Solution of Macromodels with Hansen-Sargent Robust Policies: Some Extensions*

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Abstract

We summarize some methods useful in formulating and solving Hansen-Sargent robust control problems, and suggest extensions to discretion and simple rules. Matlab, Octave, and Gauss software is provided. We illustrate these extensions with applications to the term structure of interest rates, the time inconsistency of optimal monetary policy, the effects of expectations on the variances of inflation and output, and on whether central banks should make their forecasts public.

Key words: robustness, model uncertainty, discretion, simple rules.
JEL: L61, E43, E52.

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1 Introduction

For all the abundance of competing models in economic research, the agents who populate them are, as a rule, fully devoted to the one model in which they are cast. They know everything about their model (including parameter values) and want to know nothing about any other. All their uncertainty is concentrated on the stochastic elements of the model, which, under the assumption of rational expectations (RE), coincides with the data generating process (DGP). Several approaches to relaxing these assumptions have been explored. Here we focus on one such approach, which we refer to as "Hansen-Sargent robustness". In recent contributions, Lars Hansen, Thomas Sargent, and coauthors have proposed an appealing method of designing choices under model uncertainty.[1] This method, which is based on robust control techniques adapted from engineering, encompasses RE as a special case, and has the advantage that the robust solution of a given program can be derived from a suitably modified standard RE program.

This paper is concerned with solving the Hansen-Sargent robust version of the familiar RE program in which a planner minimizes an intertemporal loss function subject to the law of motion of the economy. If the law of motion is completely backward looking, the planner's commitment technology is irrelevant. Hansen and Sargent (2002) provide a complete treatment of the robust version of this case.

In the macroeconomic literature, however, the law of motion is often a model involving expectations. It then becomes necessary to specify the commitment technology of the planner. In the RE case, there are three standard possibilities: the planner commits to the optimal policy (commitment), or to a simple linear rule (simple rule), or she cannot commit at all (discretion). Hansen and Sargent (ch. 15) give a solution approach for the robust version of the commitment case. This paper's main contribution is to suggest and implement solutions for the robust versions of discretion and simple rules.

The paper does not assume that the reader is familiar with the literature on robust control. Section 2 provides an introduction. It attempts to convey the essence of Hansen and Sargent's approach, deals with backward looking models, and then moves on to show how to solve forward looking models in the commitment case. The simple New Keynesian model of Clarida, Galí, and Gertler (1999) is solved as an example. This section also

[1] The bulk of Hansen and Sargent's work on robustness is contained in a book-length manuscript (Hansen and Sargent (2002)), which presents results from most of their articles (and more) in a comprehensive treatment. Chapters 1 and 2 give an introduction and a summary of the main results. Hansen and Sargent (2000, 2001) are relatively non-technical papers which offer a good introduction. Unless otherwise stated, the reference is to the manuscript (Hansen and Sargent (2002)).

establishes the notation and the key concepts used in free and user friendly software (Matlab, Octave, and Gauss versions) which can perform all the calculations described in this paper.

The rest of the paper proceeds as follows. Sections 3 and 4 propose an extension for the discretionary case and for simple rules respectively. We argue that these suggestions are consistent with Hansen and Sargent's rationale for robustness. They also preserve the property that the robust program can be transformed into a standard RE program. Several examples and applications illustrate the discretionary and simple rule cases. Within the context of the New Keynesian model of Clarida, Galí, and Gertler (1999), we find the following: robustness makes monetary policy more aggressive also in the discretionary solution (confirming a result often found for the commitment solution); robustness is a promising way of interpreting deviations from the expectations hypothesis of the term structure; robustness increases the inflation bias in the discretionary equilibrium; robustness in private agent's expectations increases inflation and output volatility (even if policy is non-robust). Section 5 concludes. Technical details are found in the appendices.

2 Robust Control with Commitment

2.1 Commitment in Backward Looking Models

The Ellsberg paradox motivates Hansen-Sargent robustness, as well as Epstein's ambiguity and other theories of choice under uncertainty.[2] The experiment is as follow. There are two urns, each containing one hundred balls. Balls may be either blue of red. Subjects know for certain that urn A contains 50 red balls and 50 blue balls, but they receive no further information on urn B. Subjects bet on extracting a color of their choice from an urn of their choice (one shot game). Experimental subjects mostly choose to draw from urn A, while expected utility theory predicts that they should be either indifferent between the two urns (if they have no prior on the distribution of balls in urn B), or prefer urn B (if they have a prior). The Ellsberg paradox therefore illustrates a (descriptive) shortcoming of expected utility theory in accounting for the distinction between measurable and unmeasurable uncertainty ("risk" and "uncertainty" in the terminology introduced by Knight (1921)). Hansen-Sargent robustness is one approach to modelling

[2]Epstein and co-authors (e.g. Epstein and Melino (1995) and Epstein and Wang (1994)) have developed an axiomatic theory of choice with multiple priors that, like Hansen and Sargent, draws on Gilboa and Schmeidler (1989). The differences between Hansen and Sargent robustness and Epstein's ambiguity are discussed in Epstein and Schneider (2003).

the risk between risk and uncertainty.

Like a RE agent, a Hansen-Sargent robust planner aims at minimizing a loss function and entertains a *reference model*[3] which represents the law of motion of the economy. Like a RE agent, she can formulate model consistent statements on the probability of any outcome *given* a model. However, unlike a RE agent, she is not certain that the reference model coincides with the true model. For example, exact parameter values will not be available in most circumstances.

Being uncertain about the model, the planner considers a set of them when designing an optimal policy. Faced with the same situation, a Bayesian planner would combine the data with her priors over the probability of each model being correct to arrive at a probability distribution over all models. To formulate a policy function, each model would then be weighted according to its probability and to its associated expected loss. A Bayesian agent therefore reduces all uncertainty to calculated risk. A robust agent, on the other hand, does not have her uncertainty as well organized. She is assumed to face Knightian uncertainty over a set of models, where Knightian uncertainty denotes the inability to express one's beliefs fully in terms of well defined probabilistic statements. This is not equivalent to saying that all models are considered equally likely (in which case a Bayesian solution would be straightforward). Rather, a robust agent does not have sufficient confidence in her beliefs to formulate consistent statements such as "The probability that model A is true is π" for any conceivable model.

An agent faced with multiple models needs to adopt a choice criterion, as each model will generally recommend a different course of action. For a robust agent, this criterion cannot involve a probabilistic weighting of models. Hansen and Sargent (following Gilboa and Schmeidler (1989)) adopt a min-max approach: for a proposed policy rule, the planner finds the worst model in the set (the maximum expected loss), eventually selecting the rule that minimizes the maximum expected loss. Loosely speaking, the aim of robust control is to design a policy that will work reasonably well even if the reference model does not coincide with the true model, as opposed to a policy that is optimal if they do coincide but possibly disastrous if they don't. A classical application in engineering is to program a rocket so that it will get very close to the target even if the law of motion is not correctly specified, rather than be on the target if the law of motion is exactly right but go completely astray otherwise.

Robust control in engineering is in a sense normative, because it represents engineers'

[3]Hansen and Sargent use the expression "approximating model" rather than "reference model". We depart from their terminology for reasons discussed at the end of Section 2.1.

best effort to optimize in the face of unknown misspecifications. An analogous motivation can arguably be used in economics: the complexity of real economies is so overwhelming that it is not conceivable to even formulate an exhaustive list of all possible models, much less to assign a prior probability to each. But in economic applications it is also possible to use robust control descriptively, as a tool to maintain analytical tractability and mimic certain empirical violation of expected utility theorems. In particular, robust control can rationalize agents' aversion to situations in which the odds are not obvious.[4] In a market setting, this ambiguity aversion tends to translate into a higher (with respect to RE agents with the same preferences) price of risk, a feature exploited by Hansen, Sargent, and Tallarini (1999) to show that a preference for robustness decreases the equity premium puzzle in a standard model.

Consider this example: a risk neutral firm is planning an investment which yields a discounted profit p_A in state of the world A, and a loss of p_B otherwise. The decision on whether to invest or not is obvious if the firm can confidently attach a unique probability π to the state A. However, the solution is no longer straightforward if the firm considers a range of π, say $\pi \in [\pi_L, \pi_H]$, to be plausible. If the firm is a robust decision maker, its adoption of a min-max criterion means that it will act as if the relevant probability was π_L. Some readers may infer from the example that a robust agent is observationally equivalent to a Bayesian agent with a higher degree of risk aversion and a flat prior. While it may be possible to establish this equivalence in specific circumstances, the required degree of risk aversion would not be constant, but rather vary with the level of uncertainty. For example, an agent who appears to be risk neutral in bets involving a fair coin will seem risk averse if there are doubts on the fairness of the coin.

From a technical point of view, robustness involves a switch from a minimization problem (minimizing a loss function) to an appropriately specified min-max problem. In order to set up and solve a min-max problem, it is convenient to work with a two-agent representation: the policy function selected by the planner is the equilibrium outcome of a two person game in which a fictitious evil agent, whose only goal is to maximize the planner's loss, chooses a model from the available set, and the planner chooses a policy function.

The loss function is assumed to be quadratic, and the model linear.[5] Because the evil agent is just a metaphor for the planner's cautionary behavior, he shares the planner's

[4]See Hansen and Sargent (2002) ch. 1 for an example.

