Economics 354 — Prof. Stokey

Winter 2003

INTRODUCTION TO OPTIMAL CONTROL

These notes develop the necessary and sufficient conditions for some standard control problems. Finite-horizon problems—including a variety of endpoint conditions—are studied in sections 1 - 6. Infinite-horizon problems are studied in sections 7 - 12.

FINITE HORIZONS

1. A standard control problem

A canonical control problem is to choose piecewise continuous (vector-valued) controls $\{u(t),\ 0 \le t \le T\}$ to

$$\max \int_0^T f(t, x(t), u(t))\, dt + \phi(x(T)) \tag{1}$$

$$\begin{aligned} \text{s.t.} \quad u(t) &\in U \subseteq \mathbf{R}^m, && (2) \\ x'(t) &= g(t, x(t), u(t)), \quad \text{all } t, && (3) \\ \text{given} \quad x(0) &= x_0, && (4) \end{aligned}$$

where T is fixed and $x(T)$ is free. Other terminal conditions will be discussed later on. The 'givens' are

$x(t) = [x_1(t), ..., x_n(t)]$	$n-$ vector of state variables
$u(t) = [u_1(t), ..., u_m(t)]$	$m-$ vector of controls
$U \subseteq \mathbf{R}^m$	control set
$f(t, x, u)$	return function
$g(t, x, u)$	law of motion (vector valued)
$\phi(x)$	salvage function

Assume f, g, and ϕ are continuously differentiable and strictly concave (strictly convex for a minimum). Additional constraints on x and u can also be incorporated and will be discussed later on.

2. Necessary conditions

To characterize a solution, we ask what properties the optimal path $\{u^*(t)\}$ for the controls must satisfy.

First, note that for *any* pair $\{x, u\}$ that satisfies the constraints, laws of motion, and boundary conditions, (2)-(4), and *any* continuous function $\lambda(t)$ that is piecewise continuously differentiable,

$$\begin{aligned} J\{x, u\} &= \int_0^T f(t, x(t), u(t))dt + \phi(x(T)) \\ &= \int_0^T \left[f(t, x, u) + \lambda(t) \cdot [g(t, x, u) - \dot{x}]\right] dt + \phi(x(T)). \end{aligned}$$

Integrating by parts we find that

$$-\int_0^T (\lambda \cdot \dot{x})\ dt = \int_0^T \left(\dot{\lambda} \cdot x\right)\ dt - [\lambda(T) \cdot x(T)] + [\lambda(0) \cdot x(0)],$$

so

$$\begin{aligned} J\{x, u\} &= \int_0^T \left[f + (\lambda \cdot g) + \left(\dot{\lambda} \cdot x\right)\right]\ dt + \phi(X(T)) \\ &\quad - [\lambda(T) \cdot x(T)] + [\lambda(0) \cdot x(0)]. \end{aligned} \tag{5}$$

The important thing is that (5) gives the net return $J\{x, u\}$ from *any* feasible plan $\{x, u\}$, where λ is *any* piecewise continuously differentiable (PCD) function.

Now suppose that $\{x^*, u^*\}$ is an optimal path and that $\{x, u\}$ is a feasible path. Let f^*, g^*, ϕ^* be the functions evaluated at $\{x^*, u^*\}$, and let f, g, ϕ be the functions evaluated at $\{x, u\}$. Define

$$\begin{aligned} \delta_x(t) &= x(t) - x^*(t), \\ \delta_u(t) &= u(t) - u^*(t), \quad \text{all } t. \end{aligned}$$

Then for any PCD function λ,

$$\begin{aligned} \delta_J &= J\{x, u\} - J\{x^*, u^*\} \\ &= \int_0^T \left[(f - f^*) + (\lambda \cdot (g - g^*)) + \left(\dot{\lambda} \cdot (x - x^*) \right) \right] dt \\ &\quad + (\phi - \phi^*) - [\lambda(T) \cdot (x(T) - x^*(T))], \end{aligned}$$

where we have used the fact that $x(0) = x^*(0) = x_0$. Suppose that $\{x, u\}$ is a small perturbation of $\{x^*, u^*\}$, so that we may use a first-order approximation. Then

$$\begin{aligned} \delta_J \approx & \int_0^T \left[\left(f_x + \lambda g_x + \dot{\lambda} \right) \cdot \delta_x + (f_u + \lambda g_u) \cdot \delta_u \right] dt \\ & + [\phi'(x^*(T)) - \lambda(T)] \, \delta_x(T). \end{aligned} \tag{6}$$

The important thing is that (6) must hold for *any* feasible perturbation and *any* PCD function λ.

