*}(w_2^*(w_1))\right\}_{j=1,2}\right\}$ denote an allocation for an agent with utility promise w_1, and $w_2^*(w_1)$ the associated continuation utility, in the corresponding optimal mechanism.

The market structure is the same as in the previous section, with a tax function $T\left(b_2, y_2\right)$, differentiable in b_2. We set agents' initial wealth level to: $b_1 = J_1(w_1)$. We restrict attention to $w_1 \in \left[\underline{\phi}, \overline{\phi}\right]$. Given the initial setting of b_1, the agent's period 1 budget constraint implies that the tax system must induce:

$$\begin{aligned}\hat{b}_2(b_1) &= b_2^*(w_1),\\ \hat{y}_2^j\left(\hat{b}_2(b_1)\right) &= y_2^{j*}(w_1),\end{aligned}$$

where $b_2^*(w_1) \equiv J_2(w_2^*(w_1))$ for $b_1 = J_1(w_1)$ for the optimal allocation to be implemented. Using the budget conditions to substitute for consumption, the planner's optimal allocation must satisfy:

$$\begin{aligned}\overline{y} &\in \arg\max_{y\in\{\underline{y},\overline{y}\}} u(b_2^*(w_1) - T(b_2^*(w_1), y) + y) + \theta^1 v(y)\\ \underline{y} &\in \arg\max_{y\in\{\underline{y},\overline{y}\}} u(b_2^*(w_1) - T(b_2^*(w_1), y) + y) + \theta^2 v(y)\end{aligned}$$

and

$$Qu'(c_1^*(w_1)) = \beta\left\{\left[(1 - T_b(b_2^*(w_1), \overline{y}))u'(c_2^{1*}(w_1))\right]\pi_1 + \left[(1 - T_b(b_2^*(w_1), \underline{y}))u'(c_2^{2*}(w_1))\right](1 - \pi_1)\right\},$$

for all $w_1 \in \left[\underline{\phi}, \overline{\phi}\right]$. As in the previous example, these conditions are necessary but not sufficient for implementation. Based on that example, we guess that the tax function that guarantees optimality of the planner's optimal allocation satisfies a collection of "state-by-state" intertemporal Euler equations and then verify our conjecture.

Lemma 7 *The planner's optimal allocation can be implemented with a tax function $T(b_2, y_2)$ that satisfies:,*

$$\begin{aligned}Qu'(c_1^*(w_1)) &= \beta(1 - T_b(b_2^*(w_1), \underline{y}))u'(c_2^{2*}(w_1))\\ Qu'(c_1^*(w_1)) &= \beta(1 - T_b(b_2^*(w_1), \overline{y}))u'(c_2^{1*}(w_1)),\end{aligned}$$

for $w_1 \in \left[\underline{\phi}, \overline{\phi}\right]$.

Lemma 7 implies that the expected marginal asset tax, $E_\theta T_b(b_2^*(w_1), y(b_2^*(w_1), \theta))$, is zero. The argument is identical to the proof of Lemma 6 and the intertemporal wedge is entirely generated by the negative covariance between the marginal asset tax and the agent's marginal utility of consumption. How general is this result?

A feature of this example is that, for $w_1 \in \left[\underline{\phi}, \overline{\phi}\right]$, the labor allocation as a function of θ does not depend on w_1. Hence, for all $b_2^* \in J_2\left(w_2^*\left(\mathcal{W}\right)\right)$, the tax system that implements the planner's optimal allocation should induce the same choice of y_2 conditional on θ. This is not true in general. To see this, consider the same two period economy where now agents can choose y_2 on the interval $\left[\underline{y}, \bar{y}\right]$. Then, $y_2^*\left(w_2\right)$, the level of effort in the planner's optimal allocation as a function of continuation utility, is monotone decreasing in w_2.[17] A tax system that implements the constrained efficient allocation for this economy must satisfy:

$$u'\left(c_1^*\left(w_1\right)\right) = \beta E_\theta\left(1 - T_b\left(b_2^*\left(w_1\right), y_2^*\left(J_2^{-1}\left(b_2^*\left(w_1\right)\right), \theta\right)\right)\right) u'\left(c_2^*\left(J_2^{-1}\left(b_2^*\left(w_1\right)\right), \theta\right)\right), \quad (34)$$

$$\hat{y}_2\left(b_2^*\left(w_1\right), \theta\right) = \arg\max u\left(b_2^*\left(w_1\right) + y_2 - T\left(b_2^*\left(w_1\right), y_2\right)\right) + \theta v\left(y_2\right). \quad (35)$$

Then, $\hat{y}_2\left(b_2^*\left(w_1\right), \theta\right)$ should be monotone decreasing in b_2 at $b_2^* = J_2\left(w_2^*\left(w_1\right)\right)$ under the tax system that implements the optimal allocation.