[5]Anderson, Hansen, and Sargent (2003) and Hansen and Sargent (2002) ch. 16 discuss extensions to a more general non-linear framework.

reference model and loss function (which of course he wants to maximize rather than minimize). This describes a zero sum game, and we can conveniently write a single loss function. Hansen and Sargent show that the program for the backward looking model can be formulated as

$$\min_{\{u\}_0^\infty} \max_{\{v\}_1^\infty} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t (x_t' Q x_t + u_t' R u_t + 2 x_t' U u_t), \tag{1}$$

$$\text{subject to } x_{t+1} = A x_t + B u_t + C(\epsilon_{t+1} + v_{t+1}), \tag{2}$$

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t v_{t+1}' v_{t+1} \leq \eta_0, \tag{3}$$

and where x_0 is given. In this problem, x_t is the state vector $(n \times 1)$, u_t is the planner's control vector $(k \times 1)$, ϵ_{t+1} is the vector $(n \times 1)$ of zero mean iid shocks with an identity covariance matrix, and v_{t+1} is the evil agent's control vector $(n \times 1)$. Notice that the planner's control vector is indexed by t, while the evil agent's control vector is indexed by $t+1$, in spite of the fact that it is known in t. Hansen and Sargent index v by $t+1$ in some work, and by t in other. The first convention highlights the fact that the distortions are camouflaged by the errors, the second that they are known in t. The Q and R matrices are assumed to be symmetric.

The standard RE dynamic control problem corresponds to $\eta_0 = 0$. In this case, the maximization part of the problem becomes irrelevant, and the planner simply minimizes the loss function (1), using the control vector u_t, subject to the law of motion (2) with $v_{t+1} = 0$. In the general case, the evil agent is given an intertemporal budget η_0 which defines the set of models (misspecifications) that the planner is entertaining. Therefore the set of models that the planner is considering can be interpreted as a ball around the reference model, where η_0 is the radius of the ball. Section 2.3 considers the choice of η_0; for now we take it as given.

Notice that the stochastic shocks are important for model uncertainty. As can be seen from (2), the evil agent's control vector v_{t+1} is premultiplied by the matrix C. This captures the fact that there can only be model uncertainty if the true parameters of the law of motion are (at least partially) masked by random noise ($C \neq 0$).

The constraint (3) is inserted into (1), yielding:

$$\min_{\{u\}_0^\infty} \max_{\{v\}_1^\infty} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t (x_t' Q x_t + u_t' R u_t + 2 x_t' U u_t - \theta v_{t+1}' v_{t+1}), \tag{4}$$

$$\text{subject to } x_{t+1} = A x_t + B u_t + C(\epsilon_{t+1} + v_{t+1}), \tag{5}$$

and where x_0 is given. Since the value function is monotonous and concave in η_0, there is a bijective negative function from η_0 to the Lagrange multiplier θ, so θ defines the set of models available to the evil agent, with $0 < \theta < \infty$. A very low θ allows the evil agent to wreck havoc, while $\theta = \infty$ corresponds to RE.

Misspecifications distort the reference model by modifying the errors. However, respect of the budget (3) is the only formal constraint imposed on the evil agent, and the formulation (4) enforces this constraint. This means that his choice of policy functions for v_{t+1} includes a wide range of misspecified dynamics, including wrong parameters (v_{t+1} is a linear function of x_t), autocorrelated errors (v_{t+1} is a linear function of lags of x_{1t}), and nonlinearities (v_{t+1} is a nonlinear function of x_t). At the same time, the researcher needs to specify only one additional parameter (θ) to robustify the program. This parsimony is an advantage in some cases, as it limits the number of additional parameters and the amount of prior knowledge about possible misspecifications, but it can become a drawback if the researcher wants to focus on a specific misspecification, such as distortions in a given parameter.

Other approaches to robustness, which we may call parametric (for instance, Giannoni (2002) and the Bayesian approach pioneered by Brainard (1967)) allow (but also require) the researcher to be more specific about the exact nature of the uncertainty. In a Bayesian approach, the planner uses her prior probability distribution over models (which a Hansen and Sargent robust planner does not have by assumption), so the researcher needs to specify a prior over all possible models, which can quickly become problematic. Moreover, the solution can be quite complex. Giannoni (2002) is closer to Hansen and Sargent in that the planner is solving for the min-max. However, the researcher must specify the set of possible models by setting an interval for each of the model's parameters. Onatski and Williams (2003) build a more general structure which allows the researcher to be quite specific about the type of misspecifications feared by the planner (wrong parameters, measurement errors and autocorrelated errors).

The loss function and the law of motion for the backward looking model given by equations (4)–(5) can be redefined to write the program in standard state space RE form

$$\min_{\{u\}_0^\infty} \max_{\{v\}_1^\infty} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t (x_t' Q x_t + u_t^{*\prime} R^* u_t^* + 2 x_t' U^* u_t^*), \tag{6}$$

$$\text{subject to } x_{t+1} = A x_t + B^* u_t^* + C \epsilon_{t+1}, \text{ where} \tag{7}$$

$$R^* = \begin{bmatrix} R & \mathbf{0}_{k\times n} \\ \mathbf{0}_{n\times k} & -\theta I_{n_1} \end{bmatrix}, u_t^* = \begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix}, B^* = \begin{bmatrix} B & C \end{bmatrix}, \text{ and } U^* = \begin{bmatrix} U & \mathbf{0}_{n\times n} \end{bmatrix}, \tag{8}$$

and where x_0 is given. At first the min-max form of the problem may seem intrinsically

different from a standard minimization. However, because first order conditions for a minimum are the same as for a maximum, the problem can be treated as a standard RE one, to which standard solution algorithms can be applied (for example, see Söderlind (1999) or Hansen and Sargent (2002), ch. 3 and 15).[6]

The solution of the program is that u_t and v_{t+1} are linear functions of the state x_t

$$u_t^* = -F x_t, \text{ that is, } \begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix} = -\begin{bmatrix} F_u \\ F_v \end{bmatrix} x_t. \tag{9}$$

Notice that in spite of all his freedom, with a backward looking linear model the evil agent keeps things simple, and optimally chooses to set v_{t+1} as a linear function of the state vector x_t. From a technical point of view, the linearity of the evil agent's policy function should come as no surprise: the robust program has been rewritten in standard RE form, and therefore the policy function for u_t^* must be linear, since the RE policy function for u_t is known to be linear.

Hansen and Sargent emphasize that the robust solution is not certainty equivalent: both F_u and F_v are functions of C. Intuitively, this is due to the fact that the evil agent hits harder where he can do the most damage with a given budget, which, ceteris paribus, is where the variance of the forecast error is larger. Alternatively, the planner fears misspecification the most where errors with large variance better mask the true parameters. Technically, the program (6)–(7) is still linear-quadratic: the reason why certainty equivalence does not hold is that C appears in B^* (see equation (8)).

The equilibrium dynamics of the model is found by combining the policy function with the law of motion (7). Clearly, this dynamics depends on what the true model actually is—which is captured here by the evil agent's controls, v_{t+1}. Most researchers have focused on two cases.

First, the *worst case model* defines the behavior of the economy when the planner's pessimism turns out to be fully warranted. Formally, this means using the policy functions (9) in the law of motion (7) to get

$$x_{t+1} = (A - BF_u - CF_v) x_t + C \epsilon_{t+1}. \tag{10}$$

[6] Second order conditions ensure that the evil agent is maximizing rather than minimizing. These are unlikely to be problematic. Hansen and Sargent (2002) prove that there is a θ^0 such that, for any $\theta > \theta^0$, the expected value of the loss function is finite and the second order conditions are satisfied. An easy way to check that the second order conditions are satisfied is to make sure that the expected loss is higher than in the RE solution (the value of the expected loss function is included in our software). However, if θ is chosen with the detection error probability approach (see Section 2.3), experience indicates that the second order conditions are typically satisfied for any reasonable value of θ.

This dynamics is typically also used to represent the beliefs of the agents in the model—for instance, to price assets as discounted sums of expected future payoffs.

Second, the *approximating model* is the reference model which sets $F_v = \mathbf{0}$ in (10).[7] Note that the policy is still robust, so F_u is the same as in the worst case model. Comparing the dynamics of these two models conveys information on the misspecification that the planner is fearing.

2.2 Commitment in Forward Looking Models

Forward looking models introduce another player, the private sector, who forms expectations. We consider a class of forward looking models that can be represented by the linear law of motion

$$\begin{bmatrix} x_{1t+1} \\ \mathrm{E}_t\, x_{2t+1} \end{bmatrix} = A \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + Bu_t + C(v_{t+1} + \epsilon_{t+1}), \text{ with } C = \begin{bmatrix} C_1 \\ \mathbf{0}_{n_2 \times n_1} \end{bmatrix}, \tag{11}$$

where x_{1t} is a $n_1 \times 1$ vector of predetermined (backward looking) variables with x_{10} given, and x_{2t} is a $n_2 \times 1$ vector of forward looking (or jump) variables. Only the predetermined variables have shocks, so ϵ_{t+1} is an iid $n_1 \times 1$ vector with zero mean and an identity covariance matrix—and the last n_2 rows of the C matrix are filled with zeros. The evil agent's control vector always "hides" behind the shocks, so v_{t+1} is also an $n_1 \times 1$ vector.

The planner's loss function (1) is unchanged and the evil agent's budget constraint is still given by (3), with $x_t' = (x_{1t}'\ x_{2t}')$.

Having introduced robustness in a forward looking model, we need to decide whether private sector expectations are standard or robust. If they are robust, we must specify the private sector's reference model, degree of robustness and loss function. Giannoni (2002) and Onatski (2000), who also study uncertainty in forward looking models under commitment, assume that the private sector has no uncertainty, but knows that the reference model is exactly correct, and also knows the planner's loss function and degree of robustness. On a critical stance, Sims (2001a) argues that min-max decisions are a more appropriate modeling device for the private sector than for a central bank.[8] We follow Hansen and Sargent in taking the middle ground, and assume that the private sector and planner share the same loss function, reference model and degree of robustness. These assumptions greatly simplify the solution.

In the case at hand, the planner credibly commits. Unlike the backward looking case, it matters whether or not the evil agent also commits. Hansen and Sargent assume that he does. This is intuitively appealing, considering the rationale for the existence of an evil agent: when designing a policy rule, the planner is uncertain about the model and thus designs a robust rule as if she was facing an evil agent. The evil agent is just a metaphor used to solve the min-max problem efficiently. This perspective suggests that the evil agent should optimize when and only when the planner does.