In particular, we may choose the function λ defined by

$$\dot{\lambda} = -[f_x + \lambda g_x], \quad \lambda(T) = \phi'(x^*(T)). \tag{7}$$

In this case we obtain

$$\delta_J \approx \int_0^T [f_u + \lambda g_u] \delta_u \, dt. \tag{8}$$

But (8) implies that $\delta_J \leq 0$ for all feasible δ_u if and only if

$$u^*(t) = \arg\max_{u \in U} \; [f(t, x^*(t), u) + \lambda(t) g(t, x^*(t), u)], \quad \text{all } t. \tag{9}$$

If u is in the interior of U, we get the FOC $f_u + \lambda g_u = 0$. If u^* is on the boundary of U, then the appropriate inequality must hold.

To summarize, *necessary* conditions for $\{x^*, u^*\}$ to be an optimum are that (2)-(4) and (9) hold, where λ satisfies (7).

These conditions can be generated easily by defining the Hamiltonian

$$H(t, x, u, \lambda) = f(t, x, u) + \lambda \cdot g(t, x, u). \tag{10}$$

Then the necessary conditions for an optimum are

$$\begin{aligned} u^* &= \arg\max_{u \in U} H, & & \\ \dot{x}^* &= \partial H / \partial \lambda, & x^*(0) &= x_0, \\ \dot{\lambda} &= -\partial H / \partial x, & \lambda(T) &= \phi'(x^*(T)). \end{aligned} \tag{11}$$

The first line of (11) is the optimality condition, the second and third are the laws of motion for the states and costates. One may think about solving the system by first using the optimality conditions to obtain values for the m control variables as functions of the state and costate variables. If f and g are strictly concave in u, and if U is a convex set, then this gives a unique optimum, call it $u^*(x, \lambda)$. Then substituting into the laws of motion for the state and costate variables gives a system of $2n$ differential equations with $2n$ boundary conditions: initial conditions for the states, terminal conditions for the costates. Bingo!

Note that the states and costates, $\{x^*, \lambda\}$ must be continuous functions of time, and the controls $\{u^*\}$ must be *piecewise* continuous functions of time. Thus, the controls may jump (occasionally), but the states and costates may not.

3. Sufficiency

Consider the problem in (1)-(4). Suppose that $\{u^*, x^*, \lambda\}$ satisfies (11), where H is defined in (10). Assume that f is concave jointly in (x, u) and ϕ is concave in x. Let $\{u, x\}$ be any other pair satisfying the constraints, laws of motion, and boundary conditions (2)-(4). Then

$$\delta_J \equiv \int_0^T (f - f^*)\, dt + (\phi - \phi^*)$$

$$
\begin{aligned}
&\leq \int_0^T \left[(f_x^* \cdot \delta_x) + (f_u^* \cdot \delta_u)\right] dt + \left[\phi^{*\prime} \cdot \delta_x(T)\right] \quad \text{(by concavity)} \\
&= -\int_0^T \left[\left((\dot{\lambda} + \lambda g_x^*) \cdot \delta_x\right) + (\lambda g_u^* \cdot \delta_u)\right] dt + \left[\phi^{*\prime} \cdot \delta_x(T)\right],
\end{aligned}
$$

where δ_x and δ_u are defined as before. Since

$$
\begin{aligned}
\int_0^T \left[\dot{\lambda} \cdot \delta_x\right] dt &= -\int_0^T \left[\lambda \cdot \dot{\delta}_x\right] dt + [\lambda(T) \cdot \delta_x(T)] - [\lambda(0) \cdot \delta_x(0)] \\
&= -\int_0^T \left[\lambda \cdot (g - g^*)\right] dt + [\lambda(T) \cdot \delta_x(T)],
\end{aligned}
$$

we find that

$$
\delta_J \leq \int_0^T \left\{\lambda \cdot \left[(g - g^*) - g_x^*(x - x^*) - g_u^*(u - u^*)\right]\right\} dt,
$$

where we have used the fact that $\lambda(T) = \phi'(x^*(T))$. Hence δ_J is nonpositive if $\lambda \geq 0$ and g is concave jointly in (x, u), or if $\lambda \leq 0$ and g is convex in (x, u).

