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The equivalence of lending equilibria and signalling-based insurance under asymmetric information

Bart Taub*

I present a model in which a continuum of individuals have stochastic idiosyncratic income shocks. Complete insurance is physically feasible but unattainable due to an information asymmetry; income shocks are observable only by the individuals receiving them. Any insurance institution must therefore rely on self-reporting of income innovations. Two ways of achieving incentive-compatible self-reporting are presented. The first is a debt market with an explicit lending restriction. The second is an insurance contract that linearly filters a signal transmitted by individuals. The two are then demonstrated to be identical. Equilibrium consumption fluctuates in a random walk, which is inefficient given the physical potential for complete insurance, but is efficient given the information constraints. The results are complementary to those of Green (1987) but permit more general stochastic processes of income to be analyzed. Serial correlation of income reduces the efficiency of the insurance.

1. Introduction

■ If individual income were idiosyncratically stochastic across many individuals, but observable only to the individual receiving it, a fundamental tension would arise. Insurance would be physically feasible, but how could the information about income needed to correctly insure it be elicited? This article presents a model showing how a middle ground could be attained by a contract that elicits information and partly insures income fluctuations.

As Townsend (1982) demonstrates, insurance would be impossible in a one-period framework. Individuals would report low incomes regardless of their true state, and receive insurance as if in the low state. In a bilateral framework, with a single insurer and a single insuree, the insurer would have negative expected profit and would refuse to furnish insurance. If the contract were between many agents, some of whom could suffer bad luck and some good, the reporting problem would again destroy the efficacy of an insurance program. Any contract that paid positive benefits to those reporting bad luck would encourage all individuals to report bad luck and so would be infeasible. Townsend showed that one way to overcome this problem was to have an "enduring relationship." An individual who

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* University of Illinois.

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reported low income in the first period of a contract would be required to pay back the benefits in a later period. Despite the fact that the same inefficiency as in the one-period case arises in the final period, such a contract is superior to no contract in that it is incentive compatible.

Green (1987) studied a generalized version of Townsend's structure in which there are infinitely many periods, many individuals, and with stochastic income observable only to the individual receiving it. He showed that a mutual insurance problem is equivalent to a bilateral contract. This enabled him to show that the optimal contract—that is, the contract that provided optimum insurance under the constraint that it be incentive compatible—is equivalent to a lending equilibrium with borrowing restrictions or credit rationing. Each individual's consumption and debt then follows a random walk in the manner predicted by Friedman's (1957) permanent income hypothesis. Because this arises in an informationally constrained equilibrium, the random walk of consumption in actual lending markets reflects the inability to completely insure individuals; complete insurance would eliminate idiosyncratic consumption fluctuations altogether. Thus, actual lending institutions can be reinterpreted as the working of an informationally constrained insurance equilibrium.

The analysis of this article replicates Green's findings and extends them in several ways. The primary technical difference is that individuals are assumed here to have quadratic instead of exponential utility functions. By restricting average income to lie below the bliss point, with the standard deviation of income such that it is improbable that income exceeds the bliss point, quadratic utility can be considered an approximation of the more standard framework. Quadratic utility has several advantages. First, the separability of the deterministic and stochastic components of quadratic utility means that there is no direct interaction between the idiosyncratic shocks and aggregates, and this makes analyzing the equilibrium, which uses aggregate behavior, much easier. Because the contract is explicit, it is straightforward to analyze Townsend's conjecture that a unitary discount factor yields an efficient contract, while a fractional discount factor yields incomplete insurance. Second, Green had income follow a Bernoulli process with only two discrete outcomes; here it can range over a continuum. Third, the income in Green's framework was serially uncorrelated; here, serial correlation is allowed and the efficacy of the insurance contract is shown to be reduced by such correlation. Finally, because of the linear quadratic structure, direct econometric implementation of the model is possible.

It is useful to contrast the pure insurance problem with a standard principal-agent problem. In a principal-agent problem, the principal induces the agent to undertake effort to produce profit and simultaneously reveal information about the productivity of the effort. If the information to be revealed is discrete, or if there is suitable concavity present, there is an optimum strategy that reveals information and elicits effort even if there is only one period. Thus insurance games such as that of Shavell (1979), in which care influences the hazard rate, are equivalent to principal-agent games in the sense that care is a kind of labor.

Recent research on moral hazard and principal-agent problems has focused on repeated contracts. Rubinstein and Yaari (1983) analyze an infinitely repeated insurance problem with no discounting. They show that the repetition allows punishment and reward strategies to attain perfect insurance. The insurance is not of uninsured income, but of actual hazards that are observable *ex post*. What is not observable is the care individuals take to prevent the hazard. Their central result is that optimum care is elicited if the horizon is infinite and there is no discounting.

Radner (1985) analyzes a model similar in spirit to that of Rubinstein and Yaari. There is a principal-agent structure, in which the agent is risk averse and uses the principal to obtain insurance. The principal profits by the effort of the agent's effort and the realization of the hazard. The model has an infinite horizon like that of Rubinstein and Yaari, but there is discounting. Radner shows that if the principal periodically reviews finite histories of incomes and gives rewards and punishments contingent on the histories, then for large discount factors efficiency is approached. Spear and Srivastava (1987) also have an infinite horizon principal-agent problem with discounting. The principal observes income that is produced with costly effort. The agents wish to be insured and achieve this via the intermediary. Spear and Srivastava use a dual approach like Green's; that is, the principal has as a state variable the expected utility of the agent, and this enables them to use a dynamic programming approach to the solution.

In a pure insurance game in which care is impossible, no one-period contract works because of the incentive problem. But by examining the pattern of reports over time, it may be possible to elicit accurate information and still provide some insurance. There is a mapping of individuals' reports of their incomes into contractual payments, and a contract can be structured that is incentive compatible and provides insurance. Thus the problem considered here differs from the principal-agent literature and the moral hazard literature in that only the intertemporal pattern of actions of insured agents matters. The principal could conceivably weight the intertemporal pattern in very complex ways. But when the weighting pattern is restricted to be linear, the weights are stationary and simple in form.[1]

The analysis here has two main parts, proceeding from the particular to the general. The first main part analyzes a lending equilibrium. Section 2 sets out the economy's structure. Section 3 takes advantage of the linear-quadratic structure to analyze the deterministic part of the solution and finds the equilibrium interest rate. Section 4 uses the interest rate to simplify the analysis of individuals' responses to stochastic shocks. Section 5 analyzes the efficiency properties of the equilibrium.