Technically, the program can be rewritten in state space form as a standard RE problem using the same method as in the previous section. This yields

$$\min_{\{u\}_0^\infty} \max_{\{v\}_1^\infty} \mathrm{E}_0 \textstyle\sum_{t=0}^{\infty} \beta^t (x_t' Q x_t + u_t^{*\prime} R^* u_t^* + 2x_t' U^* u_t^*), \text{ subject to} \tag{12}$$

$$\begin{bmatrix} x_{1t+1} \\ \mathrm{E}_t\, x_{2t+1} \end{bmatrix} = A \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + B^* u_t^* + C\epsilon_{t+1}, \text{ where} \tag{13}$$

$$R^* = \begin{bmatrix} R & \mathbf{0}_{k \times n_1} \\ \mathbf{0}_{n_1 \times k} & -\theta I_{n_1} \end{bmatrix}, u_t^* = \begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix}, B^* = \begin{bmatrix} B & C \end{bmatrix}, U^* = \begin{bmatrix} U & \mathbf{0}_{(n_1+n_2) \times n_1} \end{bmatrix}, \tag{14}$$

and where x_{10} is given. The numerical solution algorithm we adopt is detailed in Appendix C, and is based on the generalized Schur decomposition.

The equilibrium dynamics is more complicated than in the backward looking model: the policy functions are history dependent and the forward looking variables (x_{2t}) depend on the equilibrium expectations of future values of the other model variables.

In any case, the *worst case model* is (as before) the equilibrium dynamics of (12)–(14), that is, when the planner's pessimism turns out to be fully warranted (the evil agent is fully active). The *approximating model* uses the same policy function and expectations formation—but sets the evil agent's controls (v_{t+1}) to zero (Appendix B gives the details of these calculations). For example, in a monetary policy model with forward looking price setting (a Calvo style Phillips curve, say), the approximating model uses the same central bank interest rule and mapping from the state of the economy to the price setting as the worst case model. This means, effectively, that the approximating model uses both robust policy and robust expectations. We will return to the role expectations formation when we discuss simple policy rules in Section 4.2.

[7] HS talk of "approximating model" to indicate both what we have called "reference model", i.e. the law of motion with $F_v = 0$ prior to solving for the policy function, and the law of motion with $F_v = 0$ after solving for the policy function.

[8] Sims underlines the importance of the distinction between normative and descriptive when discussing deviations from the RE paradigm. Specifically, he argues that while it is possible that private agents' behavior may well be described as robust, it is not clear that a central bank should be advised to choose a robust policy, rather than try to specify priors and carry out an optimal Bayesian procedure.

Example: A Simple New Keynesian Model

We provide an example of how to frame a forward looking model in state space form. The model consists of an Euler/IS equation and of a Calvo style Phillips curve, as in Clarida, Galí, and Gertler (1999)

$$y_t = \mathrm{E}_t\, y_{t+1} - \gamma(i_t - \mathrm{E}_t\, \pi_{t+1}) + e_{1t}, \tag{15}$$

$$\pi_t = \beta\, \mathrm{E}_t\, \pi_{t+1} + \alpha y_t + e_{2t}, \tag{16}$$

$$e_{1t} = \rho_1 e_{1t-1} + \xi_{1t}, \text{ where } \xi_{1t} \text{ is iid } N(0, \sigma_1^2), \text{ and} \tag{17}$$

$$e_{2t} = \rho_2 e_{2t-1} + \xi_{2t}, \text{ where } \xi_{2t} \text{ is iid } N(0, \sigma_2^2). \tag{18}$$

In this model, i_t is the short interest rate controlled by the central bank, y_t is the output gap, and π_t is inflation. The central bank minimizes the loss function

$$\mathrm{E}_0 \textstyle\sum_{t=0}^{\infty} \beta^s (\pi_t^2 + \lambda_y y_t^2 + \lambda_i i_t^2). \tag{19}$$

This problem can be framed in standard state space form. Write the model in matrix form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & \beta \end{bmatrix} \begin{bmatrix} e_{1t+1} \\ e_{2t+1} \\ \mathrm{E}_t\, y_{t+1} \\ \mathrm{E}_t\, \pi_{t+1} \end{bmatrix} = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} e_{1t} \\ e_{2t} \\ y_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \gamma \\ 0 \end{bmatrix} i_t + \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{1t+1} \\ \epsilon_{2t+1} \end{bmatrix},$$

where we have ordered the predetermined variables (e_{1t} and e_{2t}) before the forward looking variables (y_t and π_t). Then, premultiply by the inverse of the matrix on the far left we get the same form as (11). Finally, the loss function matrices are

$$Q = \begin{bmatrix} \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & Q_{bb} \end{bmatrix} \text{ where } Q_{bb} = \begin{bmatrix} \lambda_y & 0 \\ 0 & 1 \end{bmatrix}, \; R = \lambda_i, \text{ and } U = \mathbf{0}_{4\times 1}.$$

Once the model is written in standard form, solving for the robust policy only requires specifying the degree of robustness (the scalar θ) and the solution strategy, which in this case is commitment.

Figure 1 provides an introduction to the effects of robustness in this model. The parameters are set as follows: $\beta = 0.99$, $\gamma = 0.5$, $\alpha = 0.645$, $\rho_1 = \rho_2 = 0.8$, $\sigma_1^2 = \sigma_2^2 = 1$, $\lambda_y = 0.5$, and $\lambda_i = 0.2$. We compute both the RE solution and the robust solution. The latter of course depends on our choice of θ. For the moment we ask our readers to think of

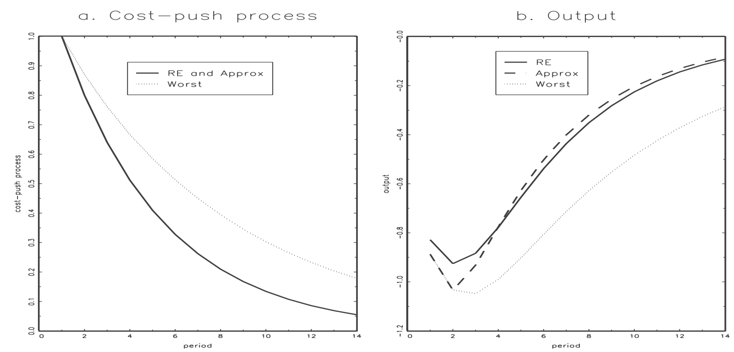

Figure 1: Impulse response functions of the cost-push process ϵ_{2t} and of output to a cost-push innovation in the model of Clarida et al. (1999), commitment solution: standard RE solution (continuous line), approximating solution (dashed line) and worst case solution (thin dots).

the amount of robustness in this example as neither trivial nor unreasonably high.[9] Figure 1.a plots the response of the cost-push process ϵ_{2t} to a one-standard-deviation innovation ($\zeta_{2t} = 1$), for the approximating model and for the worst case model. It is evident that robust agents fear that the cost-push process ϵ_{2t} will turn out to be more persistent than implied by the approximating model. In the case at hand, the predetermined variables are independent of the endogenous variables at all lags; that is, expectations cannot affect any predetermined variable, implying that the trajectory of ϵ_{2t} under the approximating model is the same as under RE.

Figure 1.b shows the response of output (a forward-looking variable) to the same cost-push shock. The contemporaneous response is the same for the approximating model and for the worst case model[10], but the dynamic paths are then different, with output feared to be more persistent than suggested by the approximating model. The RE and the approximating solution share the same underlying dynamics for the predetermined variables, but differ because of the policy function and of expectation formation.

[9] Formally, θ corresponds to a detection error probability of 20% in a sample of 150 observations. (See Section 2.3 for a discussion.)

[10] This is a general feature of the Hansen and Sargent solution, and is due to the fact that v_{t+1} is a function of variables dated t or earlier.

2.3 Choosing the Degree of Robustness, θ

In formulating a robust control problem, the choice of θ is crucial, since the evil agent's constraint is always binding in a linear-quadratic framework. In other words, the policy function chosen by a robust planner (who prepares for the worst) is tailored on a model lying on the boundary of the set from which the evil agent can choose.

This set is defined by deviations from the reference model, where the allowed deviations are decreasing functions of the parameter θ (and hence increasing functions of η_0). The choice of the parameter θ is therefore crucial, as the planner's policy function will vary with it. Svensson (2002) uses this feature of the solution to stress what seems like a weakness of robust control, at least from a Bayesian perspective: a model on the boundary of the available set shapes the policy function, yet models outside this set (including those only an epsilon away) receive no consideration. He also warns that "highly unlikely models can come to dominate the outcome of robust control" (page 7). In a linear-quadratic framework it is easy to make a robust planner look like a foolish catastrophist: her policy function will be implausible if the amount of requested robustness is sufficiently large (θ is sufficiently small).

While these warnings are appropriate, it is usually possible to define θ so that the planner looks cautious rather than foolish. As a guide to choosing θ, Hansen and Sargent adopt a detection error probability approach based on the idea that the models in the set should not be easy to distinguish with the available data. Essentially, one takes an agnostic position on whether the true data generating process is given by the approximating model or by the worst case model, and chooses a probability of making the wrong choice between the two models on the basis of in-sample fit, for a sample of given size.

The value of θ corresponding to this probability is computed by simulation, inverting the monotonous function $\pi(\theta)$

$$\pi(\theta) = \text{Probability}(L_A > L_W|W)/2 + \text{Probability}(L_W > L_A|A)/2, \tag{20}$$

where L_A and L_W are the values of the likelihood of the approximating and worst case model respectively, and the notation $(\cdot|W)$ and $(\cdot|A)$ denotes that the DGP are the worst case model and the approximating model respectively.[11] Zero robustness corresponds to a detection error probability of 50%. Hansen and Sargent suggest the range 10% to

[11]See Hansen and Sargent (2002) ch. 8. The procedure requires a distributional assumption on ϵ_t (normality in our software). For applied research, it is advisable to verify that the results of interest are not overly sensitive to reasonable variations in detection error probabilities.