This condition often fails. An alternative, due to Arrow, is much more useful. Suppose $\{x^*, u^*, \lambda\}$ satisfies the necessary conditions for a maximum. Define

$$
U\left(t, x, \lambda(t)\right) \equiv \arg\max_{u \in U} H\left(t, x, u, \lambda(t)\right), \quad \text{all } t,
$$

to be the policy function for the controls, given values for the states, costates, and time. Then define the maximized Hamiltonian

$$
H^*\left(t, x, \lambda(t)\right) \equiv H\left[t, x, U\left(t, x, \lambda(t)\right), \lambda(t)\right], \quad \text{all } t.
$$

A sufficient condition for $\{x^*, u^*\}$ to be optimal is that H^* be concave in x, all t.

4. Existence

It is not hard to construct problems for which an optimum dies not exist. Clearly U should be compact, so the "static" problem of maximizing the Hamiltonian w.r.t. u has a solution. But there are other problems. S&S give the following example.

Suppose you have a hotplate, with an initial temperature of 100°C, whose only control is an on-off switch. You want to minimize the integral of the squared deviations of the temperature from 100°C over the interval $[0, T]$. If there is no cost the turning the switch on and off, the objective can always be improved by flipping the switch faster and faster.

See Ekeland and Turnbull or Seierstad and Sydaeter for some conditions for existence.

5. Various Boundary Conditions

Suppose the terminal time T is free and the salvage function $\phi(T, x(T))$ has time as an argument. Suppose that some of the terminal stocks are fixed, some are subject to a nonnegativity constraint, and some are free. In particular, assume that

$$\begin{aligned} x_i(T) &= x_{iT,} \quad i = 1, ..., q, \\ x_i(T) & \quad \text{free}, \quad i = q+1, ..., r, \\ x_i(T) &\geq 0, \quad i = r+1, ..., n. \end{aligned}$$

For simplicity, suppose that all of the initial stocks are fixed.

Suppose $\{x^*, u^*\}$ on $[0, T]$ is an optimal path and that $\{x, u\}$ on $[0, T + \delta_T]$ is a feasible path. Define $(\delta_x, \delta_u), (f^*, g^*, \phi^*)$, and (f, g, ϕ) as before. Then for any PCD function λ defined on $[0, T]$,

$$\begin{aligned} \delta_J &= J\{x, u\} - J\{x^*, u^*\} \\ &= \int_0^T \left[(f - f^*) + \lambda(g - g^*) + \dot{\lambda}(x - x^*) \right] \, dt + \int_T^{T+\delta_T} f \, dt \\ & \quad + [\phi(T + \delta_T, x(T + \delta_T)) - \phi(T, x^*(T))] - \lambda(T) \, [x(T) - x^*(T)] \, , \end{aligned}$$

where we have used the fact that $x(0) = x^*(0) = x_0$, as before. The first-order approximation is then

$$\delta_J \approx \int_0^T \left[[f_x + \lambda g_x + \dot{\lambda}] \delta_x + [f_u + \lambda g_u] \delta_u \right] dt + [f(T) + \phi_t(T)] \, \delta_T \qquad (12)$$

$$+\phi_x(T)\left[x(T+\delta_T)-x^*(T)\right]-\lambda(T)[x(T)-x^*(T)].$$

As before, (12) must hold for *any* feasible perturbation and *any* PCD function λ. Notice that λ is defined only on $[0,T]$. Also notice that since the two terminal dates are possibly different, the two terms in the last line do not have matching terms in $\dot{x(T)}$. Define $\delta x_T \equiv x(T+\delta_T)-x^*(T)$ to be the difference in the two terminal states, and note that

$$\begin{aligned} x(T)-x^*(T) &= x(T)-x(T+\delta_T)+x(T+\delta_T)-x^*(T) \\ &= -g(T)\delta_T+\delta x_T, \end{aligned}$$

so

$$\begin{aligned} \delta_J \approx & \int_0^T \left[[f_x+\lambda g_x+\dot{\lambda}]\delta_x+[f_u+\lambda g_u]\delta_u\right] dt \\ & +\left[\phi_x(T)-\lambda(T)\right]\delta x_T+\left[f(T)+\phi_t(T)+\lambda(T)g(T)\right]\delta_T. \end{aligned} \tag{13}$$