The second main part considers the equilibrium in which a centralized contract maximizes the *ex ante* welfare of a representative individual by allocating consumption using the history of individuals' signals. The contract conditions payments on an unrestricted linear filter of those signals. Section 6 analyzes individual behavior taking the generalized linear filter as given. Section 7 finds the specific welfare-maximizing contract by choosing the specific linear filter, taking the individual reaction functions derived in Section 6 and the economy's resources as constraints, when incomes are restricted to be first-order autoregressive processes. Technically, the persona of the contract is a leader or dominant player finding a closed-loop strategy in the sense of Kydland (1975). That is, the contract is a function that allocates consumption based on the current signal from an individual. The contract is fixed at the beginning of time and there is commitment thereafter. This differs from an open-loop strategy in which all allocations would be made in the initial period with precommitment, or a feedback strategy in which a new contract would be chosen each period in the absence of precommitment. An example of an open-loop strategy would be equal sharing of income in all periods; because income is not observable, this strategy is infeasible. A feedback strategy would include a notion of nondefection, corresponding to an individual rationality constraint in static mechanism theory.[2] The framework thus corresponds to Holmstrom and Myerson's (1983) notion of *ex ante* contracts with full commitment and the corresponding *ex ante* efficiency concept.

Finding the optimal contract is equivalent to solving a benevolent planner's problem as in the optimal growth literature. The typical finding in that literature is that the planner's optimum growth path is identical to the path followed in decentralized competitive equilibrium, and that the competitive equilibrium is therefore efficient. The central result of this article is that this equivalence is found here as well; the lending equilibrium is equivalent

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[1] Holmstrom and Milgrom (1987) find in a repeated moral hazard model with exponential utility and no discounting that linear aggregation of histories of outcomes, without regard for the order of the sequences, is sufficient for optimal sharing (Holmstrom and Milgrom, equation (16)).

[2] My 1989 article explores the effects of such a constraint in a full-information setting, using an extension of the technical framework here. Other investigations of such a constraint in dynamic settings include Thomas and Worrall (1990) and Kehoe and Levine (1990).

to the optimal insurance contract when the contract is constrained to be linear. The welfare properties of the specific lending equilibrium analyzed in Section 5 thus apply to the general signalling contract. The equivalence springs from a notion of noninvertibility that is intuitively described in Section 7.

The derivations and proofs of the equivalence result in Section 7 are presented in Appendix B. Appendix A presents an intuitive description of some of the techniques used in the text and in the proofs.

2. The economy

■ There is a continuum of infinitely-lived individuals in the economy. All have identical additively separable quadratic preferences for the single-consumption good, which is perfectly perishable. Individuals have idiosyncratically risky income, but average income is constant through time.[3] The income realizations are known only to the individuals receiving them. Risk sharing must therefore be incentive compatible, in the sense that announcements are stated so that redistributions are feasible in the aggregate.

In order to prevent Ponzi-like schemes in which individuals borrow to repay old debts and to finance current consumption, some restriction on borrowing must be imposed. If the horizon were finite, requiring nonpositive net debt in the final period would be sufficient to rule out such strategies. In an infinite-horizon setting like the one here, such a requirement does not work since the terminal period never arrives. An alternative would be to impose a bound on the change of indebtedness; because that constraint would eventually be binding when large shocks to income were realized, the technical advantages of the linear-quadratic framework would be vitiated. The technical advantages can be retained, and the restriction on debt approximated, by imposing a penalty, rather than a constraint, on adjustments to debt. The transversality condition then becomes an element of optimal behavior, ruling out Ponzi schemes in the desired way.[4] Although imposed for this technical reason, modelling adjustment costs of debt seems intuitively reasonable: applying for and repaying loans takes time and effort. It turns out that an arbitrarily small adjustment cost is an adequate restriction. The adjustment cost model can be solved and the frictionless model can then be analyzed asymptotically by considering the adjustment cost model as the adjustment cost tends to zero.

Individuals thus solve the following maximization problem in each period t:

$$\max_{c_t, B_t} - E_t \sum_{s=0}^{\infty} \beta^s \{ c_{t+s}^2 + \gamma (B_{t+s} - B_{t+s-1})^2 \} \tag{1}$$

subject to the constraint

$$c_t = y_t - \rho B_{t-1} + (B_t - B_{t-1}) \tag{2}$$

with equality holding in (2) so that there is no free disposal. The notation is as follows: c_t = consumption in period t; y_t = income; B_t = the individual's stock of debt; β = the discount factor; ρ = the constant interest rate; γ = the parameter penalizing adjustment of debt; E_t = expectation conditioned on information available in period t.

The income process is autoregressive-moving average (ARMA):

$$y_t = \bar{y} + A(L)\epsilon_{zt}, \tag{3}$$

where ϵ_{zt} is a zero-mean shock, independently and identically distributed (i.i.d.) and hence covariance-stationary over time, and i.i.d. over individuals indexed by z, with a continuum

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[3] Judd (1985) shows that it is valid to make this assumption.

[4] See Sargent (1987), pp. 199–204.

of individuals. The mean income $\bar{y}$ is nonzero, and $A(L)$ is a polynomial in the lag operator,[5] with no zeroes or poles inside the disk $D(\beta) \equiv \{z:|z| \leq \beta^{1/2}\}$.

A firm is assumed to intermediate debts across individuals. The firm is assumed to have access to a market that yields a rate of return $(1-\beta)/\beta$. In the economy's initial period—the period in which Arrow-Debreu securities would be exchanged were it not for the information asymmetry—the firm sets the interest rate, ρ, that will prevail in all future periods, but takes γ, the cost of adjustment, as given.[6] Free entry into intermediation contracts in the initial period, combined with the assumption that contracts are rigidly enforced, dictate that the firm's profits are zero. This in turn dictates that one of the following conditions hold in an equilibrium contract: (i) discounted average debt is zero across individuals; (ii) if discounted average debt is positive, the interest rate is at least equal to $(1-\beta)/\beta$; (iii) if discounted average debt is negative, the interest rate is no larger than $(1-\beta)/\beta$. Condition (ii) must hold to prevent negative profit if the intermediary is a net lender to the economy. If (iii) were violated, the rate of return to intermediation would be positive and entry would occur.

Two steps are needed to solve the lending model. First, the individual's optimization problem must be solved. Next, this solution is combined with aggregate resource constraints to obtain an equilibrium. This leads to the following definition:

Definition 1. A lending equilibrium is a set of individual consumption and borrowing policy functions, $c(y_t, B_{t-1})$, $B(y_t, B_{t-1})$ that solve (1) and (2), satisfying conditions (i)–(iii).