Assume θ is distributed over a continuous support $\left[\underline{\theta}, \bar{\theta}\right]$ with a continuous density $f\left(\theta\right)$ and let $\mathcal{Y}\left(w_2\right) = y_2^*\left(w_2, \left[\underline{\theta}, \bar{\theta}\right]\right) \subseteq \left[\underline{y}, \bar{y}\right]$. Assume that the tax system $T\left(b_2, y_2\right)$ is differentiable in y_2 and that it satisfies (34), (35) and in addition:

$$u'\left(b_1 - Qb_2\right) = \beta\left(1 - T_b\left(b_2, y_2\right)\right) u'\left(y_2 + b_2 - T\left(b_2, y_2\right)\right), \quad (36)$$

for $y_2 \in \mathcal{Y}\left(w_2\right)$, which implies $E_\theta T_b\left(b_2, y_2\right) = 0$. Then, differentiating the right hand side of this expression with respect to y_2 yields:

$$\begin{aligned} 0 &= \left(1 - T_b\left(b_2, y_2\right)\right)\left(1 - T_y\left(b_2, y_2\right)\right) u''\left(y_2 + b_2 - T\left(b_2, y_2\right)\right) \\ &\quad - T_{by}\left(b_2, y_2\right) u'\left(y_2 + b_2 - T\left(b_2, y_2\right)\right). \end{aligned}$$

However, notice that by (35), $\hat{y}_2\left(b_2, \theta\right)$ satisfies:

$$u'\left(y_2 + b_2 - T\left(b_2, y_2\right)\right)\left(1 - T_y\left(b_2, y_2\right)\right) + \theta v'\left(y_2\right) = 0,$$

which implies:

$$\frac{\partial \hat{y}_2}{\partial b_2}\left(b_2, \theta\right) = \frac{\left(1 - T_b\right)\left(1 - T_y\right) u''\left(c_2\right) - T_{by} u'\left(c_2\right)}{-\left[\left(1 - T_y\right)^2 u''\left(c_2\right) + \theta v''\left(y_2\right)\right]} = 0,$$

[17] For a proof of this see Albanesi and Sleet (2003).

under (36). It follows that a tax system which satisfies (36) cannot be used to implement an allocation in which an agent's effort depends upon her continuation utility. Thus, the state by state Euler equations and our earlier proof that expected marginal asset tax equals zero, do not hold in general. Moreover, the agent's budget constraint in period 2 implies that, for all b_2 above the debt limit:

$$\int_{\underline{\theta}}^{\overline{\theta}} T(b_2, \widehat{y}_2(b_2, \theta)) f(\theta) d\theta = 0.$$

Totally differentiating this with respect to b_2 leads to the formula

$$\int_{\underline{\theta}}^{\overline{\theta}} \left[T_b(b_2, \widehat{y}_2(b_2, \theta)) + T_y(b_2, \widehat{y}_2(b_2, \theta)) \frac{\partial \widehat{y}_2}{\partial b_2}(b_2, \theta) \right] f(\theta) d\theta = 0.$$

Thus, if $T_y(b_2, \widehat{y}_2(b_2, \theta))$ is strictly positive and wealth effects on effort are strictly negative over some positive measure interval of θ's, then a strictly positive expected marginal asset tax will occur. To analyse these effects further, we turn to numerical examples in section 7.

6.3 Utility bounds and debt limits

The previous examples focus on the implications of the incentive compatibility constraint for the tax system. However, the lower bound on continuation utilities in the planner's problem also has important implications both for taxes and the structure of asset markets. Since agents' lifetime utilities are monotonically linked to transfer values in the dual component mechanism, the lower bound on continuation utilities can be implemented with a "debt limit" in the market economy. This limit binds on those agents with after-tax resources, $b_2 + y_2 - T(b_2, y_2)$, below a critical level, restricting their ability to borrow When it binds, it also generates a wedge in the intertemporal Euler equations of agents, though one that runs in the opposite direction to the wedge generated by the tax system. Specifically,

$$u'(c_t) = \beta E_t \left[(1 - T_{t,b}(b_{t+1}, y_{t+1})) u'(c_{t+1}) \right] + \widehat{\varphi}_t,$$

where $\widehat{\varphi}_t$ denotes the multiplier on the debt limit. Consequently, the intertemporal wedge of an agent can be decomposed into two components:

$$\underbrace{\frac{\beta E_t u'(c_{t+1}) - Q_t u'(c_t)}{\beta E_t u'(c_{t+1})}}_{\text{Intertemporal Wedge}} = \underbrace{\frac{E_t T_{t,b}(b_{t+1}, y_{t+1}) u'(c_{t+1})}{E_t u'(c_{t+1})}}_{\text{Tax component}} - \underbrace{\frac{\widehat{\varphi}_t}{Eu'(c_{t+1})}}_{\text{Limit Component}}. \quad (37)$$

The first of these components, on the right hand side of (37), is induced by the tax system, the second by the multiplier on the debt limit. Obviously, the second component is only non-zero when the debt limit binds. However, the lower utility bound in the planner's problem has a broader set of implications for the optimal tax system. These stem from its interaction with the incentive compatibility constraint. Specifically, it restricts the planner's ability to use continuation utilities to provide incentives for truthful revelation. Thus, the planner is forced to rely more heavily on variations in current consumption to provide incentives. Close to the lower bound, the constrained efficient allocation exhibits greater variability in consumption and larger insurance and effort wedges. These characteristics translate into a tax function that has more variation in T_b across efforts and higher values of T_y close to the debt limit.