The FOC for u and the laws of motion for λ are as before. The terminal conditions for the state variables are

$$\begin{array}{lll} x_i(T) \text{ is fixed,} & \lambda_i(T) \text{ is free} & i=1,...,q, \\ x_i(T) \text{ is free,} & \lambda_i(T)-\partial\phi/\partial x_i=0, & i=q+1,...,r, \\ x_i(T)\geq 0, & \lambda_i(T)-\partial\phi/\partial x_i\geq 0, & i=r+1,...,n, \\ & x_i(T)\left[\lambda_i(T)-\partial\phi/\partial x_i\right]=0. & \end{array}$$

If T is free, then

$$H(T)+\phi_t(T)=f(T)+\lambda(T)g(T)+\phi_t(T)=0.$$

If $T\leq\hat{T}$, then

$$H(T)+\phi_t(T)\geq 0, \quad \text{with equality if } T<\hat{T}.$$

If there is a constraint of the form $K(x(T))=0$ or ≥ 0 on the terminal state, then

$$\lambda_i(T)-\partial\phi/\partial x_i=pK_i(x),$$

where $p \geq 0$, with eq. if the constrain does not bind.

If $T \leq \hat{T}$, and K depends on T, then the terminal date must satisfy

$$H(T) + \phi_t(T) + pK_t(T, x) \geq 0, \quad \text{with equality if } T < \hat{T}. \tag{14}$$

If T is free, then (14) must hold with equality.

6. Calculus of Variations

If $\dot{x}(t) = g(t, x, u) = u(t)$, then we can write $f(t, x, \dot{x})$ and eliminate the laws of motion. In this case we have a classical Calculus of Variations problem. Notice that if the solution is interior, we get

$$\begin{aligned} f_u + \lambda &= 0, \\ \dot{x} &= u, \qquad x(0) = x_0, \\ \dot{\lambda} &= -f_x, \quad \lambda(T) = \phi'(x(T)). \end{aligned}$$

so

$$-f_u = \lambda = \lambda(T) - \int_t^T \dot{\lambda}\, ds = \phi'(x(T)) + \int_t^T f_x\, ds$$

or

$$\int_t^T f_x\, ds + f_u = -\phi'(x(T))].$$

This is the Euler equation in integral form. One may also write it in the more usual form

$$f_x = df_u/dt.$$

If $\dot{x}(t) = g(t, x, u) = u(t) - \eta x$, and the solution is interior, the same method still applies. In this case

$$H = f(t, x, u) + \lambda[u - \eta x],$$

so the conditions for an optimum are

$$\begin{aligned} f_u + \lambda &= 0, \\ \dot{x} &= u - \eta x, \quad x(0) = x_0, \\ \dot{\lambda} - \eta\lambda &= -f_x, \qquad \lambda(T) = \phi'(x(T)). \end{aligned}$$

Hence we can integrate the costate equation to get

$$\lambda(t)e^{-\eta t} = \int_t^T f_x(s, x, u)\, e^{-\eta s}\, ds + \phi'(x(T))e^{-\eta T},$$

and then substitute from the optimality condition to get

$$\int_t^T f_x(s, x, u)\, e^{-\eta s}\, ds + f_u(t, x, u)e^{-\eta t} = \phi'(x(T))e^{-\eta T}.$$

For $\eta = 0$, this is exactly the Euler equation in integral form.

INFINITE HORIZONS

The next sections deal with infinite-horizon problems.

7. A standard control problem

The canonical infinite horizon is

$$\max \int_0^\infty f(t, x(t), u(t))\, dt \tag{15}$$

$$\text{s.t.} \quad u(t) \in U \subseteq \mathbf{R}^m, \tag{16}$$

$$x'(t) = g(t, x(t), u(t)), \quad \text{all } t, \tag{17}$$

$$\text{given } x(0) = x_0. \tag{18}$$

The necessary conditions for a maximum are the same as before, except for the terminal conditions. There are no longer any necessary terminal (tranversality) conditions.