3. The equilibrium interest rate

■ Because the problem is a linear-quadratic one, the optimum borrowing and consumption behavior will be expressed as linear functions of predicted future values of income as well as of current and past income. Whittle (1983) and Whiteman (1985) show that the objective (1) can be decomposed into two components, one deterministic and one stochastic, and since the behavior in each component does not affect the other, they can be maximized separately. Since the stochastic components have a zero cross-sectional mean, the aggregate demand for borrowing will be determined by the deterministic component alone. The deterministic component of (1)–(3) is

$$\max_{c_t, B_t} -\sum_{s=0}^{\infty} \beta^s \{c_{t+s}^2 + \gamma(B_{t+s} - B_{t+s-1})^2\} \tag{4}$$

subject to

$$c_t = y_t - \rho B_{t-1} + (B_t - B_{t-1}) \tag{5}$$

$$y_t = \bar{y} \tag{6}$$

$$B_{t-1} \quad \text{given.} \tag{7}$$

The first-order condition is

$$c_t - \beta(1+\rho)c_{t+1} + \gamma\{(B_t - B_{t-1}) - \beta(B_{t+1} - B_t)\} = 0. \tag{8}$$

Because the first-order condition is a difference equation with constant coefficients,

[5] When applied to a time-subscripted variable, the lag operator, L, reduces the time subscript by one: $Lx_t = x_{t-1}$. Conversely, L^{-1} is the forward-shift operator: $L^{-1}x_t = x_{t+1}$. A polynomial in the lag operator, $A(L) = \sum_{j=0}^{n} A_j L^j$, when applied to a time-subscripted variable, yields a weighted sum of current and past realizations of that variable.

[6] A deeper theory might leave the adjustment cost under the firm's influence and so connect it to the interest rate. I do not do this here, but in Appendix B I examine how the firm would behave if it could choose a generalized pattern of adjustment penalties.

then after substituting the budget constraint (5) to eliminate consumption, it can be restated as

$$[\beta(1+\rho)^2(1-\beta^{-1}(1+\rho)^{-1}L)(1-(1+\rho)^{-1}L^{-1}) + \gamma(1-L)(1-\beta L^{-1})]B_t = -(1-\beta(1+\rho)L^{-1})y_t. \quad (8a)$$

Following the procedure in Sargent (1987) and factoring the left-hand side of (8a) yields the formulation

$$\lambda_0(1-\lambda_1 L)(1-\lambda_2^{-1}L^{-1})B_t = (1-\beta(1+\rho)L^{-1})y_t \quad (8b)$$

with conjugate roots: $\lambda_1 = \beta^{-1}\lambda_2^{-1}$. Moreover, assuming without loss of generality that $\lambda_1 < \lambda_2$, then the roots are bracketed by $\lambda_1 < 1 < \beta^{-1} < \lambda_2$. The behavior of the solution depends on the magnitudes of the roots of the system, λ_1 and λ_2, and these depend in turn on the interest rate, ρ.

The general solution of the difference equation is

$$B_{t+1} = \lambda_1 B_t - \lambda_1\lambda_0^{-1}(1-\lambda_2^{-1}L^{-1})^{-1}(1-\beta(1+\rho)L^{-1})y_{t+1} + c_1\lambda_1^t + c_2\lambda_2^t, \quad (9)$$

where λ_1 is presumed to be the stable root, and c_1 and c_2 are constants.

Proposition 1. The equilibrium interest rate is $\rho = (1-\beta)/\beta$.

Proof. See Appendix B.

Corollary 1. Average consumption equals average income each period, that is, $\bar{c}_t = \bar{y}_t$.

Proof. Substitute $B_t = 0$ in the budget constraint (5). Q.E.D.

The corollary shows that the intermediary's role is simply to redistribute income each period.

4. The stochastic component of equilibrium behavior

■ With the interest rate fixed by the deterministic component of the equilibrium, individuals will respond to the stochastic component of their income by increasing or decreasing their debts in order to smooth consumption. Because of the linear-quadratic framework, the stochastic component of debt, $\tilde{B}_t$, will be a linear function of the innovations: $\tilde{B}_t = B(L)\epsilon_t$. The deterministic component of debt analyzed in the previous section made use of the prediction of the future course of income, as expressed in the discounted income term of (9). By definition the course of the stochastic part of income cannot be predicted, and so optimal behavior involves prediction of income as well as its intertemporal redistribution via borrowing, and this makes characterizing $B(\cdot)$ different from the exercise of the previous section. In particular the filter $B(L)$ must be nonanticipative; that is, it can act only on current and past stochastic innovations, ϵ_t.

Substituting $B(L)\epsilon_t$ into the constraint (2) and the constraint into the objective (1), the objective can then be z-transformed, as detailed in Whiteman (1985). (The reader unfamiliar with time series and z-transform concepts should read Appendix A before proceeding further.) Assume at this point that all innovations prior to the initial period are normalized to zero: $0 = \epsilon_0 = \epsilon_{-1} = \ldots$. This assumption is useful later on in the welfare analysis, since it renders all individuals *ex ante* identical. The stochastic component of the objective is

$$\max -\frac{\beta\sigma_\epsilon^2}{1-\beta}\frac{1}{2\pi i}\oint\{(A(z)+(1-(1+\rho)zB(z))(A(\beta z^{-1}) + (1-(1+\rho)\beta z^{-1})B(\beta z^{-1})) + \gamma(1-z)(1-\beta z^{-1})B(z)B(\beta z^{-1})\}\frac{dz}{z}, \quad (10)$$

where z is an element of the complex plane and the integration is around the unit circle. The integrand has two parts. The first part arises from substituting the budget constraint (2) into (1) and expressing the choice of borrowing as a function of the innovations: $B_t = B(L)\epsilon_t$. The second part comes from writing the adjustment cost $B_t - B_{t-1}$ as $(1 - L)B_t$. Since the argument of the maximand is a function, Whiteman observed that a variational method can be used, resulting in an Euler equation which is a Wiener-Hopf equation.

The first step in the solution procedure is to express the policy function, $B(z)$, as the sum of the (as-yet unknown) optimal filter, $\tilde{B}(z)$, and a "variation," $\alpha h(z)$, where α is a real number and $h(z)$ is feasible. To be candidates for feasible filters, both $\tilde{B}(z)$ and $h(z)$ must contain only nonnegative powers of z—that is, they must be analytic—to reflect the fact that only current and past stochastic innovations, ϵ_t, can be observed and filtered. Substituting the sum $\tilde{B}(z) + \alpha h(z)$ into the objective (10), taking the derivative with respect to α, and setting both the derivative and α to zero as in a calculus of variations problem yields an Euler condition.[7] The Euler condition can be expressed as a Wiener-Hopf equation:

$$(1 - (1 + \rho)\beta z^{-1})A(z) + (1 - (1 + \rho)z)(1 - (1 + \rho)\beta z^{-1})B(z) + \gamma(1 - z)(1 - \beta z^{-1})B(z) = \sum_{-\infty}^{-1}, \quad (11)$$

where $\sum_{-\infty}^{-1}$ denotes an arbitrary function with only negative powers of z in its Laurent expansion inside $D(\beta)$. Equation (11) can be restated as follows:

$$C(z)C(\beta z^{-1})B(z) = \sum_{-\infty}^{-1} -(1 - (1 + \rho)\beta z^{-1})A(z), \quad (12)$$

where $C(z)$ is defined by the equation

$$C(z)C(\beta z^{-1}) = (1 - (1 + \rho)z)(1 - (1 + \rho)\beta z^{-1}) + \gamma(1 - z)(1 - \beta z^{-1}). \quad (13)$$