We illustrate this reasoning in the context of the following three period example. The first and second periods are identical to those of the earlier examples. In period three, the agent consumes, but does not supply labor. In addition, the agent must obtain utility above a lower bound $\underline{U}$. Let w_2 denote a second period utility promise to the agent. The planner's period 2 continuation cost function, J_2, has the form:

$$J_2(w_2) = \left[X_2(w_2 - E\theta v(\underline{y}) - \theta^1 \Delta v) - \underline{y}\right]\pi + \left[X_2(w_2 - E\theta v(\underline{y})) - \overline{y}\right](1-\pi),$$

where X_2 is an interim cost function implied by the problem:

$$\begin{aligned} X(d_2) &\equiv \min C(u_2) + qC(u_3) \\ \text{s.t. } d_2 &= u_2 + \beta u_3. \\ u_3 &\geq \underline{U}. \end{aligned}$$

The incentive compatibility constraint implies that the planner must set:

$$d_2^1 - d_2^2 = -\theta^1 \Delta v > 0, \quad (38)$$

where $d_2^j = u_2^j + \beta u_3^j$, for $j = 1, 2$ denotes the utility from current and future consumption awarded to an agent with shocks θ^j at time 2. If w_2 is low enough that the constraint $u_3^2 \geq \underline{U}$ binds, generating the spread in (38) requires a reduction in u_2^2 and an increase in d_2^1 relative to a value of w_2 for which the lower bound on continuation utility is not binding. As a result, the planner's optimal allocation will feature greater dispersion in date 2 consumption across agents with different values of θ.

From Lemma 7, we know that, when the agent's debt limit is not binding, a tax function that implements the planner's optimal allocation satisfies:

$$Qu'(c_1^*) = \beta(1 - T_b(b_2^*, y_2^{*j}))u'(c_2^{j*}).$$

Consequently, the greater variability of consumption in response to θ close to the lower bound translates into a greater responsiveness of marginal asset taxes to y_2 for agents close to the debt limit.[18] The greater variability in consumption in response to θ also increases the effort wedge $\left[\frac{\theta v'(y_2^{i*}(w_1))}{u'(c_2^{i*}(w_1))} + 1\right]$. If agents have a continuous labor choice, this will translate in a larger marginal tax on output, $T_y = \partial T\left(b, y\right)/\partial y$, at wealth values close to the debt limit. In the problem, with a discrete output choice, the marginal output tax is not defined.

7 Numerical Analysis

To shed further light on the properties of the optimal tax system, we study several representative numerical examples. Our benchmark example uses numerical parameters adopted in recent calibrations of Bewley economies with endogenous labour supply. Other examples provide sensitivity analysis by altering those parameter values that have received most attention in the public finance literature, namely those that influence labor supply elasticity and the distribution of skill shocks. However, overall the examples in this section are intended to be illustrative rather than fully calibrated quantitative exercises. The key qualitative properties of the optimal tax function that we emphasise are common not only to the examples presented below, but also to many others that we have computed..

[18] For a given initial wealth level, there will also be some adjustments in period 1 consumption relative to average consumption in period 2.

7.1 Calibration and numerical procedure

We adopt the utility function:

$$u(c, y; \theta) = \alpha \frac{c^{1-\sigma}}{1-\sigma} + (1-\alpha)\frac{(\bar{l} - \theta y)^{1-\gamma}}{1-\gamma}. \tag{39}$$

Here, θ may be interpreted as a cost of effort shock or the reciprocal of a productivity shock. This preference specification is common in macroeconomics[19].

The numerical parameters for this economy are $\left\{\alpha, \sigma, \gamma, \bar{l}, \beta, \overline{U}, \Theta, \pi, \{G_t\}\right\}$. We select preference parameters for our benchmark parameterization following Heathcote, Storesletten and Violante (2003) (HSV). They set $(1-\alpha)/\alpha = 1.184$ with $\bar{l}$ to 1, σ to 1.461, γ to 2.54, to match the empirical share of hours worked, the wage-hours correlation and a Frisch elasticity of labour supply of 0.5[20]. In addition, we choose β to 0.94. In the benchmark case, we assume that $1/\theta$ is distributed uniformly on the interval $[0.2, 1.2]$. We set $\underline{U}$ to -3.48, which translates into a debt limit of -0.214 or about 56% of average output at this minimal wealth. Note that this value of $\underline{U}$ lies between the lifetime utility that an agent would attain if she were at her "natural" debt limit[21] in a Bewley economy without taxes, which is clearly $-\infty$, and the lifetime utility under autarchy without taxes and markets, equal to -2.74[22]. Government consumption is constant over time and equal to 1.0 in each period, which amounts to approximately 30% of period aggregate output. Recent work in public finance has emphasised the role of the labour supply elasticity and the shock distribution in shaping the pattern of marginal income taxes in static models. In the sensitivity analysis, we also consider $\gamma = 1.5$ and $\gamma = 4$, which correspond, respectively, to a higher and lower Frisch elasticity of labor supply. We also consider an example with a log-normal

[19] In a departure from previous assumptions, the preferences in (39) are not multiplicatively separable in the shock and agent disutility of output and they are not bounded. We used separability in θ to establish the concavity of the planner's problem. We check concavity numerically and find that it is satisfied in all of our numerical examples. Boundedness is required in the proof of proposition 1. We assume this to be the case below.

[20] Treating $1/\theta$ as a productivity shock and defining labour supply as θy, the formula for the Frisch labour supply elasticity is $\frac{1}{\gamma}\frac{1-l}{l}$, in this case.

[21] The natural debt limit is the maximal debt that an agent can service. Given the bound on the agent's per period output, this debt limit is finite, but it translated into a utility bound of $-\infty$.

[22] A more stringent debt limit increases the curvature of the optimal tax function.

distribution for $1/\theta$.