But if the problem is concave (satisfies Arrow's condition), then adding the following conditions is sufficient (for a present value Hamilton):

$$\lim_{t\to\infty} \lambda(t) \geq 0, \quad \text{and} \quad \lim_{t\to\infty} \lambda(t)x(t) = 0.$$

For an autonomous problem with discounting at the rate $\rho > 0$, and a current value Hamiltonian, the conditions are

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t) \geq 0, \quad \text{and} \quad \lim_{t\to\infty} e^{-\rho t}\lambda(t)x(t) = 0.$$

A useful application of these TC's is the following. Consider a discounted, autonomous problem, formulated in current value terms. Let $U^*(x, \lambda)$ be the optimal policy function, defined by maximizing the Hamiltonian. Suppose the system has a steady state (x^s, λ^s). Suppose, too, that for any initial state x_0, an initial value λ_0 for the costates can be chosen so that the system of $2n$ ODE's

$$x'(t) = g\left(x(t), u(t)\right),$$

$$\lambda'(t) = \rho\lambda(t) - \left[f_x\left(x(t), u(t)\right) + \lambda g_x\left(x(t), u(t)\right)\right],$$

$$x(0) = x_0, \quad \lambda(0) = \lambda_0,$$

converges to this steady state, where

$$u(t) = U^*\left(x(t), \lambda(t)\right), \quad \text{all } t.$$

Then clearly the TC's hold.

Notice that for infinite horizon problems we have the following facts:

1. For a strictly concave problem, *if* an optimum exists, it is unique.

2. For an autonomous problem with discounting, any path that converges to a steady state satisfies the TC's.

Suppose we have an autonomous, concave problem with discounting. A path that satisfies the necessary conditions and converges to a steady state is optimal (by 2). Therefore, any other path that satisfies the necessary conditions must violate the TC's (by 1). [You can check this directly, too, if you wish, but it seems redundant.]

8. Optimal growth: Cass-Koopmans

Consider the problem

$$\max \int_0^{\infty} e^{-\rho t} U\left(c(t)\right) dt$$

$$\begin{aligned}
\text{s.t.} \quad k'(t) &= f(k(t)) - \delta k(t) - c(t), \\
0 &\leq c(t) \leq f\left(k(t)\right), \\
\text{given} \quad k(0) &= k_0 > 0.
\end{aligned}$$

Assume that U and f are strictly increasing and strictly concave, with

$$\begin{aligned}
\lim_{c \to 0} U'(c) &= +\infty, \quad \text{and} \quad \lim_{c \to \infty} U'(c) = 0, \\
\lim_{k \to 0} f'(k) &= +\infty, \quad \text{and} \quad \lim_{k \to \infty} f'(k) = 0.
\end{aligned}$$

In addition, assume that k_0 is small, so the upper bound on consumption can also be ignored, or else that capital goods can be consumed (so gross investment can be negative).

The (current value) Hamiltonian is

$$H(k, c, \lambda) = U(c) + \lambda \left[f(k) - \delta k - c \right],$$

so the conditions for a maximum are

$$\begin{aligned}
H_c &= 0 = U'(c) - \lambda, \\
-H_k &= \lambda' - \rho \lambda = -\lambda \left[f'(k) - \delta \right],
\end{aligned}$$

$$\lim_{t \to \infty} e^{-\rho t} \lambda(t) \geq 0, \qquad \text{and} \qquad \lim_{t \to \infty} e^{-\rho t} \lambda(t) k(t) = 0.$$

The first two are necessary. These three, together with the concavity of the objective and the law of motion, are sufficient.

Define $\gamma = U'^{-1}$, and note that

$$\gamma'(\lambda) = \frac{1}{U''(\gamma(\lambda))} < 0, \quad \lim_{\lambda\to 0}\gamma(\lambda) = \infty, \quad \text{and} \quad \lim_{\lambda\to\infty}\gamma(\lambda) = 0.$$

Hence the policy function $c = \gamma(\lambda)$ is well defined and continuous.