Hansen and Sargent's (1981, p. 153) extension of Rozanov's (1967, p. 47, Th. 10.1) factorization theorem states the following: there exists a $C(z)$ that satisfies the factorization equation (13) and that is both analytic and has no zeroes inside the disk $D(\beta)$. The no-zeroes condition makes $C(z)$ invertible inside the disk; that is, $C(z)^{-1}$ will also have only nonnegative powers of z. A secondary consequence of this is that $C(\beta z^{-1})^{-1}$ will have nonpositive powers of z. Multiplying both sides of (12) by $C(\beta z^{-1})^{-1}$ therefore leaves the left-hand side with only nonnegative powers of z, and the identity $C(\beta z^{-1})^{-1} \sum_{-\infty}^{-1} = \sum_{-\infty}^{-1}$ on the right-hand side because products of negative and nonpositive powers of z are negative.

[7] The Euler condition is an integral equation in the optimal policy filter for borrowing, $\tilde{B}(z)$. The equation can be simplified using β-symmetry: for arbitrary functions $f(z)$ and $g(z)$,

$$\frac{1}{2\pi i}\oint f(z)g(\beta z^{-1})\frac{dz}{z} = \frac{1}{2\pi i}\oint f(\beta z^{-1})g(z)\frac{dz}{z}.$$

The integral equation has the form

$$\frac{1}{2\pi i}\oint a(z)h(\beta z^{-1})\frac{dz}{z} = 0,$$

where $h(z)$ is the analytic variation described in the text and $a(z)$ is the left-hand side of equation (11). A necessary condition for this equality to hold for every analytic $h(z)$ is that $a(z)$ have only negative powers of z.

The annihilator operator, $[\cdot]_+$, sets to zero all linear terms with negative powers of z; thus $[\sum_{-\infty}^{-1}]_+ = 0$. Applying the annihilator to both sides of (12) thus leaves the left-hand side unaffected and eliminates the $\sum_{-\infty}^{-1}$ term; multiplying by $C(z)^{-1}$ yields the solution

$$B(z) = -C(z)^{-1}[C(\beta z^{-1})^{-1}(1 - (1 + \rho)\beta z^{-1})A(z)]_+. \tag{14}$$

An annihilate is by definition analytic, and because $C(z)^{-1}$ is analytic, and products of analytic functions are analytic, $B(z)$ is therefore analytic on the disk $D(\beta)$, which is all that is needed for (10) to be defined. The analyticity of $B(z)$ implies stationarity of $\tilde{B}_t$, and thus for any interest rate ρ, the stochastic component of debt will follow a stationary process. Indeed, this will follow even if there are no adjustment costs. Since the innovations are i.i.d. across individuals, the average stochastic component of debt across individuals is zero.

Now use the finding that the equilibrium interest rate equals the internal rate of discount: $\rho = (1 - \beta)/\beta$. The factorization in (13) can now be carried out directly, yielding

$$C(z) = (\beta^{-1} + \gamma)^{1/2}(1 - z). \tag{15}$$

Substituting into (14) yields

$$B(z) = -(\beta^{-1} + \gamma)^{-1}(1 - z)^{-1}[(1 - \beta z^{-1})^{-1}(1 - z^{-1})A(z)]_+. \tag{16}$$

The annihilate is just the annihiland less its principal part,[8] so

$$[(1 - \beta z^{-1})^{-1}(1 - z^{-1})A(z)]_+ = (z - \beta)^{-1}[zA(z) - \beta A(\beta) - (A(z) - A(\beta))],$$

and

$$\begin{aligned} B(z) &= -(\beta^{-1} + \gamma)^{-1}(z - \beta)^{-1}(1 - z)^{-1}[(z - 1)A(z) - (\beta - 1)A(\beta)] \\ &= (\beta^{-1} + \gamma)^{-1}(z - \beta)^{-1}[A(z) - (1 - z)^{-1}(1 - \beta)A(\beta)]. \end{aligned} \tag{17}$$

□ **The consumption function.** Substituting the solution of $B(z)$, (17), into the z-transform of the consumption constraint

$$c(z) = A(z) + (1 - (1 + \rho)z)B(z) \tag{18}$$

yields the stochastic part of the consumption function,

$$\begin{aligned} c(z) &= A(z) + (1 - \beta^{-1}z)B(z) \\ &= \beta\gamma(1 + \beta\gamma)^{-1}A(z) + (1 - \beta)A(\beta)(1 + \beta\gamma)^{-1}(1 - z)^{-1}. \end{aligned} \tag{19}$$

Converting to time-domain notation yields an expression for the first difference of consumption:

$$c_t = \beta\gamma(1 + \beta\gamma)^{-1}y_t + (1 - \beta)A(\beta)(1 + \beta\gamma)^{-1}(1 - L)^{-1}\epsilon_t,$$

which says that the stochastic part of consumption has two parts: the first part is a constant fraction of current income, and the second part is a random walk process, just as Green found. Moreover, it is expressed in a framework more amenable to econometric interpretation because the support of the innovation process is a continuum.

[8] The principal part of a function is the part of the Laurent expansion with negative powers of z. If P is the principal part of an arbitrary function $f(z)$, then $[f(z)]_+ = f(z) - P$. The annihiland in (16) can be written $(z - \beta)^{-1}(z - 1)A(z)$, which has a pole at β. The principal part of the annihiland is therefore

$$\lim_{z \to \beta} (z - 1)A(z) = (\beta - 1)A(\beta).$$

As the adjustment cost penalty tends to infinity, consumption tends toward current income, just as it would be expected to if individuals were completely liquidity-constrained. As the adjustment cost falls to zero, the consumption function reduces even further to

$$c_t = (1 - \beta)A(\beta)(1 - L)^{-1}\epsilon_t, \tag{20}$$

that is, consumption is a pure random walk, the behavior predicted by the permanent income hypothesis.[9]

5. Efficiency

■ Since income innovations before the initial period are normalized to zero for all individuals, all individuals are *ex ante* identical, and welfare is simply the discounted utility of the representative individual. Consider welfare as the adjustment cost parameter, γ, tends toward zero. The deterministic side of the equilibrium can be ignored since it is unaffected by the magnitude of γ, as long as it is positive. Substituting (19) into (10), the stochastic component of discounted utility is

$$-\frac{\beta^2\sigma_\epsilon^2}{1-\beta}\frac{1}{2\pi i}\oint (1-\beta)^2 A(\beta)^2(1-z)^{-1}(1-\beta z^{-1})^{-1}\frac{dz}{z} = -\beta\sigma_\epsilon^2 A(\beta)^2, \tag{21}$$

which is not zero as it would be if complete insurance were available to individuals. Evidently the absence of transaction costs is insufficient to attain complete insurance via a lending contract. Because of the incentive compatibility problem, there is necessarily a lag between the acquisition and discharge of debt, and less debt is incurred than would be needed to attain complete insurance.