We numerically solve for a steady state component mechanism. In the steady state, the price of one period risk-free claims is constant at Q, the component planner's cost and policy functions are time invariant, and the cross sectional distribution of utility promises, Φ, is a fixed point of the Markov operator implied by the policy function for continuation utilities. The algorithm proceeds by solving the recursive component planner problem using numerical dynamic programming techniques at each intertemporal price[23]. The component planner policy functions are then used to obtain an approximation to the limiting distribution over utility promises. The intertemporal price is adjusted until this distribution is consistent with resource feasibility. The solution to the component planner problems imply a pattern of transfers to agents across (w, θ) states. The steady state tax function T can then be recovered from this pattern by mapping each $(b, y) \in \text{Graph}\widehat{\mathcal{Y}}$ to a corresponding utility promise $J^{-1}(w)$ and shock $y^{*-1}(J^{-1}(w))(y)$ and extending T linearly from Graph$\widehat{\mathcal{Y}}$ onto the whole of $[J(\underline{U}), \infty) \times \mathcal{Y}$.

7.2 Numerical Results

The optimal tax function T for the benchmark parameterisation is illustrated in Figure 2 below on the set $B \times \mathcal{Y}$. The striking feature of the tax function is the high curvature in the neighbourhood of the debt limit. Immediate inspection of the graph reveals that T_y is large for b small, and that T_b is high for y and b small, but low and negative for y large and b small. This conforms with the discussion in section 6.3.Figure 3 shows the marginal labour income tax as a function of y. Each curve corresponds to a different asset level b. The solid line is drawn for an agent at the debt limit, the dash dot line for an agent at a higher wealth, the dashed line for a still higher wealth and so on. Marginal labour income taxes are strongly decreasing in *wealth*. We find to this to be a robust feature of the optimal tax function across alternative numerical parameterisations (see the sensitivity analysis below). In contrast, we find that the dependence of the marginal labour income tax on labour income is highly sensitive to the choice of utility function and shock distribution.

[23] Θ is discretised with $1/\theta$ taking 51 values over the interval defined above. The value functions are approximated with schumaker shapre preserving splines.

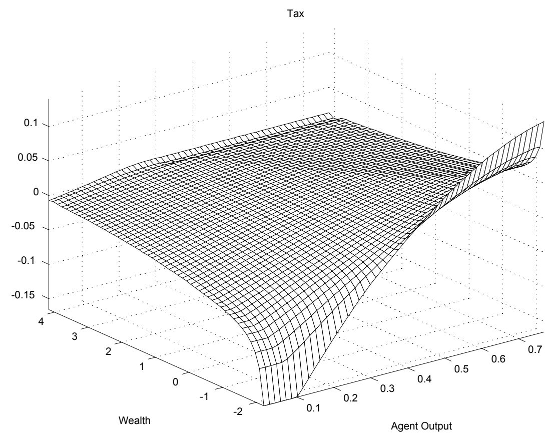


Figure 2: Tax function, T

For the benchmark case, at all wealth levels marginal income tax curves have an inverted U-shape over y's, with zero marginal taxes at the lowest and highest labour incomes attained at each asset level, and positive taxes at intermediate y's. It is interesting to compare these findings with results in the static non-linear taxation literature. That literature has analysed the dependence of optimal marginal income taxes on income only. It has found few general results and, like us, sensitivity of the optimal tax schedule to the specification of preferences and the underlying shock distribution. In his seminal contribution, Mirrlees (1971) obtained marginal income tax rates that are low and slightly declining in income, while recent work by Diamond (1998) and Saez (2001) find marginal income taxes that are high and sharply declining in income at low income levels[24]. The Diamond and Saez result has attracted some attention in the literature, where it has been interpreted as consistent with the phasing out of social benefits at low incomes. Our result, that marginal income taxes should be high at low wealth levels rather than, or as well as, low income levels,

[24]The low value of marginal income taxes in Mirrlees (1971) stems from his choice of utility function: $\log c + \log(1 - l)$, which implies a high labour supply elasticity. The monotonically declining pattern of rates in income stems from his assumption of a log-normal distribution of shocks. Diamond (1998) and Saez (2001) assume lower labour supply elasticities and a (calibrated) Pareto shock distribution, and obtain higher marginal income taxes with a U-shape pattern in income.