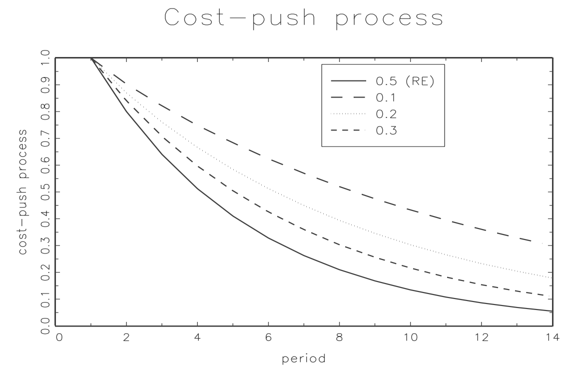

Figure 2: Impulse response functions of the cost-push process ϵ_{2t} to a cost-push innovation in the model of Clarida et al. (1999), commitment solution, for different detection error probabilities.

20%. The larger the sample (for a given probability), the higher θ, so the uncertainty surrounding the reference model disappears as the sample goes to infinity. However, it is assumed that the planner makes no attempt to incorporate learning in a dynamic fashion: when solving for a policy function at t, she does not consider the reduction in uncertainty that future observations may provide (see Hansen and Sargent (2000) for a discussion).

Example: Persistence of Cost-Push Shocks

As an example of the effects of varying θ, consider the simple New Keynesian model of Section 2.2 (with the same parametrization), where it was noticed that the cost-push shock ϵ_{2t} is more persistent in the worst case scenario. Figure 2 plots the response of ϵ_{2t} to a one-standard-deviation innovation for varying degrees of robustness (i.e. for varying detection error probabilities). As the detection error probability becomes smaller, ϵ_{2t} becomes more persistent. The result is intuitive, since a more persistent process results in larger inflation and output variance, and thus higher expected loss for the planner.

3 Robust Control with Discretion

The discretionary solution coincides with the commitment solution in a backward looking model, which has already been treated in Section 2.1.

3.1 Discretion in Forward Looking Models

When working with forward looking models (particularly in the field of monetary policy) it is often assumed that the planner (the central bank) cannot commit. Since this case is of crucial interest, it seems important to extend the robust methods.[12] In this section we propose solution concepts and algorithms for dynamic models which preserve the property of transforming the problem to a RE form.

In order to illustrate our solution approach to the robust case, it is useful to review the main steps involved in the RE solution (see Backus and Driffil (1986) or the summary in Appendix C for a more detailed description of the solution procedure).

1. At time t, the private sector observes x_{1t} and decides on a matrix K_{t+1} to use in formulating expectations $\mathrm{E}_t^a x_{2t+1} = K_{t+1} \mathrm{E}_t^a x_{1t+1}$, where the notation E_t^a denotes agents' expectations in period t. The planner moves after the private sector, so the matrix K_{t+1} incorporates a guess of the planner's policy function.

2. At time t, the planner observes x_{1t} and K_{t+1} and chooses a policy function $u_t = -F_{ut}x_{1t+1}$ to minimize the loss function (1) subject to the law of motion (11) (the same as in the commitment case), but also subject to the expectation formation process $\mathrm{E}_t^a x_{2t+1} = K_{t+1} \mathrm{E}_t^a x_{1t+1}$.

3. The equilibrium solution is found when the matrix K_{t+1} of the private sector's expectations coincides with the mathematical expectation. This happens when the policy function F_{ut} implied by K_{t+1} is also the policy function that solves the planner's problem given K_{t+1}. In equilibrium K_{t+1} and F_{ut} are constant.

Our proposal for dealing with the discretionary case is to extend the principle that, robustness being a metaphor for the planner's concerns for model misspecification at the time of choosing a policy function, the evil agent should optimize when and only when the planner does. When applied to the commitment case, this results in Hansen

[12] Hansen and Sargent (2002) ch. 5 discuss the robust discretionary solution of the static model in Kydland and Prescott (1977).

and Sargent's solution. In the discretionary case, this principle suggests that since the planner reoptimizes at every period (taking expectations as given), the evil agent should be allowed to do the same. The interpretation is that every time the planner considers a policy, she will have to deal with uncertainty and design a robust rule.

We maintain the assumptions (used by Hansen and Sargent in the commitment case) that the private sector's loss function, reference model, and θ are the same as the planner's.

The main steps involved in the RE solution are therefore modified as follows to find the robust discretionary policy. *First*, K_{t+1} now implies a guess of the policy functions of both the planner and the evil agent (private agents share the planner's concern for robustness). *Second*, the evil agent chooses a policy function $v_{t+1} = -F_{vt}x_{1t}$ (at the same time as the planner) in order to maximize the loss function, subject to the same constraints as the planner, but also subject to the budget $\mathrm{E}_t \Sigma_{s=1}^{\infty} \beta^s v_{t+s} v_{t+s} \leq \eta$. *Third*, in equilibrium both policy rules are constant and consistent with the private sector's expectations.

This formulation of the robust discretionary case seems quite natural. Moreover, since η does not depend on t, the size of the deviations from the reference model contemplated by the planner is constant through time.

This formulation is also convenient, since it allows us to handle the discretionary case by augmenting the law of motion and the loss function just like in the commitment case. In practice, this means finding the discretionary solution to the problem detailed in equations (12)–(14). We use algorithms developed for the standard RE discretionary case (see Appendix C), because they solve for the first order conditions (which are the same for the minimum and maximum).

In equilibrium, the predetermined variables (x_{1t}) follow a VAR(1) process

$$x_{1t+1} = Mx_{1t} + C_1\epsilon_{t+1}, \tag{21}$$

and the forward looking variables and the controls are linear functions of the predetermined variables

$$\begin{bmatrix} x_{2t} \\ u_t \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} N \\ -F_u \\ -F_v \end{bmatrix} x_{1t}. \tag{22}$$

The only difference between the *worst case model* and the *approximating model* is in terms of the M matrix in (21) (see Appendix B for details). The difference between the two M matrices is therefore a useful indicator of the misspecification that the planner fears.

Since (22) is the same, it is clear that the approximating model uses both robust policy and robust expectations (the mapping from the predetermined variables to the forward looking variables is very closely tied to expectations).

Application: The Simple New Keynesian Model

The model defined by equations (15)–(19) is written in state space form exactly as for the commitment solution. Only the solution algorithm changes. The robust policy function takes the form given in equation (22) and is therefore not history dependent.

Consider the following application. We wish to derive the central bank policy function and the behavior of the economy as the degree of robustness goes from zero (RE) to a θ which implies a 20% probability of error detection in a sample of 150 observations. The other parameters are set as in Section 2.2. Figure 3 shows the results. Each quadrant plots the response of a variable to a cost push shock (ξ_{2t}) for three cases: standard RE ($\theta = \infty$), the approximating model, and the worst case model.

Robustness leads to higher reactions of all variables at all horizons. The response of the short interest rate is also higher for the approximating model (when policy is robust but v_{t+1} is always zero) than for the standard RE case. Finally, the robust monetary policy function is more aggressive: the policy vector F_u is $-(3.0, 1.9)'$ for the RE solution, and $-(3.5, 2.4)'$ for the robust solution. This result is not new, as other papers conclude that robustness lead to more aggressive policies under commitment.[13] However, it is of some independent interest that we reach the same conclusion in the discretionary case.

A recurrent feature of the solution is the evil agent's common choice of increasing the persistence of the driving processes. In Figure 3 the responses of all variables are in fact more persistent in the worst case than in both the standard RE and the approximating case. More persistent processes imply higher variances and therefore a greater loss for the risk averse planner.[14] Fearing this outcome, a robust agent typically has a stronger reaction to shocks than a standard agent.[15]

This feature of the robust solution, namely the worst case model displaying more persistence than the approximating model, suggests than we can often expect robustness

[13]See, for instance, Hansen and Sargent (2001), and, with a different approach, Giannoni (2002). However, this result is not general (Hansen and Sargent (2002), ch.8, provide a counter-example): the outcome will depend on the model and on the loss function parameters.

[14]Hansen and Sargent (2003, 2000) analyze this point at length through spectral analysis, showing how the evil agent often accentuates the low frequency components of the exogenous processes.

[15]Kasa (2001) proves that a robust forecaster, whose loss function is the mean squared error, revises the forecast by more than a standard forecaster following new information, because she is more vulnerable when she underestimates the persistence of the driving processes.

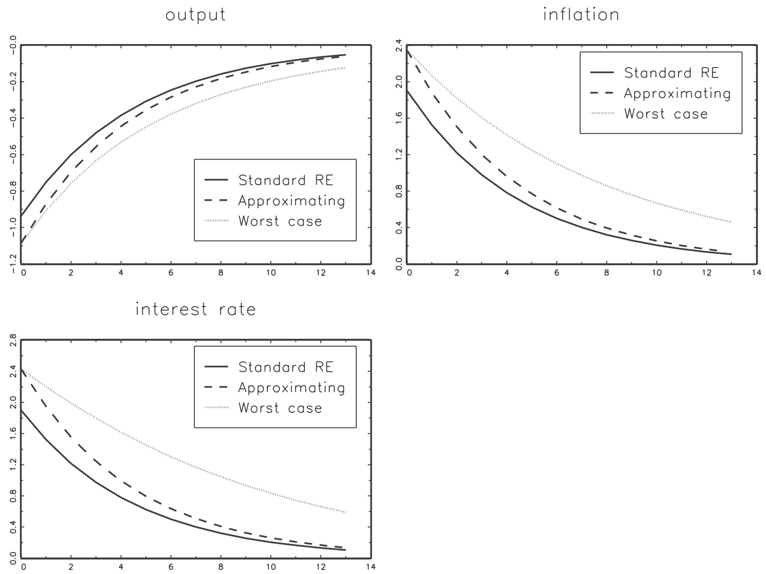

Figure 3: Impulse response functions of inflation, output and short interest rate to a cost-push shock in the model of Clarida et al. (1999), discretionary solution: standard RE solution (continuous line), approximating solution (dashed line) and worst case solution (thin dots).

to make forward looking prices overreact to news. This implies that robustness makes asset prices more volatile and more forecastable, as illustrated in the applications of sections 4.1 and 3.1.