The finding that lending markets can provide only incomplete insurance can be better understood by examining the stochastic component of the consumption that would result if the equilibrium interest rate were zero. It would then be the case that

$$C(z) = (1 + \gamma)^{1/2}(1 - z),$$

and so

$$B(z) = -(1 + \gamma)^{1/2}(1 - z)A(z). \tag{22}$$

Substituting in the consumption function yields

$$c(z) = \gamma(1 + \gamma)^{-1}A(z),$$

which is zero when the adjustment cost, γ, is zero. In other words, if individuals could costlessly rearrange current and future stochastic income, consumption would not fluctuate at all; borrowing would provide complete insurance. It was seen in Proposition 1, though, that a zero interest rate is not an equilibrium. At the equilibrium interest rate $(1 - \beta)/\beta$, individual consumption responds "too much" to income fluctuations. The technical reason for this will be explored in Section 7.

□ **The effect of serial correlation on efficiency.** Implicit in the penalization of debt adjustment is the observability of the adjustment by the intermediary, or at least the knowledge of the presence of adjustment costs, $\gamma > 0$. This results in an analytic debt adjustment function, $(1 - z)B(z)$, and therefore a stationary debt adjustment process. If the income innovations are white noise—that is, $A(z) = 1$—as in Green's model, the debt adjustment process is invertible, allowing the recovery of the individual income innovations. Invertibility

[9] A random walk is a random variable x_t such that $x_t = x_{t-1} + e_t$, where e_t is a serially uncorrelated shock. In lag operator notation, $(1 - L)x_t = e_t$. Thus (20) can be written $c_t = c_{t-1} + (1 - \beta)A(\beta)\epsilon_t$.

holds for serially correlated income processes as well, as long as they are not too serially correlated. This condition is not always met: in particular, if the income process is a random walk, i.e., $A(z) = (1 - z)^{-1}$, then $(1 - z)B(z)$ is identically zero. That is, from (16),

$$[(1 - \beta z^{-1})^{-1}(1 - z^{-1})A(z)]_+ = [z^{-1}(1 - \beta z^{-1})^{-1}]_+ = 0,$$

and therefore $(1 - z)B(z) = 0$. No lending will occur, which clearly is inefficient.

This can be generalized by comparing welfare under autarky and in a lending equilibrium when income is a first-order autoregressive process, that is,

$$A(z) = (1 - \alpha z)^{-1}, \qquad |\alpha| < \beta^{-1/2}.$$

In that case the stochastic part of discounted utility under autarky is

$$-\frac{\beta^2\sigma_\epsilon^2}{1-\beta}\frac{1}{2\pi i}\oint (1 - \alpha z)^{-1}(1 - \beta\alpha z^{-1})^{-1}\frac{dz}{z} = -\frac{\beta^2\sigma_\epsilon^2}{1-\beta}(1 - \beta\alpha^2)^{-1},$$

in the limit as the adjustment cost, γ, tends to zero. Substituting into (21) yields utility under the debt equilibrium:

$$-\beta\sigma_\epsilon^2 A(\beta)^2 = -\beta\sigma_\epsilon^2(1 - \beta\alpha)^{-2}. \tag{23}$$

Subtracting these two expressions yields the value of the insurance,

$$\frac{\beta^2\sigma_\epsilon^2}{1-\beta}(1 - \alpha^2)(1 - \beta\alpha^2)^{-1}(1 - \beta\alpha)^{-2}. \tag{24}$$

The limiting case of maximum serial correlation occurs as $\alpha \to 1$, that is, as the stochastic process of income tends toward a random walk. As α approaches one, the value of the insurance diminishes to zero. The intuition is that current innovations affect future income as well as current income, and so an individual's borrowing power will consequently fall if he suffers a bad realization. This persistent effect reaches its extreme when income is a random walk; wealth and income move in unison and no borrowing is possible. This is not surprising since the effect of the lending equilibrium is to generate a random walk of consumption. By definition any attempt to insure the random walk further would meet with failure.

□ **Efficiency with no discounting.** The finding that perfection results in principal-agent problems when there is no discounting, as in Rubinstein and Yaari (1983) and Radner (1985) can be extended to this setting. Perfection in the sense of eliminating all risk is impossible here, but it is immediate from (24) that the insurance value of the lending equilibrium becomes infinite as the discount factor tends to unity. There are two opposing effects. The first is that the residual risk from the uninsured random walk is discounted less as β approaches unity, reducing utility. The second effect results from the insurance. Under autarky, the disutility of uninsured risks tends to infinity as β tends to unity. Thus the fact that discounted risk remains finite in the lending equilibrium makes the value of the insurance infinite.

6. Signalling to an insurance contract

■ Now consider the equilibrium in which an insurance contract is chosen in order to maximize *ex ante* welfare in the initial period of the economy, and is perfectly enforced thereafter. The contract attempts to insure the heterogeneous shocks individuals face, but cannot observe the shocks directly. Instead, it observes a signal, B_t, transmitted by individuals. The contract responds to the signal in two ways: first, it transfers or removes income to individuals by linearly filtering the current and past values of the signal; second, it penalizes the current and past values of the signal in order to maintain aggregate feasibility.[10] The aggregate feasibility condition is that economywide average consumption equal average income in each period.

The individual's problem. Individuals solve the following problem:

$$\max_{c_t, B_t} - E_t \sum_{s=0}^{\infty} \beta^s \{ c_{t+s}^2 + \gamma(\phi(L)B_{t+s})^2 \} \tag{25}$$

subject to

$$c_t = y_t + \delta(L)B_t. \tag{26}$$

$$B_{t-1}, B_{t-2}, \ldots \qquad \text{given.} \tag{27}$$

$$\delta(0) = 1, \qquad \phi(0) = 1, \tag{28}$$

where $\delta(L)$ is the filter on the signal and $\phi(L)$ generates disutility of the signal. The restrictions (28) are a normalization. In the previous section, the restrictions $\delta(L) = 1 - (1 + \rho)L$, and $\phi(L) = 1 - L$ were imposed *a priori*. By leaving the forms of $\delta(\cdot)$ and $\phi(\cdot)$ unrestricted it is possible to discover whether an insurance contract that dominates the lending equilibrium exists. The techniques used to analyze the lending equilibrium can be used here as well.