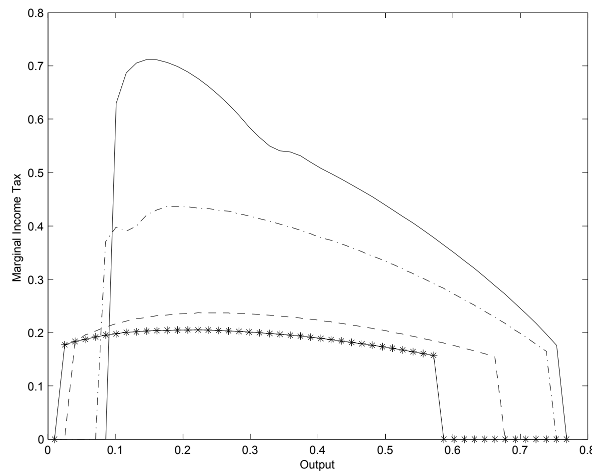


Figure 3: Marginal Labour Income Tax

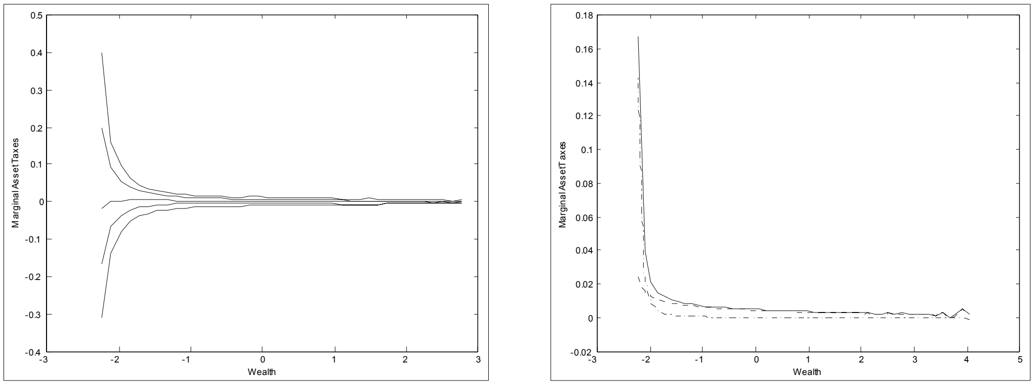
complements this.

Figure 4 illustrates the implications of the intertemporal wedge for optimal asset taxation. The left hand side panel of the figure shows the marginal asset tax, T_b. Each curve plots $T_b(\cdot, y)$ across wealth levels b for a fixed labour income level y. Curves for five different income levels are plotted, with the highest curve corresponding to the lowest income level, the next highest curve corresponding to the second lowest income level and so on. As the Figure indicates, marginal asset taxes are highest at low income, and lowest, and negative, at high income levels. Moreover, the figure reveals that the average marginal asset tax must be small over most of the wealth range, since it must lie between the outer most curves in the figure. The right hand side panel of the figure elaborates further. Recall that the contribution of the tax function to the intertemporal wedge can be written decomposed as:

$$\frac{E_t T_{t,b}\left(b_{t+1}, y_{t+1}\right) u'\left(c_{t+1}\right)}{E_t u'\left(c_{t+1}\right)} = E_t T_{t,b}\left(b_{t+1}, y_{t+1}\right) + \frac{Cov_t\left\{T_{t,b}\left(b_{t+1}, y_{t+1}\right), u'\left(c_{t+1}\right)\right\}}{E_t u'\left(c_{t+1}\right)}. \quad (40)$$

The total contribution, on the left hand side of the equality, is equal to the intertemporal wedge for agents with a non-binding debt limit. It can also be regarded as a naive measure of the marginal asset tax since. The right hand panel of 4 shows this contribution (solid line), the expected marginal asset tax component (dashed line) and the covariance component

Figure 4: Asset taxation



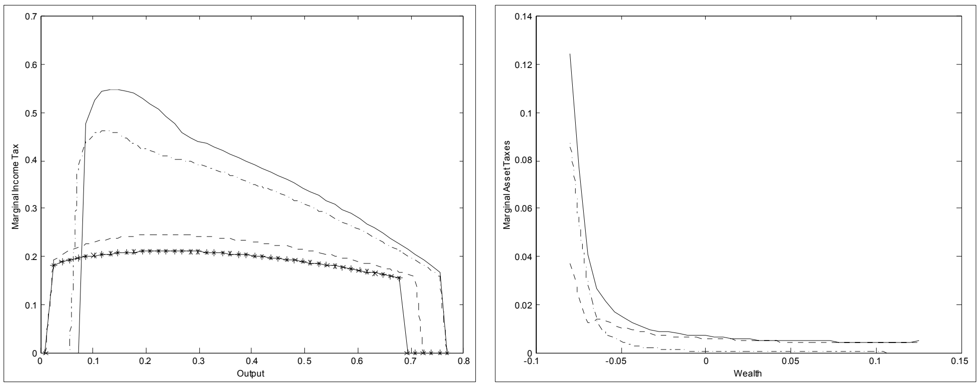
Marginal asset taxes, T_b Intertemporal wedge: ET_b, $\frac{\mathrm{Cov}(T_b,u_c)}{Eu_c}$

(dash-dot) (corresponding to the two terms to the right of the equality in (40)) as a function of wealth. The figure indicates that the covariance component is always positive. Then, the total contribution acts as an upper bound for the expected marginal asset tax. The total contribution is small away from the debt limit. Over most of the wealth range it is less than 1% in value, but that close to the debt limit it becomes much larger rising to about 16%. The expected marginal asset tax peaks at a little over 2% at the debt limit, and then falls steadily with wealth. The covariance component is also decreasing in wealth, but it is much larger close to the limit and falls off much more quickly as wealth increases. Consequently, the covariance component plays the major role in generating the total contribution of the tax system to the intertemporal wedge only when the agent's wealth is small and the total contribution is high.