Application: The Term Structure of Interest Rates

The literature considered in this paper is young, and yet offers several interesting empirical applications, including consumption/saving decisions (Hansen, Sargent, and Tallarini (1999)), asset pricing (Anderson, Hansen, and Sargent (2003), Hansen, Sargent, and Tallarini (1999), Tornell (2000), Hansen, Sargent, and Wang (2002)), and monetary policy rules (Hansen and Sargent (2001), Giannoni (2002), Onatski (2000)). In the latter case the focus is on the behavior of the short interest rate (the policy instrument). We suggest a natural extension, namely to consider the implied behavior of multiperiod interest rates. We continue to work with the model of Section 2.2.[16] The exercise could be carried out assuming commitment, but discretion is arguably more realistic, so we opt for the latter.

Let i_t—the policy instrument—be the one period interest rate (not annualized). We assume that multiperiod rates obey the expectations hypothesis of the term structure. The h-period interest rate (denoted $i_{t,h}$) then follows

$$i_{t,h} = \mathrm{E}_t^* \sum_{i=0}^{h-1} i_{t+i}, \tag{23}$$

where E^* denotes robust expectations, that is, expectations which condition on the worst case model. We also define a 'fundamental' rate, computed substituting the mathematical expectation E for the robust expectation E^* in equation (23). The fundamental rate therefore guarantees that no expected excess profits are available, whereas the actual rate does not. Referring to Figure 3, the actual rate and the fundamental rate are derived by plugging into equation (23) the path of the short interest rate for the worst case model and for the approximating model respectively.

Figure 4 shows the difference—at the time of a unit shock to the inflation equation—between a long ($h = 4$) interest rate, and the corresponding fundamental rate, for different degrees of robustness (represented by error detection probabilities). This difference can be considered an *overreaction* in the classical sense that the price of the multiperiod bond reacts to a shock by jumping beyond its new equilibrium value (we are assuming, of course, that the approximating model is in fact the DGP). The overreaction is around 1.25% at a 20% detection probability, and grows monotonously with the degree of robustness.

A rather large empirical literature on the term structure has found that actual changes in short interest rates are smaller than predicted by the slope of the yield curve.[17] Our examples show that robust expectations can contribute to an interpretation of this finding.

Application: The Inflation Bias

A well known example of how the presence or absence of a commitment technology can affect an economic outcome is the time inconsistency of optimal monetary policy first studied by Kydland and Prescott (1977) and Barro and Gordon (1983). They assume that the planner is targeting a level of output above potential output, and then show that

[16]Parameter values are the same as in Section 2.2, and θ again implies a detection error probability of 20% in a sample of 150 observations.

[17]See, for example, Walsh (1998) ch. 10 and Campbell, Lo, and MacKinlay (1997) ch. 10.

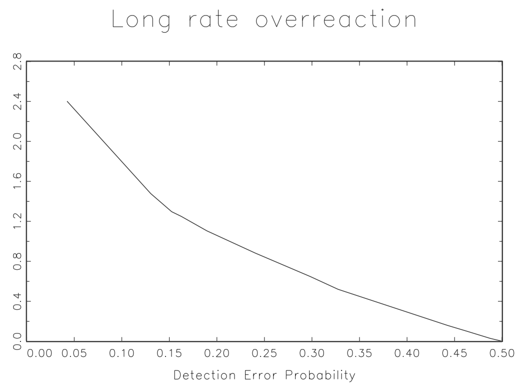

Figure 4: Over-reaction of the long interest rate to a CP shock as a function of the degree of robustness (in terms of the detection error probability) in the model of Clarida et al. (1999) with discretionary solution. 0.01 corresponds to one basis point.

the discretionary solution involves an inflation bias (inflation is higher than the optimal level). The model of Kydland and Prescott is static and involves expectations about the control variable, so it cannot be cast into the form of equation (11). However, an analytical solution is available. Hansen and Sargent (2002) show that fear of misspecification increases the inflation bias. The intuition is that the planner fears that the expected value of output is lower than in the reference model, which increases the distance between desired and expected output. Thus a preference for robustness has the same effect as an increase in target output: higher inflation.

With our solution approach for discretion in dynamic models, we can recast Hansen and Sargent's exercise in more general settings. Here we study the inflation bias in the dynamic model of Clarida, Galí, and Gertler (1999) used in the previous section (equations (15)–(19)), except that we allow for an output target level $y^* > 0$ in the loss function, which becomes

$$\mathrm{E}_t \textstyle\sum_{s=0}^{\infty} \beta^s [\pi_{t+s}^2 + \lambda_y (y_{t+s} - y^*)^2 + \lambda_i i_{t+s}^2]. \tag{24}$$

Technically, this requires adding the constant 1 to the vector of predetermined variables. Average inflation in the standard RE solution is then a positive function of y^*. In our proposed solution for the discretionary case, the evil agent's control vector, v_{t+1}, is a

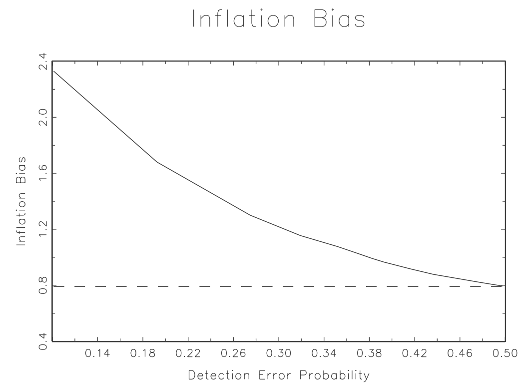

Figure 5: Inflation bias as a function of the detection error probability in the model of Clarida et al. (1999). The horizontal line gives the inflation bias for the RE solution.

linear function of the predetermined variables, just like the central bank's policy rule.

It turns out that the evil agent decides to affect both the constant and the autoregressive parameters, thus increasing both the mean of inflation and its variance. The constant is negative because y^* is positive. Intuitively, since the loss function is symmetric in y around y^* while the evil agent's cost is symmetric in y around zero, it is cheap and effective for the evil agent to set a negative constant to output. The result is that robustness leads to an increase of average inflation in the discretionary solution, for the same reason as in the Kydland-Prescott example.

Figure 5 shows how the inflation bias varies with the degree of robustness in the model of Clarida, Galí, and Gertler (1999). The calibration is the same as in Section 2.2. We set y^* to 0.4. The inflation rate is defined as the growth rate in prices during one period. Therefore, if we think of the model as applying to quarterly data, an inflation bias of 0.8% translates into an annual bias of approximately 3.2%.

4 Robust Control with Simple Rules

The monetary policy literature has paid a good deal of attention to the properties of *simple rules*, defined as commitment rules that set the policy instrument as a linear function of

the system variables. Examples include Taylor type rules and rules for money growth. Simple rules are typically not optimal. In some cases they are motivated as good empirical approximations to actual policy.

In other cases simple rules are justified as an attempt to identify rules that work well in a variety of models. A prominent proponent of robustness in this sense is McCallum (1988, 1999). An interesting example is Levin, Wieland, and Williams (2001), who focus on simple monetary policy rules that work well in models that incorporate rather different views of the transmission mechanism.[18]

This literature uses the term robust for a rule that performs well across models. Hansen and Sargent propose, instead, to design rules that perform well for deviations around a single model. Sims (2001a) argues that a Hansen-Sargent robust solution to a single reference model may in fact not be robust in the sense of McCallum. There is of course no reason why the two concepts should be substitutes rather than complements: one could try to identify a rule that is robust in a Hansen-Sargent's sense for several reference models, or, when possible, merge the competing models and thus reduce model uncertainty to parameter uncertainty. A preliminary requirement for this is to specify solution concepts for simple rules in a Hansen-Sargent robust framework. A solution for backward looking models is already available, and one for forward looking models is proposed in this section.

Another reasons why we are interested in simple rules in a robust framework is that they allow us to isolate the effects of the private sector deviations from RE. In effect, a planner who has committed to a given simple rule is no longer involved in any decision, so any change in economic outcome between the RE and the robust solution is entirely due to the role of private sector expectations.

4.1 Simple Rules in Backward Looking Models

Managing a simple rule in a backward looking model is a straightforward application of the robust pure prediction problem analyzed in Hansen and Sargent (2002) and Kasa (2001). The planner commits to a specific F_u in setting $u_t = -F_u x_t$ (where x_t can be augmented by any variables that are important for policy decisions). Then the evil agent

[18] Leitemo and Söderström (2003) evaluate the performance of simple monetary policy rules (compared to optimal rules) in several variations of a baseline model.

is left with the following program

$$\max_{\{v\}_1^\infty} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t (x_t' Q x_t + u_t' R u_t + 2 x_t' U u_t - \theta v_{t+1}' v_{t+1}) \tag{25}$$
$$\text{subject to} \;\; x_{t+1} = (A - BF_u) x_t + C(v_{t+1} + \epsilon_{t+1}),$$

and where x_0 is given. This is a standard RE problem of finding the optimal policy rule in a backward looking model. The evil agent will therefore choose to set v_{t+1} as a linear function of the state x_t.

Example: A Simple Forecasting Problem As an illustration, consider a simple robust forecasting problem. Let the loss function be the mean squared forecast error $\mathrm{E}_t(x_{t+i} - x^e_{t+i,t})^2$, where $x^e_{t+i,t}$ denotes the forecast of x_{t+i} made at time t. Suppose that x_t is the amount of dividends. The reference model of the dividend is an AR(1) process (A in (25) is the autoregressive coefficient and $B = 0$).