Consider the system of differential equations

$$\begin{aligned}\lambda'(t) &= [\rho + \delta - f'(k(t))]\,\lambda(t),\\ k'(t) &= f(k(t)) - \delta k(t) - \gamma(\lambda(t)).\end{aligned}$$

Notice that

$$\begin{aligned}\lambda' &= 0 \quad \text{if} \quad f'(k) = \rho + \delta,\\ k' &= 0 \quad \text{if} \quad f(k) - \delta k = \gamma(\lambda).\end{aligned}$$

Define (k^s, λ^s) by

$$f'(k^s) = \delta + \rho, \quad f'(k^s) - \delta k^s = \gamma(\lambda^s).$$

This is the unique steady state for the system. Note that $\lambda^s = U'(c^s)$, where $c^s = f(k^s) - \delta k^s$.

To study transitional dynamics, define $\overline{k} > k^g > k^s$ by

$$f(\overline{k}) = \delta\overline{k}, \quad f'(k^g) = \delta.$$

Note that $\overline{k}$ is the largest maintainable capital stock, and that the concave function $f(k) - \delta k$ reaches a maximum at k^g. Define $\Lambda(k)$ by

$$f(k) - \delta k \equiv \gamma(\Lambda(k)),$$

so $(k, \Lambda(k))$ is the locus where $k' = 0$. Note that the steady state pair (k^s, λ^s) is one point on this locus. To find the slope of this locus, differentiate to get

$$\Lambda'(k) = \frac{f'(k) - \delta}{\gamma'(\Lambda(k))}.$$

Since $\gamma' < 0$, it follows that $\Lambda'(k) < 0$ for $k < k^g$, and $\Lambda'(k) > 0$ for $k > k^g$. See Figure 1 for the phase diagram.

9. Linearizing around the steady state

To study transitional dynamics near the steady state, linearize around the steady state to get

$$\begin{aligned} k'(t) &\approx [f' - \delta]\,(k - k^s) - \gamma'\,(\lambda - \lambda^s)\,, \\ \lambda'(t) &\approx -f''\lambda^s\,(k - k^s) + \underbrace{[\delta + \rho - f']}\,(\lambda - \lambda^s)\,, \end{aligned}$$

where all functions are evaluated at the SS. Note that the marked term is zero at the steady state. Hence we have

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \approx \begin{pmatrix} \rho & -\gamma'(\lambda^s) \\ -f''(k^s)\lambda^s & 0 \end{pmatrix} \begin{pmatrix} x \\ y^s \end{pmatrix},$$

where $x = (k - k^s)$, and $y = (\lambda - \lambda^s)$. This a linear homogeneous system, with constant coefficients. The characteristic equation is

$$(\rho - R)\,(0 - R) - \gamma' f''\lambda^s = 0,$$

so the roots are

$$R = \frac{1}{2}\left[\rho \pm \sqrt{\rho^2 + 4\gamma' f''\lambda^s}\right].$$

Since $4\gamma' f''\lambda^s > 0$, it follows that the roots are real and of opposite sign.

Given any small $x_0 \equiv k_0 - k^s$ (of either sign), an approximation to the (unique) solution can be constructed as follows. The solution is of the form

$$\begin{aligned} x(t) &= a_1 e^{R_1 t} + a_2 e^{R_2 t}, \\ y(t) &= b_1 e^{R_1 t} + b_2 e^{R_2 t}, \end{aligned}$$

where (a_i, b_i) is a scalar multiple of the eigenvector v_i associated with $R_i,\ i = 1, 2$. Let $R_1 < 0$ be the negative root. The transversality conditions imply that the positive root must have zero coefficients: $a_2 = b_2 = 0$. The initial condition for capital implies that $a_1 = x_0$. The initial condition $b_1 = y_0$ for y can be calculated by computing the eigenvector. (If you are using Matlab, this is very convenient.) Or, note that

$$y'(0) = R_1 y_0 = -f'' \lambda^s x_0,$$

so

$$y_0 = \frac{-f'' \lambda^s x_0}{R_1}.$$

The equation for the characteristic roots can be used to study the effect of various parameters on the speed of convergence. Raising the discount rate ρ slows down convergence. To study the effect of curvature in the utility function, note that

$$\gamma'(\lambda^s)\lambda^s = \frac{U'(c^s)}{U''(c^s)}.$$

Consider a change in U that leaves the steady state unchanged. This can be accomplished by taking a concave transformation of U that leaves $U(c^s)$ and $U'(c^s)$ unchanged. Thus, $U''(c^s)$ increases in absolute value, slowing down convergence. Similarly, consider a concave transformation of the production function f that leaves $f(k^s)$ and $f'(k^s)$ unchanged. Then $f''(k^s)$ increases in absolute value, increasing the speed of convergence.