7.2.1 Sensitivity Analysis

Figures 5 to 7 show marginal labour income tax rates and intertemporal wedge decompositions for several other numerical parameterisations. The first parameterisation is one with a lower γ (equal to 1.5), and, hence, a higher labour supply elasticity. The second uses a higher value of γ (equal to 4). The third assumes a log-normal distribution for $1/\theta$

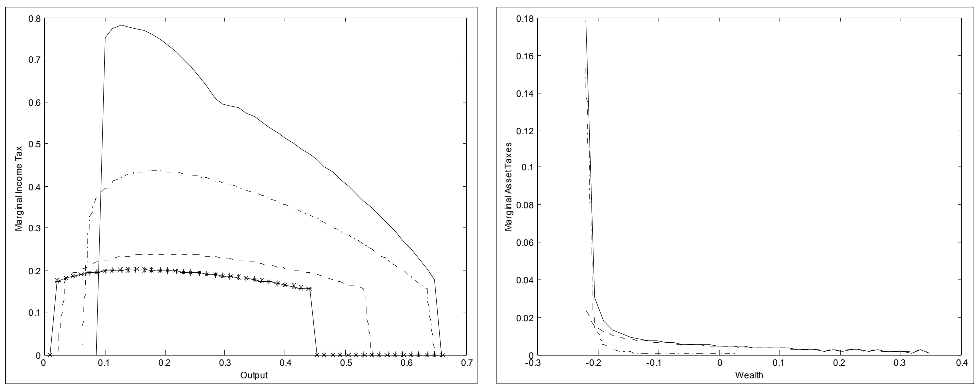
Figure 5: Low γ Case



Marginal Income Tax Intertemporal Wedge

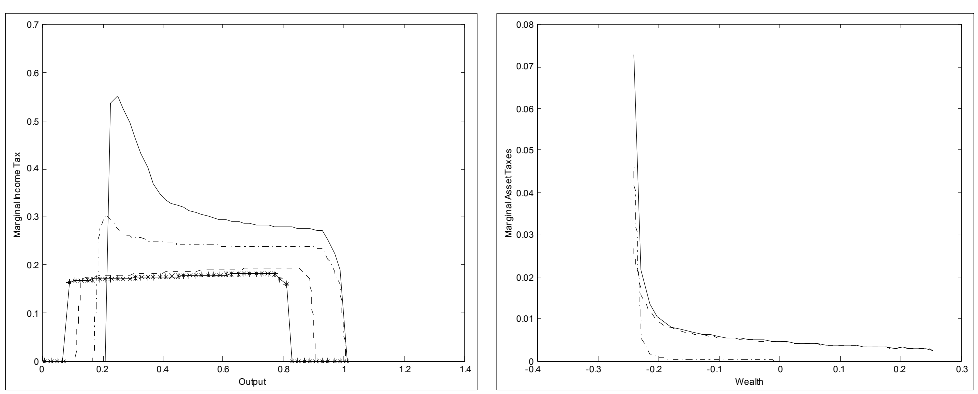
shocks. As expected, the pattern of marginal income taxes across labour income levels (at low wealths) is sensitive to the choice of utility and shock distribution function. Marginal labour income taxes are lower when the labour supply elasticity is high and higher when it is low. The log-normal distribution places more mass on lower shock values and, consistent with work in numerical public finance, this translates into high marginal income taxes at low incomes (and low wealths). However, other features are robust across the examples. Marginal labour income taxes are always higher at lower wealth levels. Moreover, as wealth increases, the tax functions converge to one that is approximately linear in labour income, at a low marginal rate. Additionally, all of the tax functions exhibit a value of the intertemporal wedge that is monotone in agent wealth, with a sharp decrease from a high value close to the debt limit (between 7% and 18%) to a much lower value outside a fairly small neigbourhood of this limit. As in the benchmark case, the covariance term makes a large contribution to the intertemporal wedge close to the lower limit and a very small contribution elsewhere.

Figure 6: High γ case



Marginal Income Tax | Intertemporal wedge

Figure 7: Log Normal Case



Marginal Income Tax | Intertemporal Wedge

8 Concluding remarks

We conduct an exploratory analysis of optimal dynamic taxation in economies where agents are subject to idiosyncratic, privately observed, skill shocks. We characterize the optimal fiscal policy by implementing the constrained efficient allocation in a market economy with taxes. The market structure is identical to that in Bewley (1986): agents trade current output and risk-free claims to future output subject to a budget constraint and a debt limit. The government supplements this market arrangement with a simple tax system that depends only upon an agent's wealth and her labour income. Incentive compatibility conditions endogenously restrict the set of fiscal instruments available to the government and shape the features of the resulting tax system. We analytically derive implications for both income and asset taxation and further explore them in numerical examples.

A critical assumption in our set up is that skill shocks are independently distributed over time (as well as across agents). The empirical literature on the structure of individual wages and skills suggests that plausible skill processes should incorporate both a fixed and an autoregressive component. Hence, a quantitative study of the optimal dynamic tax system in this class of economies should allow for persistence in skill shocks. This is a challenging problem, since an agent utility promise is not sufficient to render the planner's problem recursive. Fernandes and Phelan (2001) and Doepke and Townsend (2001) propose recursive formulations that restore recursivity by enlarging the state-space to include off-the-equilibrium path utility promises. These work when the cardinality of the shock set is low. We plan to extend our implementation strategy and derive the corresponding implications for optimal tax systems in economies with persistence and limited record keeping in future work.

The decentralization we study in this paper embeds specific assumptions about the relative roles of markets and government policy. In particular, no private insurance contracts are allowed with the current market structure. In practice, government welfare programs and private credit and insurance contracts are complementary in providing incentives and determining the extent of risk-sharing supported in a competitive equilibrium. Exploring this complementarity could provide important insight in cross-country differences in government policies.