It is straightforward to show that the robust forecast of x_{t+i} at time t, denoted by $\mathrm{E}^*_t\, x_{t+i}$, is $\mathrm{E}^*_t\, x_{t+i} = (A^*)^i x_t$, where $A^* > A$ so the investor forecasts as if the process driving dividends was more persistent than in the reference model. The investor thus fears that the process has high persistence. The intuition is that more persistence gives larger uncertainty of long horizon forecasts (as future shocks are propagated more).

If the asset price is the discounted (at rate β) sum of expected dividends, then we get the price $x_t/(1 - A^*\beta)$. Since A^* is a positive function of the degree of robustness, so too is the price variance.[19] A small degree of robustness can have large effects on the behavior of prices. For example, let $\beta = 0.98$, and $A = 0.99$, and $A^* = 1$. This relatively small degree of robustness implies an increase of around 50% in the standard deviation of the asset price.

4.2 Simple Rules in Forward Looking Models

The forward looking case is less straightforward. We propose to be more specific about the set of models from which the evil agent can choose (that is, the type of misspecification feared), by imposing that he sets his instruments v_{t+1} as a linear function of predetermined variables. That is, we allow for misspecifications of the form

$$v_{t+1} = -F_v x_{1t}, \tag{26}$$

[19] See Hansen and Sargent (2002), from which this example is adapted.

and leave the evil agent free to choose the coefficients of the $(n_1 \times n_1)$ matrix F_v (within the limits of the budget defined by θ).

For the moment we concentrate on the technical aspects of our proposed solution, postponing its motivation to the end of this section. Formally, we suggest to set up the problem as

$$\max_{F_v} \; \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t (x_t' Q x_t + u_t' R u_t + 2 x_t' U u_t - \theta v_{t+1}' v_{t+1}), \tag{27}$$
$$\text{subject to (11), (26) and } u_t = -F_u x_t,$$

and where x_{10} is given. The constraint $v_{t+1} = F_v x_{1t}$ has been imposed and the maximization is explicitly in terms of F_v. The interpretation is that the planner is fearing errors in the coefficients which relate predetermined variables to lags of predetermined variables. As in previous cases, the problem can be written as a standard RE problem

$$\max_{F_v} \; \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t (x_t' Q x_t + u_t^{*\prime} R^* u_t^* + 2 x_t' U^* u_t^*),$$
$$\text{subject to (13) and } u_t^* = \begin{bmatrix} -F_u \\ -F_v \;\; \mathbf{0}_{n_1 \times n_2} \end{bmatrix} x_t, \tag{28}$$

where x_{10} is given and the starred $(^*)$ matrices are defined in (14).

For given F_u and F_v, the solution concept is that of a simple rule in a forward looking model: private sector expectations are consistent with the evolution of the economy in the worse case model. The solution to (28) is then found by letting a numerical maximization routine search over F_v (the policy rule F_u is kept constant). The solution algorithm is outlined in Appendix C.

The formal representation of the equilibrium can be written in the same form as in the discretionary case (21)–(22) where the predetermined variables (x_{1t}) follow a VAR(1) process and the other variables are linear functions of the predetermined variables (see Appendix B for details).

Example: The Simple New Keynesian Model The state space form is as for the commitment and discretionary solution, except that the researcher must provide a matrix F_u (and of course a value of θ).

Motivation of Our Robust Simple Solution

We will now motivate of our proposed solution for the simple rule in a forward looking model. Recall that we are constraining the v_{t+1} to be a linear function of predetermined variables. Why this constraint? The problem is that an evil agent free to commit to any rule uses agents' expectations to his advantage, and therefore makes the set of plausible models dependent on the expectation formation. By strategically exploiting expectations, an agent free to commit can drive the loss function to infinity for any degree of robustness, for example by committing to an exponentially increasing or decreasing series of v_{t+1}, a highly improbable misspecification to fear. This does not happen when the planner is allowed to choose a robust policy (in the commitment or discretionary case)—but it happens with a simple rule since the policy maker is bound to follow a given rule. To put it simply, the planner is defenceless against the evil agent.

On the other hand, the choice of constraining v_{t+1} to be linear in the predetermined variables ensures that the set of misspecifications that the planner considers plausible is given exogenously, in the sense that it does not depend on how expectations are formed, and that there is a finite θ for which a model that has stable dynamics under RE remains stable in the robust solution.

The following example illustrates the argument. Let the planner's loss function depend on squared inflation rates, $L_0 = \mathrm{E}_0 \Sigma_{t=0}^{\infty} \beta^t \pi_t^2$, and the law of motion of the economy be given by the simplified Calvo style Phillips curve

$$\pi_t = \beta \, \mathrm{E}_t \, \pi_{t+1} + \epsilon_t, \text{ where } \epsilon_t \text{ is iid with } \mathrm{E}\,\epsilon_t = 0 \text{ and } \mathrm{Var}(\epsilon_t) = 1.$$

In this case, the planner has no effect at all on inflation—but the more general point is that he cannot revise his policy to defend against the evil agent. Assume that the evil agent can commit to any policy rule. He therefore solves[20]

$$\max_{\{v\}_1^{\infty}} \ \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \pi_t^2 \tag{29}$$
$$\text{subject to } \pi_t = \beta \, \mathrm{E}_t \, \pi_{t+1} + v_{t+1} + \epsilon_t \text{ and } \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t v_{t+1}^2 \leq \eta_0.$$

It is straightforward to prove that an evil agent who is able to commit will choose a non-stationary (exponentially increasing or decreasing) v_{t+1} (see Appendix D), which makes the loss function unbounded for any strictly positive η_0. The misspecification feared is then a trend increase (or decrease) of inflation, which is a case of limited interest.

[20]We write the evil agent's constraint explicitly rather than in terms of the Lagrange multiplier θ. A high η_0 corresponds to a low θ. We also set $c = std(\epsilon_t) = 1$.

In contrast, this problem has a well defined solution under our proposed approach to the simple rule case, which here forces the evil agent to set v_{t+1} as a function of ϵ_t (the only predetermined variable), say $v_{t+1} = a\epsilon_t$. The robust expectations are therefore formed as if the standard deviation of the errors was $1 + a$ rather than 1.

Application: Output and Inflation Volatility

The differences between the robust and RE solutions illustrated in Figure 3 are due to deviations from rational expectations of both the planner and the private sector. The solution approach to simple rules in forward looking models developed in Section 4.2 opens up the possibility of isolating the effects of private sector deviations from RE. We might then ask how a preference for robustness on the part of the private sector affects macroeconomic variables and asset prices, keeping the behavior of the planner fixed by assuming that she has committed to a simple rule.

To illustrate, we continue to consider the simple model of Section 2.2. The goal is to establish a link between the degree of robustness and the volatility of inflation and output.

The central bank is assumed to commit to the Taylor rule

$$i_t = 1.5\pi_t + 0.5y_t.$$

The solution takes the form

$$\begin{bmatrix} x_{2t} \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} N \\ -F_v \end{bmatrix} x_{1t}. \tag{30}$$

As in the other equilibria, N is a function of θ and does not depend on whether the planner's fears are founded or not (that is, the approximating model and the worst case model share the same way of forming expectations).

We assume that the approximating model is correct, and therefore that x_{1t} evolves as (see Appendix B)

$$x_{1t+1} = M_a x_{1t} + C_1 \epsilon_{1t}, \tag{31}$$

from which we obtain the variance of the predetermined shocks (e_{1t} and e_{2t}). The matrix N in (30) is then used to compute the variance of the corresponding series of output and inflation. This is done for the standard RE case ($\theta = \infty$) and for the robust case.

The variance of output and inflation is found to be a monotonous function of the degree of robustness (an inverse function of θ). At $\theta = 850$, which corresponds to a 20% error detection probability for a sample of 150 observations, the variances of output and inflation are some 3% and 46% higher than in the standard RE case respectively.

To gain some intuition for this result, it is useful to compare the matrix M_a, which actually generates the predetermined variables in (31), and M, which corresponds to the worst case model and therefore to agents' expectations (see Appendix B). The parameters in M_a are ρ_1 and ρ_2

$$M_a = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix},$$

while M (the worst case model) is

$$M = \begin{bmatrix} 0.86 & 0.04 \\ 0.02 & 0.82 \end{bmatrix}.$$

Comparing M_a and M, we notice that the private sector is fearing that the exogenous processes are more persistent than in the approximating model. Expecting persistent dynamics of the driving processes (the predetermined variables), consumers and price setters overreact to news, in the sense that output and inflation have a stronger response to shocks than in the RE case.

Using the same model we can isolate the effects of robust expectations on the term structure. The results (not reported) are similar to those of the discretionary solution, shown in Figure 3 and 4: the public fears more persistent effects of shocks and multiperiod interest rates overreact.[21]

Application: Publishing Central Bank Forecasts

Sims (2001a) argues that a min-max approach to robustness is better suited to capture agents' near rational behavior than as a normative course of action for central banks. The following example embraces his point of view and departs from the assumption that the private sector and the central bank share the same information and the same taste for robustness. We assume that the central bank has a good and precisely estimated model, while the private sector nurtures strong doubts about the behavior of the economy.

To simplify the problem as much as possible, we make the following assumptions. First, the private sector does not know the model (or simply the coefficients) estimated by the central bank, but it knows that the bank has an accurate representation of the economy. Second, the central bank follows the Taylor rule $i_t = 1.5\pi_t + 0.5y_t$. Third, the private sector's reference model is the same as the central bank's and it is also the DGP.

[21] We obtained similar results on the behavior of long interest rates in the model of Fuhrer (1997), with the parameters estimated in Söderlind (1999).

The second assumption means that we can adopt the simple rule solution developed in Section 4.2. Altogether, these three assumptions are admittedly unrealistic, but they help us make the following point: the central bank can reduce the variance of inflation and output by releasing information to the public, for example in the form of forecasts. To verify the statement, notice that if the central bank does not release information, the setup and outcome is exactly as in the previous application, where robustness increases business cycle volatility. However, if the central bank announces its forecasts of the predetermined variables, the private sector makes these forecasts their own (by the first assumption), taking the economy back to the superior RE solution.