Deterministic part of the individual's problem. Substituting (26) into (25) yields the problem

$$\max_{B_t} E_t \sum_{s=0}^{\infty} \beta^s \{ (y_t + \delta(L)B_t)^2 + \gamma(\phi(L)B_{t+s})^2 \}$$

with first-order condition

$$0 = \delta_0(y_t + \delta(L)B_t) + \beta\delta_1(y_{t+1} + \delta(L)B_{t+1}) + \beta^2\delta_2(y_{t+2} + \delta(L)B_{t+2}) + \ldots + \gamma(\phi_0\phi(L)B_t + \beta\phi_1\phi(L)B_{t+1} + \beta^2\phi_2\phi(L)B_{t+2} + \ldots).$$

Collecting terms yields

$$\begin{aligned} 0 &= \sum_{s=0}^{\infty} \delta_s \beta^s L^{-s}(y_t + \delta(L)B_t) + \gamma(\sum_{s=0}^{\infty} \phi_s \beta^s L^{-s} \phi(L)B_t) \\ &= \delta(\beta L^{-1})\delta(L)B_t + \delta(\beta L^{-1})y_t + \gamma\phi(\beta L^{-1})\phi(L)B_t. \end{aligned}$$

This can be arranged as

$$(\delta\delta_* + \gamma\phi\phi_*)B_t = -\delta_* y_t,$$

using the shorthand notation $f(z) = f$, $f(\beta z^{-1}) = f_*$. This resembles the stochastic first-order condition (11), except that on the right-hand side there is an exogenous process that goes off into the future, and it is not possible to apply the annihilator to it since it is deterministic. The left-hand side can be factored just as in the stochastic case; define $\kappa(\cdot)$ by

$$\kappa\kappa_* \equiv \delta\delta_* + \gamma\phi\phi_*, \tag{29}$$

choosing $\kappa(\cdot)$ to have no poles or zeroes in $D(\beta)$. The solution is then

$$\kappa(L)B_t = -\kappa_*^{-1}\delta_* y_t,$$

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[10] It is not evident *a priori* that the optimal filter is linear. Varian (1980) finds nonlinear taxation to be optimal in a linear-quadratic setting. My conjecture is that linear filtering is in fact optimal here, but I have not as yet been able to prove this conjecture.

recalling that $B_{t-1}, B_{t-2}, \ldots$ are given. It is possible to express the solution as

$$B_t = -\kappa_0^{-1}\left[\frac{\kappa(L)}{L}\right]_+ B_{t-1} - \kappa_0^{-1}\kappa_*^{-1}\delta_* y_t, \tag{30}$$

that is, as convergent sums of past and future exogenous variables.[11]

□ **Stochastic part of individual's problem.** The solution of the stochastic part of the individual's problem follows from a straightforward application of Whiteman's methods. The stochastic part of the objective is

$$-\frac{\beta\sigma_\epsilon^2}{1-\beta}\frac{1}{2\pi i}\oint\Big\{(A(z)+\delta(z)B(z))(A(\beta z^{-1})+\delta(\beta z^{-1})B(\beta z^{-1}))$$
$$+\gamma\phi(z)B(z)\phi(\beta z^{-1})B(\beta z^{-1})\frac{\delta z}{z}\Big\}. \tag{31}$$

The Wiener-Hopf equation is

$$\delta(\beta z^{-1})A(z)+\delta(z)\delta(\beta z^{-1})+\gamma\phi(z)\phi(\beta z^{-1}) = \sum_{-\infty}^{-1},$$

or

$$(\delta\delta_* + \phi\phi_*)B = \sum_{-\infty}^{-1} - \delta_* A.$$

Factoring as in the deterministic case, the solution is

$$B(z) = -\kappa(z)^{-1}[\kappa(\beta z^{-1})^{-1}\delta(\beta z^{-1})A(z)]_+. \tag{32}$$

The deterministic and stochastic solutions have separate roles in the equilibrium insurance contract. The deterministic part of the solution determines feasibility, the stochastic part optimality.

7. The optimal contract

■ Since there is no production, the contract must choose the filters, $\delta(L)$ and $\phi(L)$, so as to leave average consumption equal to endowment:

$$\bar{c}_t = \bar{y}_t.$$

The stochastic part of consumption averages to zero across individuals, and so average consumption is the same as the deterministic part of consumption. Using (30), the feasibility condition is equivalent to

$$c_t = y_t + \delta(L)\left(\kappa_0^{-1}\left[\frac{\kappa(L)}{L}\right]_+ B_{t-1} - \kappa_0^{-1}\kappa_*^{-1}\delta_* y_t\right) = y_t,$$

or

$$\delta(L)\left(\kappa_0^{-1}\left[\frac{\kappa(L)}{L}\right]_+ B_{t-1} - \kappa_0^{-1}\kappa_*^{-1} y_t\right) = 0.$$

[11] See Sargent (1987) pp. 199–204.

Requiring that $\delta(L) = 0$ would leave an obviously inefficient autarkic contract in which no insurance was provided. The feasibility condition satisfies

$$\delta_* \bar{y}_t = \kappa_* \left[\frac{\kappa(L)}{L} \right]_+ \bar{B}_{t-1}. \tag{33}$$

The signal, B_t, need not satisfy some condition such as zero average debt at this point. However, notice also that from (30), (33) is equivalent to requiring that the deterministic part of the current signal, B_t, be zero. In a stationary equilibrium, therefore, the average signal will be zero. Imposing such a condition, and in addition assuming that deterministic income is constant, i.e.,

$$B_{t-1} = 0, \qquad B_{t-2} = 0, \ldots \qquad y_t \equiv \bar{y},$$

then (33) reduces to

$$\delta(\beta) = 0.$$

Recall that in the lending equilibrium the budget constraint was equivalent to setting $\delta(z) = 1 - (1 + \rho)z$. In the lending equilibrium $\rho = (1 - \beta)/\beta$, and so

$$\delta(\beta) = 1 - (1 + \rho)\beta = 0,$$

which indeed satisfies the feasibility constraint.

The feasibility condition (33) can be reinterpreted as follows. A risk-neutral principal with access to a capital market or technology with a rate of return equal to the internal rate of discount would require that the filtering of the deterministic signal, as on the right-hand side of (33), equal a weighted discounted sum of current and future deterministic income, as on the left-hand side. Since the contract is chosen to maximize the welfare of a representative individual, the analysis could be reinterpreted as a more familiar principal-agent problem. The approach here is to solve the primal problem rather than the dual problem solved by Spear and Srivastava and by Green.

□ **Deterministic component of reduced-form objective.** Now substitute the policies and constraints into the individual's objective to obtain the *ex ante* welfare function. The feasibility condition makes the deterministic part of the objective easy to find. The deterministic part is simply

$$-\sum_{s=0}^{\infty} \beta^s \{ y_t^2 + \gamma(\phi(L) B_{t+s})^2 \},$$

where B_{t+s} is determined by (30). It is now apparent why having zero aggregate debt matters. If it is not zero, a penalty is paid, despite no information being transmitted, since the deterministic part of debt is predictable. Nevertheless, the structure will be left as is in order to see if there is a tradeoff between eliciting information about the stochastic state and the average signal, B_t.