9 Appendix: Proofs

Proof of Lemma 1: Part 1 follows directly from Theorem 1, Atkeson and Lucas (1992) p. 443. For part 2, notice that the component problem can be recast in terms of utility allocations with $u_t^{w_0}(\theta^t) = u(c_t^{w_0}(\theta^t))$ and $v_t^{w_0}(\theta^t) = v(a_t^{w_0}(\theta^t))$. This renders all constraints linear for the transformed problem and the component planner's objective strictly convex. Strict monotonicity is elementary. Let $\overline{u}$ and $\underline{v}$ denote the bounds on u and v. $\mathcal{W}$ is clearly bounded above by $\overline{w} = (\overline{u} + E\theta\overline{v})/(1-\beta)$. Since u and $-v$ are strictly increasing, $\lim_{w\uparrow\overline{w}} J_0(w, q^\infty) = \infty$. Thus, since $J_0(., q^\infty)$ is strictly convex it must be continuous except, possibly, at $\underline{U}$. Suppose $J_0(\underline{U}, q^\infty) < \lim_{w\downarrow\underline{U}} J_0(w, q^\infty) - \varepsilon$ some $\varepsilon > 0$. Let $\{w_n\}$ be a sequence converging to $\underline{U}$ from above. With out loss of generality assume that the solution in period 0 for an agent with promise $\underline{U}$ is such that $s \equiv \int u(c_0(\theta)) + \theta v(y_0(\theta))d\pi < (1-\beta)\overline{w}$. (If this is not true find a period such that it is). Let $(\widehat{c}, \widehat{y})$ be such that $u(\widehat{c}) + E\theta v(\widehat{y}) = s + \Delta$. Define $\{c_n, y_n, \delta_n\}$ so that $u(c_n) = (1-\delta_n)u(c_0) + \delta_n u(\widehat{c})$, $v(y_n) = (1-\delta_n)v(y_0) + \delta_n v(\widehat{y})$, $\delta_n \in (0,1)$ and the $\underline{U}$-agent's allocation with these alterations delivers a lifetime utility of w_n. This is feasible for n large enough. Then the cost of delivering this allocation converges towards $J_0(\underline{U}, q^\infty)$ and must be below $J_0(w_n, q^\infty)$ for n large enough - a contradiction. Since J_0 is continuous and $\lim_{w\uparrow\overline{w}} J_0(w, q^\infty) = \infty$, J_0 has range $[\underline{b}_0, \infty)$.

For the last part, the component dual problem can be rewritten as

$$\begin{aligned}\mathcal{L}(w_0; q_0^\infty) &= \inf_{\widetilde{\Omega}} \sum_{t=0}^{\infty} q_t \int_{\Theta^t} [c_t^{w_0}(\theta^t) - y_t^{w_0}(\theta^t)]\Pi(\theta^t)d\theta^t \\ &\quad + \mu \left\{ w_0 - \sum_{t=0}^{\infty} \beta^t \int_{\Theta^t} [u(c_t^{w_0}(\theta^t)) + \theta_t v(y_t^{w_0}(\theta^t))]\Pi(\theta^t)d\theta^t \right\}\end{aligned}$$

where $\widetilde{\Omega}$ captures the other constraints and $\mu \in \mathbb{R}_+$ is the Lagrange multiplier. Equivalently, the component dual problem implies a component primal problem:

$$\begin{aligned}-\mathcal{L}(w_0; q_0^\infty) &= \sup_{\widetilde{\Omega}} \mu \sum_{t=0}^{\infty} \beta^t \int_{\Theta^t} [u(c_t^{w_0}(\theta^t)) + \theta_t v(y_t^{w_0}(\theta^t))]\Pi(\theta^t)d\theta^t \\ &\quad - \sum_{t=0}^{\infty} q_t \int_{\Theta^t} [c_t^{w_0}(\theta^t) - y_t^{w_0}(\theta^t)]\Pi(\theta^t)d\theta^t.\end{aligned} \tag{41}$$

Applying the envelope theorem (Luenberger, Theorem 1 p. 222), $\mu \in \partial J_0(w_0; q^\infty)$. Set $\gamma(w_0) = \mu$. By the convexity of J_0, γ is monotone increasing and, hence, measurable. Thus, a primal allocation can be constructed, by associating each $\{c_t^{w_0}, y_t^{w_0}\}_{t,\theta^t}$ with the Pareto-Negishi weight $\gamma(w_0)$ and an initial distribution over weights recovered using the rule $\Psi_0(\Phi_0, \gamma)$. This allocation is clearly feasible for the primal problem at $\Psi_0(\Phi_0, \gamma)$. Since the allocation solves the collection of component primal problems (41), a primal version of the argument in Atkeson and Lucas (1992), ensures its optimality for the primal problem. ■

Proof of Lemma 2: Given the multiplicative nature of the shocks, existence can be established by recasting the problem in terms of utilities and applying the arguments of Kahn (1993). By a standard argument, all incentive-compatible contracts, (x, y), are monotone in the shock, and, hence, $(x^*(w, \cdot), y^*(w, \cdot))$ are monotone. That J is strictly convex and increasing follows from the arguments used to establish the same properties for J_0. ■

Proof of Lemma 3: For each $b \in B$, $y \in \widehat{\mathcal{Y}}(b)$, simply set $T(b, y) = b + y - x^*(J^{-1}(b), y^{*-1}(J^{-1}(b))(y))$. (Here, $y^{*-1}(J^{-1}(b))(y)$ denotes a selection from the preimage of $y^{*-1}(J^{-1}(b))(y)$). For other (b, y) set the tax function arbitrarily. Thus, the set of allocations available to an agent with wealth b in the static economy with taxes that coincides with those available to an agent with utility promise $J^{-1}(b)$ under the optimal mechanism. ■