Whether central banks should release explicit forecasts is a matter of current debate (see Svensson (2001) for a list of central banks that do, and arguments in favor of the practice). While our example cannot be a serious attack on the issue, it does lead us to believe that a more thorough investigation would be worth the effort.

5 Conclusions

The approach to dealing with model uncertainty proposed by Lars Hansen and Thomas Sargent seems promising from both a normative (designing a rule that works well in a neighborhood of the reference model) and a descriptive (replicating the behavior of actual agents) perspective. In the discussion of fiscal and monetary policy, much of the debate is centered around two non history dependent sets of policies: simple rules and discretionary solutions, which Hansen and Sargent do not consider. This paper proposes solution approaches for simple rules and discretion. These extensions preserve the property that the robust program can be written and solved as a suitably modified rational expectations program. Some applications show that these extensions can be interesting for applied work. The analysis of the term structure of interest rates complements previous research and suggests that standard models give a better empirical description of asset returns if agents are attributed a taste for robustness. The result that the inflation bias increases with robustness can be interpreted as saying that the gains from commitment increase if potential output is imprecisely estimated. The application to the variance of inflation and output for a given policy function is perhaps the most interesting. It suggests that a robust private sector may amplify those same fluctuations in inflation and output that it fears, and can provide a theoretical motivation for central banks to be transparent about their forecasts.

Appendix A Software

Our software can be downloaded freely at http://home.tiscalinet.ch/paulsoderlind. A user's manual and example programs are provided. The procedures follow the notation of this paper, and their syntax is therefore immediately understood. The user needs to write the loss function (that is, specify β, Q, R, U), write the model in state space form (that is, specify A, B, C, n_1), select a θ, and decide on a solution algorithm (commitment, discretion, simple rule). Advanced users can fine tune the optimization algorithms. Bayesian error detection probability is also implemented (assuming normally distributed errors) to assist the user in selecting θ. We follow Hansen and Sargent in plotting the probability (of selecting the wrong model) against $\sigma = -1/\theta$, rather than against θ. To find the θ corresponding to a probability of 0.2 in the Euler+Calvo model (discretionary solution), we solved the model 10,000 times. This took around 20 seconds on a Pentium III PC using Gauss, and a few seconds more using Matlab.

Appendix B Reduced Form Dynamics of the Forward Looking Models

This appendix shows how to compute the dynamics of the worst case and of the approximating model for the forward looking commitment case. The dynamics of the forward looking discretionary and simple rule cases have the same general form, but with ρ_{2t} being an empty vector (or a vector of zeros).

The solution can be written in the following general form (see Appendix C)

$$\begin{bmatrix} x_{1t+1} \\ \rho_{2t+1} \end{bmatrix} = M \begin{bmatrix} x_{1t} \\ \rho_{2t} \end{bmatrix} + C\epsilon_{t+1} \tag{32}$$

$$\begin{bmatrix} x_{2t} \\ u_t \\ v_{t+1} \\ \rho_{1t} \end{bmatrix} = N \begin{bmatrix} x_{1t} \\ \rho_{2t} \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \begin{bmatrix} x_{1t} \\ \rho_{2t} \end{bmatrix} \tag{33}$$

with $\rho_{20} = \mathbf{0}_{n_2 \times 1}$. In the discretionary and simple rule solutions, $\rho_{2t} = \mathbf{0}$ for all t. Equations (32) and (33) give the dynamics of the worst case model. To retrieve the dynamics of the approximating model, we rewrite (32) as

$$\begin{bmatrix} x_{1t+1} \\ \rho_{2t+1} \end{bmatrix} = M \begin{bmatrix} x_{1t} \\ \rho_{2t} \end{bmatrix} + C\epsilon_{t+1}, \tag{34}$$

where

$$M = P^{-1}(A - BF_u - BF_v)P \tag{35}$$

$$P = \begin{bmatrix} I_{n_1} & \mathbf{0}_{n_1 \times n_2} \\ & N_1 \end{bmatrix} \Longrightarrow \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = P \begin{bmatrix} x_{1t} \\ \rho_{2t} \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} u_t \\ v_{t+1} \end{bmatrix} = \begin{bmatrix} N_2 \\ N_3 \end{bmatrix} P^{-1} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} = \begin{bmatrix} -F_u \\ -F_v \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}.$$

We then set $F_v = \mathbf{0}_{n_1 \times n}$ in (34), so

$$\begin{bmatrix} x_{1t+1} \\ \rho_{2t+1} \end{bmatrix} = M_a \begin{bmatrix} x_{1t} \\ \rho_{2t} \end{bmatrix} + C\epsilon_{t+1}, \tag{36}$$

where $M_a = P^{-1}(A - BF_u)P$. The values x_{1t} and x_{2t} are then determined by (36) and (33). In applications, M_a is sometimes not a function of θ. This happens when the predetermined variables are block exogenous, that is, independent of lagged values of the forward looking variables.

Appendix C Solution Algorithms

This appendix summarizes the solution algorithms by adapting some material in Söderlind (1999). The forward looking models are in the main focus, but we also comment on how backward looking models can be handled.

Appendix C.1 Optimal Policy under Commitment

The Lagrangian of the forward looking model (12)–(13) is

$$L_0 = \mathrm{E}_0 \textstyle\sum_{t=0}^{\infty} \beta^t [x_t' Q x_t + 2x_t' U^* u_t^* + u_t^{*\prime} R^* u_t^* + 2\rho_{t+1}'(Ax_t + B^* u_t^* + \xi_{t+1} - x_{t+1})], \tag{37}$$

where $\xi_{t+1} = (C\epsilon_{t+1}, x_{2t+1} - \mathrm{E}_t\, x_{2t+1})$. Let $n = n_1 + n_2$ be the number of elements in x_t and let $q = k + n_1$ be the number of elements in u_t^*.

The first order conditions with respect to ρ_{t+1}, x_t, and u_t^* are

$$\begin{bmatrix} I_n & \mathbf{0}_{n \times q} & \mathbf{0}_{n \times n} \\ \mathbf{0}_{n \times n} & \mathbf{0}_{n \times q} & \beta A' \\ \mathbf{0}_{q \times n} & \mathbf{0}_{q \times q} & -B^{*\prime} \end{bmatrix} \begin{bmatrix} x_{t+1} \\ u_{t+1}^* \\ \mathrm{E}_t\, \rho_{t+1} \end{bmatrix} = \begin{bmatrix} A & B^* & \mathbf{0}_{n \times n} \\ -\beta Q & -\beta U^* & I_n \\ U^{*\prime} & R^* & \mathbf{0}_{q \times n} \end{bmatrix} \begin{bmatrix} x_t \\ u_t^* \\ \rho_t \end{bmatrix} + \begin{bmatrix} \xi_{t+1} \\ \mathbf{0}_{n \times 1} \\ \mathbf{0}_{q \times 1} \end{bmatrix}. \tag{38}$$

Take conditional expectations of (38), expand x_t and ρ_t as (x_{1t}, x_{2t}) and (ρ_{1t}, ρ_{2t}) respectively, and reorder the rows by placing ρ_{2t} after x_{1t}. Write the result as

$$G\,\mathrm{E}_t \begin{bmatrix} k_{t+1} \\ \lambda_{t+1} \end{bmatrix} = D \begin{bmatrix} k_t \\ \lambda_t \end{bmatrix}, \text{ where } k_t = \begin{bmatrix} x_{1t} \\ \rho_{2t} \end{bmatrix} \text{ and } \lambda_t = \begin{bmatrix} x_{2t} \\ u_t^* \\ \rho_{1t} \end{bmatrix}. \tag{39}$$

The elements in k_t have initial conditions: the initial state vector x_{10} is given and the forward looking variables can be chosen freely in the initial period so their shadow prices ρ_{20} are zero (see Currie and Levine (1993)).

Given the square matrices G and D in (39), the generalized Schur decomposition gives the square complex matrices Q (not the same Q as in (37)), S, T, and Z such that $G = QSZ^H$ and $D = QTZ^H$, where Z^H denotes the transpose of the complex conjugate of Z. Q and Z are unitary ($Q^HQ = Z^HZ = I$), and S and T are upper triangular—see Golub and van Loan (1989). Reorder the decomposition (see Sims (2001b) and Klein (2000)) so the block corresponding to the stable (modulus less than one) generalized eigenvalues (the diagonal of T divided by the corresponding elements in S) come first.

Introduce the auxiliary variables

$$\begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix} = Z^H \begin{bmatrix} k_t \\ \lambda_t \end{bmatrix}, \tag{40}$$

where θ_t corresponds to the stable eigenvalues.

Use the generalized Schur decomposition in (39), premultiply by Q^H, use the definitions in (40), and partition S and T conformably with θ_t and δ_t to get

$$\begin{bmatrix} S_{\theta\theta} & S_{\theta\delta} \\ \mathbf{0} & S_{\delta\delta} \end{bmatrix} \mathrm{E}_t \begin{bmatrix} \theta_{t+1} \\ \delta_{t+1} \end{bmatrix} = \begin{bmatrix} T_{\theta\theta} & T_{\theta\delta} \\ \mathbf{0} & T_{\delta\delta} \end{bmatrix} \begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix}. \tag{41}$$

Since the lower right block contains the unstable roots, δ_t will diverge unless $\delta_0 = \mathbf{0}$. From (41) we see that any stable solution will therefore have $\delta_t = \mathbf{0}$ and

$$\mathrm{E}_t\,\theta_{t+1} = S_{\theta\theta}^{-1} T_{\theta\theta}\theta_t, \tag{42}$$

since $S_{\theta\theta}$ is invertible (follows from the way we have ordered the eigenvalues).