□ **Stochastic component of reduced-form objective.** I demonstrate in Appendix B that the stochastic component of the objective of the contract is

$$-\frac{\beta \sigma_\epsilon^2}{1 - \beta} \frac{1}{2\pi i} \oint \{ A A_* - \delta \kappa^{-1} A_* [\delta_* \kappa_*^{-1} A]_+ \} \frac{dz}{z}. \tag{34}$$

The reduced-form objective of the contract is therefore

$$-\sum_{s=0}^{\infty} \beta^s \{ y_t^2 + \gamma(\phi(L)B_{t+s})^2 \} - \frac{\beta\sigma_\epsilon^2}{1-\beta}\frac{1}{2\pi i}\oint \{ AA_* - \delta\kappa^{-1}A_*[\delta_*\kappa_*^{-1}A]_+ \} \frac{dz}{z},$$

subject to (30) and (33). The expression can be interpreted as the difference between autarkic consumption and consumption that cannot be smoothed by predicting future income with the filtered signal.

At this point recall that feasibility of the contract requires that $\bar{B}_{t+s} = 0$, $s = 0, 1, \ldots,$ regardless of initial conditions. Therefore the stationary level of the average signal or debt will be zero, and the calculations will be profoundly simplified by considering only the stationary objective; this has already been done with the stochastic part. The objective then simplifies to the expression

$$-\sum_{s=0}^{\infty} \beta^s y_t^2 - \frac{\beta\sigma_\epsilon^2}{1-\beta}\frac{1}{2\pi i}\oint \{ AA_* - \delta\kappa^{-1}A_*[\delta_*\kappa_*^{-1}A]_+ \} \frac{dz}{z}.$$

The objective of the contract is to maximize this subject to the constraint

$$\delta(\beta L^{-1})\bar{y}_t = 0.$$

If y_t is not a constant, then the contract chosen at time t would not necessarily be ratified at time $t + 1, t + 2, \ldots,$ due to the changes in this constraint; there would be a vector of objectives indexed by the state of average income, and separate $\delta(\cdot)$ and $\phi(\cdot)$ functions indexed by the y-state would be chosen. The contract's objective in the constant-income case is therefore

$$-\frac{\beta\sigma_\epsilon^2}{1-\beta}\frac{1}{2\pi i}\oint \delta\kappa^{-1}A_*[\delta_*\kappa_*^{-1}A]_+ \frac{dz}{z} \tag{35}$$

subject to

$$\delta(\beta) = 0, \tag{36}$$

and $\delta(0) = 1$, $\phi(0) = 1$.

□ **Incompleteness of insurance under the optimal contract.** From (36), $\delta(\cdot)$ is not invertible, so it is necessary to define $\hat{\delta}(\cdot)$ as the invertible representation of $\delta(\cdot)$. Then from Lemma B1 in Appendix B, which states that $\delta\delta_* = \phi\phi_*$,

$$\kappa(z) = (1+\gamma)^{1/2}\hat{\delta}(z).$$

Now examine the stochastic component of the optimal consumption policy, (32). Substituting for $\kappa(z)$,

$$c_t = y_t - \delta(L)(1+\gamma)^{-1/2}\hat{\delta}(L)^{-1}y_t = (1 - (1+\gamma)^{-1}\delta(L)\hat{\delta}(L)^{-1})y_t. \tag{37}$$

It is now possible to ask whether consumption fluctuations tend to zero as the adjustment cost parameter tends to zero. The following proposition states that they do not.

Proposition 2. Complete insurance is unattainable under the contract.

Proof. The proof is presented in Appendix B. The intuition of the proof is that because $\delta(\beta) = 0$ is a necessary condition for feasibility, and since $\beta < \beta^{1/2}$, $\delta(z)$ is not invertible on $D(\beta)$.

The reason a noninvertible filter results in an incentive-compatible (or equivalently, feasible) equilibrium can perhaps be better understood with a finite example. Suppose there are T goods, and individuals have additive utility over consumption of those goods. The distribution of goods across individuals is random and independent. Exchange possibilities therefore exist. If an individual's final allocation depends on his reported endowments, an incentive mechanism must be imposed. Suppose the allocation vector an individual receives, c, depends on a vector of reports, B, and is transformed into a consumption allocation by a matrix δ:

$$c = \delta B.$$

Then it is necessary that δ not be invertible. If it were, individuals could choose any consumption bundle, c^*, by choosing the appropriate report vector:

$$B = \delta^{-1} c^*.$$

This is what an interest rate of zero allowed in Section 5; $\delta(z) = (1 - z)$ is invertible on $D(\beta)$. In the aggregate there is some feasibility constraint, perhaps expressible as a vector constraint:

$$\Gamma c = \bar{y}.$$

Thus it must be the case that

$$\Gamma \delta B = \bar{y}.$$

For example, if $T = 3$ and if each individual's goods follow the constraint $\sum y_i = \bar{y}$, then $\Gamma = [1, 1, 1]$. The three-dimensional space of consumption vectors is thus restricted to a plane by Γ. This in turn requires δ to have rank two, that is, to reduce the space of signals to a two-dimensional space (a plane) of consumption vectors. Of course in the infinite-dimensional analogue, the goods vector is indexed by time, and its discounted value must sum to the average. Thus

$$\Gamma = [1, \beta, \beta^2, \ldots], \qquad \delta = [1, \beta^{-1}, \beta^{-2}, \ldots].$$

In the appropriate sense, δ is still not invertible.

□ **Equivalence to lending: an AR(1) example.** It is hard to solve the general case of (35). To approach it, consider an example in which income follows a first-order autoregressive process, and apply Lemma C.3 in Taub (1986).[12] Thus for $A(z) = (1 - \alpha z)^{-1}$, (35) becomes

$$\max \delta(\beta\alpha)^2 \kappa(\beta\alpha)^{-2} A(\beta\alpha). \tag{38}$$

The optimum $\delta(\cdot)$ and $\phi(\cdot)$ can be found in two steps. The first step is to find the form of $\phi(\cdot)$ in relation to $\delta(\cdot)$, holding $\delta(\cdot)$ fixed, and the second is to find the optimum $\delta(\cdot)$.

Holding $\delta(\cdot)$ fixed, solving (38) is equivalent to minimizing $\kappa(\beta\alpha)^2$, subject to $\phi(0) = 1$. This latter condition is needed to reflect that the average level of the penalty is γ, and is not a choice variable of the contract due to the fact that there is no feasible equilibrium when γ is zero.

Proposition 3. When income is first-order autoregressive, the optimal signalling contract is equivalent to the debt contract, that is,

$$\delta(z) = 1 - \beta^{-1} z, \qquad \phi(z) = 1 - z$$

solves (38).