The savings rate along the optimal path is

$$s(t) = 1 - \frac{c(t)}{f(k(t))},$$

so

$$s'(t) = \frac{c}{f}\left[\frac{f'k'}{f} - \frac{c'}{c}\right],$$

so the sign is indeterminate: the savings rate may rise or fall along the optimal path.

The analysis here has been in terms of the pair (k, λ). Since there is a one-to-one relationship between λ and c, the pair (k, c) could have been used instead.

Exercise. Suppose there is exogenous population growth at the constant rate $n > 0$. Show how the formation above can be reinterpreted to incorporate population growth. Explain why the restriction $\rho > n$ is needed.

Exercise. Suppose there is exogenous, labor augmenting technical change at the constant rate $g > 0$. That is

$$Y(t) = F\left[K(t), e^{(g+n)t}L_0\right].$$

Assume that the utility function has the CIES form

$$U(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}.$$

Formulate the problem so that the previous analysis applies. Is any restriction on g needed?

10. Log-linear approximation

When approximating around the SS, it is sometimes more convenient (and more accurate) to linearize in the log space. Instead of using the linear approximation

$$f(k) \approx f(k^s) + f'(k^s)(k - k^s),$$

define

$$z = \ln\left(\frac{k}{k^s}\right), \quad \text{so} \quad \dot{z} = \frac{\dot{k}}{k},$$

$$k = k^s e^z \approx k^s(1+z), \quad \text{and} \quad k - k^s \approx k^s z$$

and

$$f(k) \approx f(k^s) + f'(k^s)k^s z.$$

This is especially useful when the functions being approximated are of the form $f(k) = Ak^\alpha$, so

$$f(k) \approx f(k^s)\left[1 + \alpha z\right].$$

To see this, consider the system of ODE's

$$\frac{\dot{x}_i}{x_i} = \sum_{j=1}^{n} a_{ij} x_j^{\alpha_{ij}}, \quad i = 1, ..., n.$$

Suppose this system has a steady state at $(\overline{x}_1, ...\overline{x}_n)$. Define the log deviations from the steady state to be $z_i = \ln(x_i/x_i^s)$, all i. Then we have

$$\dot{z}_i \approx \sum_{j=1}^{n} a_{ij}\overline{x}_j^{\alpha_{ij}} \left[1 + \alpha_{ij} z_j\right], \quad i = 1, ..., n.$$

The sum of the first terms is zero, by the definition of a steady state, so

$$\dot{z}_i \approx \sum_{j=1}^{n} a_{ij}\overline{x}_j^{\alpha_{ij}} \alpha_{ij} z_j, \quad i = 1, ..., n.$$

Numerical approximation to power functions are generally more accurate of the linearization is done in the log space.

Exercise. Consider the two-dimensional system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x - x^s \\ y - y^s \end{pmatrix}.$$

Let $X = \ln(x/x^s)$ and $Y = \ln(y/y^s)$. Show that

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} \approx \begin{pmatrix} a & by^s/x^s \\ cx^s/y^s & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Hence both systems have the same characteristic roots. If an eigenvector for the system that is linear in levels is (v_{i1}, v_{i2}), then the log system has an eigenvector $(V_{i1}, V_{i2}) = (v_{i1}/x^s, v_{i2}/y^s)$.

Exercise. For the Cass-Koopmans model, suppose the utility and production functions are

$$U(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}, \quad \sigma > 0,$$

and

$$f(k) = Ak^{\alpha}, \quad 0 < \alpha < 1.$$

What is the linear approximation to the laws of motion in the log space?