From (13) $x_{1t+1} - \mathrm{E}_t\,x_{1t+1} = C_1\epsilon_{t+1}$ and Backus and Driffil (1986) show that $\rho_{2t+1} - \mathrm{E}_t\,\rho_{2t+1} = \mathbf{0}$. Using (39), stack these expressions as $k_{t+1} - \mathrm{E}_t\,k_{t+1}$. Invert (40), partition conformably with k_t, λ_t, θ_t and δ_t, and use the fact that $\delta_t = \mathbf{0}$

$$\begin{bmatrix} k_t \\ \lambda_t \end{bmatrix} = \begin{bmatrix} Z_{k\theta} & Z_{k\delta} \\ Z_{\lambda\theta} & Z_{\lambda\delta} \end{bmatrix} \begin{bmatrix} \theta_t \\ \delta_t \end{bmatrix} = \begin{bmatrix} Z_{k\theta} \\ Z_{\lambda\theta} \end{bmatrix} \theta_t. \tag{43}$$

It follows that $k_{t+1} - \mathrm{E}_t\,k_{t+1}$ can be written

$$Z_{k\theta}(\theta_{t+1} - \mathrm{E}_t\,\theta_{t+1}) = \begin{bmatrix} C\epsilon_{t+1} \\ \mathbf{0} \end{bmatrix}. \tag{44}$$

We can solve for θ_{t+1} in (44) if $Z_{k\theta}$ is invertible. A necessary condition is that the number of backward looking variables (the number of rows in $Z_{k\theta}$) equals the number of stable roots (the number of columns in $Z_{k\theta}$)—this is the saddlepoint condition in proposition 1 of Blanchard and Kahn (1980). Assuming this is satisfied, solve for θ_{t+1} and substitute for $\mathrm{E}_t\,\theta_{t+1}$ from (42)

$$\theta_{t+1} = S_{\theta\theta}^{-1} T_{\theta\theta}\theta_t + Z_{k\theta}^{-1} \begin{bmatrix} C\epsilon_{t+1} \\ \mathbf{0} \end{bmatrix}. \tag{45}$$

The last step is to combine this expression with (43) and the definitions in (40) to write the dynamics as

$$\begin{bmatrix} x_{1t+1} \\ \rho_{2t+1} \end{bmatrix} = M \begin{bmatrix} x_{1t} \\ \rho_{2t} \end{bmatrix} + \begin{bmatrix} C\epsilon_{t+1} \\ \mathbf{0} \end{bmatrix}, \text{ where } M = Z_{k\theta} S_{\theta\theta}^{-1} T_{\theta\theta} Z_{k\theta}^{-1}, \tag{46}$$

$$\begin{bmatrix} x_{2t} \\ u_t^* \\ \rho_{1t} \end{bmatrix} = N \begin{bmatrix} x_{1t} \\ \rho_{2t} \end{bmatrix}, \text{ where } N = Z_{\lambda\theta} Z_{k\theta}^{-1}, \tag{47}$$

and where x_{10} is given and $\rho_{20} = \mathbf{0}$.

This algorithm can also be used for the backward looking case (although there are simpler algorithms that also work): x_{2t} and ρ_{2t+1} are then empty vectors and $n = n_1$.

Appendix C.2 Optimal Policy under Discretion

This section summarizes the iterative algorithm for the discretionary case used by Backus and Driffil (1986). The policy maker reoptimizes every period by taking the process by which private agents form their expectations as given. The solution in $t+1$ gives a value function which quadratic in the state variables, $x'_{1t+1}V_{t+1}x_{1t+1} + v_{t+1}$, and a linear relation between the forward looking variables and the state variables, $x_{2t+1} = K_{t+1}x_{1t+1}$. Private agents form expectations about x_{2t+1} accordingly.

Partition the matrices A, B^*, Q, U^*, and C in (12)–(13) conformably with x_{1t} and x_{2t}.

The Bellman equation for the optimization problem can then be written

$$x_{1t}'V_t x_{1t} + w_t = \min_{u_t^*}\max[x_{1t}'\tilde{Q}_t x_{1t} + 2x_{1t}'\tilde{U}_t u_t^* + u_t^{*\prime}\tilde{R}_t u_t^* + \beta\,\mathrm{E}_t(x_{1t+1}'V_{t+1}x_{1t+1} + w_{t+1})]$$

$$\text{s.t. } x_{1t+1} = \tilde{A}_t x_{1t} + \tilde{B}_t u_t^* + C_1\epsilon_{t+1} \text{ and } x_{1t} \text{ given,} \tag{48}$$

where the matrices with a tilde ($\sim$) are defined as

$$\begin{aligned} D_t &= (A_{22} - K_{t+1}A_{12})^{-1}(K_{t+1}A_{11} - A_{21}) \\ G_t &= (A_{22} - K_{t+1}A_{12})^{-1}(K_{t+1}B_1^* - B_2^*) \\ \tilde{A}_t &= A_{11} + A_{12}D_t \\ \tilde{B}_t &= B_1 + A_{12}G_t \\ \tilde{Q}_t &= Q_{11} + Q_{12}D_t + D_t'Q_{21} + D_t'Q_{22}D_t \\ \tilde{U}_t &= Q_{12}G_t + D_t'Q_{22}G_t + U_1^* + D_t'U_2^* \\ \tilde{R}_t &= R^* + G_t'Q_{22}G_t + G_t'U_2^* + U_2^{*\prime}G_t. \end{aligned} \tag{49}$$

The first order condition of (48) with respect to u_t^* are

$$u_t^* = -F_{1t}x_{1t},\ F_{1t} = (\tilde{R}_t + \beta\tilde{B}_t'V_{t+1}\tilde{B}_t)^{-1}(\tilde{U}_t' + \beta\tilde{B}_t'V_{t+1}\tilde{A}_t). \tag{50}$$

Combining with (48) gives

$$x_{2t} = K_t x_{1t}, \text{ with } K_t = D_t - G_t F_{1t}, \text{ and} \tag{51}$$

$$V_t = \tilde{Q}_t - \tilde{U}_t F_{1t} - F_{1t}'\tilde{U}_t' + F_{1t}'\tilde{R}_t F_{1t} + \beta(\tilde{A}_t - \tilde{B}_t F_{1t})'V_{t+1}(\tilde{A}_t - \tilde{B}_t F_{1t}). \tag{52}$$

The algorithm involves iterating until convergence ("backwards in time") on (49)–(52). It should be started with a symmetric positive definite V_{t+1} and some K_{t+1}. If F_{1t} and K_t converge to constants F_1 and K, the dynamics of the model is (rewrite the first n_1 equations of (13))

$$x_{1t+1} = Mx_{1t} + C\epsilon_{t+1}, \text{ where } M = A_{11} + A_{12}K - B_1^*F_1, \tag{53}$$

$$\begin{bmatrix} x_{2t} \\ u_t^* \end{bmatrix} = Nx_{1t}, \text{ where } N = \begin{bmatrix} K \\ -F_1 \end{bmatrix}. \tag{54}$$

For a backward looking model, the discretionary solution coincides with the commitment solution (see above for a solution algorithm).

Appendix C.3 An Optimal Simple Rule

We start by finding the equilibrium for a given "decision rules" of the evil agent, F_v, and the private sector, F_u. Use the expression for u_t^* in (28) in (13) and take conditional expectations to get

$$\mathrm{E}_t\begin{bmatrix} x_{1t+1} \\ x_{2t+1} \end{bmatrix} = D\begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}, \text{ where } D = A + B^*\begin{bmatrix} & -F_u \\ -F_v & \mathbf{0}_{n_1\times n_2} \end{bmatrix}. \tag{55}$$

This has the same form as (39), so the same solution method can be used: the solution (46)–(47) still applies. The only difference is that u_t^*, ρ_{1t+1}, and ρ_{2t+1} are empty vectors here.

It is straightforward to show that the loss function value for given F_u and F_v matrices is

$$J_0 = x_{10}'Vx_{10} + \mathrm{trace}(VC_1C_1')\beta/(1-\beta), \tag{56}$$

where the matrix V is the fixed point in the iteration ("backwards in time") on

$$V_s = P'\begin{bmatrix} Q & U^* \\ U^{*\prime} & R^* \end{bmatrix}P + \beta M'V_{s+1}M, \text{ where } P = \begin{bmatrix} I_n \\ -F_u \\ -F_v \end{bmatrix}\begin{bmatrix} I_{n_1} \\ N \end{bmatrix}, \tag{57}$$

where M and N are from the solution (46)–(47). The evil agent maximizes the loss function (56) by choosing the optimal elements in the decision rule F_v. This rule is found by a non-linear optimization algorithm.

For a backward looking model, the solution is trivial, but the algorithm here can still be used if we set x_{2t} to an empty matrix.

Appendix D The Stability Problem of Robust Simple Rules

Proposition 1 *If η_0 is strictly positive, the loss function is unbounded. This outcome can be achieved by the evil agent committing to an ever increasing (or decreasing) constant in the law of motion.*

Proof. *Assume that the evil agent commits to the sequence $v_{t+1} = \alpha\gamma^t$, $\alpha > 0$. Then*

$$\mathrm{E}_0\,\pi_t = \textstyle\sum_{s=0}^{\infty}\beta^s v_t = \frac{\alpha\gamma^t}{1-\beta\gamma}. \tag{58}$$

The constraint is binding, so $\eta_0 = \alpha^2(1-\beta\gamma^2)$ *and the problem can be written as*

$$\max_\gamma \frac{\alpha^2}{(1-\beta\gamma^2)(1-\beta\gamma)^2}, \ \ s.t. \quad 1-\beta\gamma^2 > 0. \tag{59}$$

This problem does not have a maximum: $\gamma = \sqrt{1/\beta}$ *is a supremum. However, the evil agent can pick a* γ^* *such that* $1 < \gamma^* < \sqrt{1/\beta}$*, which in turn makes the loss unbounded by (59), and* $\lim_{t\to\infty} E_0\pi_t = \infty$ *by* (58). ■

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