Proof of Lemma 7: We aim to show that for each $w \in [\underline{\phi}, \overline{\phi}]$, the desired (optimal planner) allocation, $\alpha(w_1) = \left\{ c_1^*(w_1), \left\{ c_2^{j*}(w_1), y_2^{j*}(w_1) \right\}_{j=1,2} \right\}$, is budget feasible and optimal for an agent in the decentralised economy when given initial wealth $b_1(w_1)$. We start with budget feasibility. By (4), $\alpha(\underline{\phi})$ is clearly budget feasible at $b_1(\underline{\phi})$.Now, for $b_1(w_1)$, $w_1 > \underline{\phi}$. The desired savings choice $b_2^*(w_1)(= J_2(w_2^*(w_1)))$ and the initial consumption bundle $c_1^*(w_1)$ is budget feasible in period 1 by construction. For period 2, notice that:

$$\begin{aligned}(1-T_b(b_2^*(w_1),\underline{y})) &= \frac{qu'(c_1^*(w_1))}{\beta u'(c_2^{2*}(w_1))}\\ &= \frac{1}{u'(c_2^{2*}(w_1))}\frac{1}{J_2'(w_2^*(w_1))}\\ &= \frac{C'(u(c_2^{2*}(w_1))}{J_2'(w_2^*(w_1))}\\ &= \frac{\partial c_2^{2\circ}(b_2^*(w_1))}{\partial b_2}\end{aligned}$$

where $c_2^{2\circ}(b_2) = c_2^{2*}(b_2^{*-1}(b_2))$. And so,

$$\begin{aligned}&T(b_2^*(w_1),\underline{y}) - b_2^*(w_1) + c_2^{2*}(w_1) - \underline{y}\\ &= \underline{b} + \underline{y} - c_2^{2*}(\underline{\phi}) - \underline{b} - \int_{\underline{b}}^{b_2^*(w_1)} \big(1 - T_b(b_2,\underline{y})\big)\, db_2 + c_2^{2*}(w_1) - \underline{y}\\ &= -c_2^{2*}(\underline{\phi}) - \int_{\underline{b}}^{b_2^*(w_1)} \frac{\partial c_2^{2\circ}(b_2)}{\partial b_2} db_2 + c_2^{2*}(w_1) = 0\end{aligned}$$

Thus, the allocation ($c_2^{2*}(w_1)$,$\underline{y}$) is budget feasible for the agent when she enters period 2 with savings $b_2^*(w_1)$ and is confronts the tax function T. An identical argument ensures that $(c_2^{1*}(w_1), \overline{y})$ is budget feasible at at $b_2^*(w_1)$ as well.

We now turn to optimality. Consider an agent with initial wealth $b_1(w_1)$, $w_1 \in [\underline{\phi}, \overline{\phi}]$. The agent's problem has a well defined optimum. Suppose that such an optimum involves a savings level $\widetilde{b} \neq b_2^*(w_1)$ and an effort level equal to $\underline{y}$ regardless of shock. Suppose $\widetilde{b} = b_2^*(\widetilde{w}_1)$, $\widetilde{w}_1 \neq w_1$. If $\widetilde{w}_1 > w_1$, then the agent's first period consumption is $\widetilde{c}_1 = c_1^*(\widetilde{w}_1) + b_1(w_1) - b_1(\widetilde{w}_1) <.c_1^*(\widetilde{w}_1)$ and so

$$qu'(\widetilde{c}_1) > \beta(1 - T_b(\widetilde{b},\underline{y}))u'(c_2^{2*}(\widetilde{w}_1)).$$

Hence, this can not be an optimum: the agent can raise her payoff by cutting her savings level. Similarly, if $\widetilde{w}_1 < w_1$,

$$qu'(\widetilde{c}_1) < \beta(1 - T_b(\widetilde{b},\underline{y}))u'(c_2^{2*}(\widetilde{w}_1)).$$

So that the agent can do better by raising her savings level. If $\widetilde{b} > \overline{b}$, $\widetilde{c}_1 = c_1^*(\overline{\phi}) + b_1(w_1) - b_1(\overline{\phi}) - (\widetilde{b} - \overline{b}) <.c_1^*(\overline{\phi})$. and $\widetilde{c}_2 = c_2^*(\overline{\phi}) + (\widetilde{b} - \overline{b}) >.c_2^*(\overline{\phi})$. So,

$$qu'(\widetilde{c}_1) > \beta(1 - T_b(\overline{b},\underline{y}))u'(\widetilde{c}_2).$$

and again, this can not be an optimum. A similar argument holds for the case in which the agent picks $\widetilde{b} < \underline{b}$. Hence, if an agent does deviate to the low output level regardless of state she will choose $\widetilde{b} = b_2^*(w_1)$. But then the allocation she obtains is equal to the one she would obtain under the mechanism if she always announced the high cost shock. and, by incentive compatibility of the mechanism, she does no better than if she made the choices associated with the desired allocation. If she deviates to an allocation with output set to $\overline{y}$ regardless of shock, then by a similar argument to that given above, it can be shown that it is optimal for the agent to set her savings level to $b_2^*(w_1)$. But then the allocation she obtains is strictly worse than that if she selected the appropriate output levels. ■

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