11. Rebelo's Ak model

Suppose the production function is linear in capital, and consider the problem

$$\max \int_0^\infty e^{-\rho t} \frac{c(t)^{1-\sigma} - 1}{1-\sigma} dt$$

$$\text{s.t.} \quad \dot{k}(t) = Ak(t) - c(t),$$

$$\text{given} \quad k(0) = k_0 > 0.$$

The (current value) Hamiltonian is

$$H(k, c, \lambda) = \frac{c(t)^{1-\sigma} - 1}{1-\sigma} + \lambda \left[Ak - c \right],$$

so the conditions for a maximum are

$$\begin{aligned} c(t)^{-\sigma} &= \lambda, \\ \frac{\dot{\lambda}}{\lambda} &= \rho - A, \end{aligned}$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t) \geq 0, \quad \text{and} \quad \lim_{t\to\infty} e^{-\rho t}\lambda(t)k(t) = 0.$$

Note that

$$\frac{\dot{c}}{c} = -\frac{1}{\sigma}\frac{\dot{\lambda}}{\lambda} = \frac{A-\rho}{\sigma},$$

so consumption grows at a constant rate, and that rate is positive if and only if $A > \rho$. Clearly, constant consumption growth implies that $c/k = \gamma$ is constant, so capital growth is also constant. Hence

$$\gamma = A - \frac{\dot{k}}{k} = A - \frac{\dot{c}}{c} = \frac{\rho}{\sigma} - \frac{1-\sigma}{\sigma} A.$$

The second TC holds if and only if

$$\lim_{t\to\infty} e^{-\rho t} c(t)^{-\sigma} k(t) = 0.$$

Since consumption and capital grow at the same rate, we need

$$\rho > \frac{1-\sigma}{\sigma}\left(A - \rho \right),$$

or

$$\rho > (1 - \sigma) A.$$

Thus, the model implies sustained growth along a unique optimal path if and only if $A > \rho > (1 - \sigma) A$. Note that if all output is saved, $k(t) = k_0 e^{At}$. If all output along this path is *also* consumed, then $c(t) = Ak_0 e^{At}$. Thus, total utility along this path is bounded if the TC holds.

12. A perverse example

This example is from Takayama, who got it from Arrow and Kurz, who got it from H. Halkin, who thought it up (I guess). It is an infinite horizon problem with many optimal paths where the TC's do not hold. The problem is

$$\max \int_0^\infty [1 - x(t)]\, u(t) dt$$

$$\text{s.t.} \quad x'(t) = [1 - x(t)]\, u(t), \quad x(0) = 0,$$

$$-1 \leq u(t) \leq +1.$$

For any $T > 0$,

$$\begin{aligned} \int_0^T [1 - x(t)]\, u(t) dt &= \int_0^T x'(t) dt \\ &= x(T) \\ &= 1 - e^{-U(T)}, \end{aligned}$$

where

$$U(T) = \int_0^T u(t) dt.$$

Hence any path for which $\lim_{T \to \infty} U(T) = +\infty$ is optimal. For example, let $u(t) = \hat{u}$, where $0 < \hat{u} < 1$. Since solutions of this sort are interior, $\partial H / \partial u = 0$, where

$$H(x, u, \lambda) = (1 + \lambda)(1 - x)\, u.$$

Hence it must be the case that $\lambda(t) = -1$, all t. [Note that $\lambda' = -\partial H/\partial x = (1+\lambda)\,u = 0$, if $\lambda = -1$.] Hence

$$\lim_{t\to\infty} \lambda(t) = -1, \quad \text{and} \quad \lim_{t\to\infty} \lambda(t)x(t) = -1.$$

In general, TC's for infinite horizon problems are tricky. See Ekeland and Teman (1976) and Araujo and Scheinkman.

13. Other caveats

A problem may have no feasible solution. For example, consider

$$\max \int_0^T e^{-\rho t} U\,(c(t))\,dt$$

$$\text{s.t.} \quad k'(t) = f(k(t)) - \delta k(t) - c(t),$$

$$0 \le c(t) \le f\,(k(t))\,,$$

$$k(0) = k_0 > 0, \quad k(T) = k_T.$$

For k_T sufficiently large and/or k_0 and T sufficiently small, the feasible set is empty.

If the problem has constraints, the extremum may not be a regular point.

Or, the objective function may get zero weight in the Hamiltonian. The following example is from Kamien and Schwartz (KS), p. 137.

$$\max \int_0^T u(t)dt$$

$$\text{s.t.} \quad x' = u^2, \quad x(0) = x(T) = 0.$$

There is only one feasible path, $u(t) \equiv 0$. Let

$$H = \lambda_0 u + \lambda_1 u^2,$$

The correct answer is obtained by setting $\lambda_0 = 0$. Setting $\lambda_0 = 1$ does not work. [Why?] See KS, Section 14 and references.

References

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