[12] Since analytic functions can be approximated by rational functions (see Conway, 1978, p. 195), and they in turn can be represented in partial fraction form, the example might serve as a foundation for a more general attack.

The proof uses functional analysis methods and is given in Appendix B. The intuition of the proof is as follows. The optimal filter must have a zero at β because of the feasibility constraint (36). The simplest function that satisfies this condition is $\delta(z) = 1 - \beta^{-1}z$. The filter could be made more complicated by multiplying it by analytic factors with or without zeroes; poles are forbidden because they would make the filter forward looking. Factors without zeroes could be defeated by individuals multiplying their reaction functions by the inverse factors, which are also analytic, and this is not incentive compatible. Adding factors with zeroes restricts consumption unnecessarily and is suboptimal; in the finite-T example, this would be like restricting δ to be of rank one, mapping consumption into a line instead of a plane.

The equivalence of behavior in the lending and contractual equilibria, combined with the fact that an incentive constraint prevents complete insurance in the contract, means that the lending equilibrium is efficient when the information asymmetry is recognized as a constraint. In expression (22) of the analysis of the lending equilibrium, the invertible factorization of $\delta(\cdot)$, $\hat{\delta}(\cdot)$, would realize perfect consumption smoothing, but would not be an equilibrium. A noninvertible filter was required to satisfy the equilibrium conditions, and this noninvertibility of $\delta(\cdot)$ resulted in a nontrivial annihilate. The only difference here is the interpretation: feasibility and incentive compatibility are equivalent.

8. Conclusion

■ The equivalence of debt and insurance is connected to the need to penalize individuals for failing to communicate truthfully via their signals. If individuals are endowed with reputations, then it must be that these reputations are eroded by continual reports of bad realizations, as in Rubinstein and Yaari (1983).[13] Because reputations are like stocks of capital, they can be built up only slowly. Individuals will insure against true bad realizations by maintaining their "stock" of reputation at a high level. This stock behaves no differently from a stock of wealth.

Because the costs and benefits of altering the stock of the signal appear in the future and hence are discounted, individuals would not match current responses of the contract to reports of deterministic income with future costs and benefits if they could exchange them one-for-one. It is the contract's need to overreact to the resulting tendency toward exaggeration (by penalizing adjustments to debt and charging interest) that results in the imperfection.

Wherever asymmetric information prevents insurance of true hazards, the expression of the constraint in lending markets will therefore be relevant. An example is predation by firms to induce exit or deter entry. If lenders cannot fully measure the long-run profitability of a firm suffering from a predatory attack, the firm will be unable to borrow enough to weather the attack. This would tip the theoretical scales toward an expectation of industrial concentration as a persistent outcome.[14]

Stochastic shocks permeate the world, and many are idiosyncratic. The hunger for insurance will impel individuals to seek close and enduring relationships—in families and firms—that promote observability and hence complete insurance. The problem with close and enduring relationships is that they must be exclusive, and this opens the Pandora's box of inefficient strategic behavior. The avoidance of such inefficiency will leave residual unobservable shocks that spill over into lending markets. There will be a margin at which individuals will be indifferent between strengthening their close and enduring relationships and seeking insurance in lending markets. This margin, particularly its influence on the serial correlation of residual income, seems an object worthy of further study.

[13] Rubinstein and Yaari find an efficient outcome, but there is no discounting in their framework.

[14] Fudenberg and Tirole (1986, pp. 373–375) note this and discuss the literature.

Appendix A

■ Below is an intuitive description of z-transform techniques.

Consider a serially correlated stochastic process a_t that can be expressed as weighted sum of i.i.d. innovations:

$$a_t = \sum_{k=0}^{\infty} A_k \epsilon_{t-k}.$$

If it is stationary the innovations change through time, but the weights, A_k, remain fixed. The stochastic process can therefore be written succinctly as a function of the lag operator, L:$a_t = A(L)\epsilon_t$, with the serial correlation properties fully captured by $A(L)$. Functions of linear operators such as the lag operator correspond to functions of a complex variable. Thus, the properties of $A(L)$ are fully captured by the corresponding function of a complex variable, $A(z)$. The properties of such functions depend on their domain; the disk $D(\beta)$ discussed in the text is topologically equivalent to the closed-unit disk.

Functions can be *analytic* on the disk. This means they can be expressed as a Taylor expansion in the usual sense: an infinite series with coefficients, and with the powers of the independent variable ranging from zero to infinity.

Functions can also be nonanalytic. Such functions have *poles.* Poles result in negative powers of z, the independent variable, in the generalized Taylor expansion, known as the Laurent expansion. The sum of the negative powers is the *principal part.* A function with poles corresponds to a function of negative powers of the lag operator, which in turn means it operates on future values of a variable. If a variable is deterministic, this is permissible. If it is stochastic, it is not permissible, as it means the future is predictable, contradicting its stochastic aspect. Part of the solution process of the optimization problems is the elimination of poles. In engineering parlance, the solution cannot be forward looking.

A nonanalytic function can be converted into an analytic function by lopping off its poles. The operator that does this is the *annihilator,* $[\cdot]_+$. A function with both backward- and forward-looking parts is converted to one with only backward-looking parts by the application of the annihilator.

A second property of a function are its *zeroes.* An analytic function that takes on a value of zero at a point (a zero) inside the domain is said to be *noninvertible* on the domain. The inverse of such a function is not analytic. Conversely, a function that has no zeroes is invertible. This means that if a serially correlated stochastic process is representable by an invertible operator, the innovations can be completely and exactly recovered by observing the history of the process. That is, the inverse of the operator applied to the vector of realizations of the process yields the vector of innovations, exactly as it would if a finite vector of innovations were converted into a finite vector of realizations by an invertible matrix. Conversely, this recovery of innovations is not possible if there is a zero, because inverting a function with a zero results in a function with a pole, and hence is forward looking. Recovery of the innovations then depends on future knowledge of the observed process.

Any process described by a z-transform with either poles or zeroes can be converted into an equivalent one with neither poles (backward looking) nor zeroes (invertible) by the *factorization* theorem of Rozanov. The equivalent stochastic process will have innovations that have a higher variance.

The stochastic component of a quadratic utility function is essentially a conditional variance. If the innovations are i.i.d., the expectation of cross-products of random variables yields the sum of variances. For white noise innovations $\{\epsilon_t\}$,

$$E_{t-k}\epsilon_{t-r}\epsilon_{t-s} = \begin{cases} 0, & r \neq s \\ \sigma_\epsilon^2, & r = s \end{cases}$$

because of the independence of the innovations, where $k > s$, $k > r$. Expressed in lag operator notation, this is

$$E_{t-k}(L^r\epsilon_t)(L^s\epsilon_t) = \begin{cases} 0, & r \neq s \\ \sigma_\epsilon^2, & r = s \end{cases}.$$

Notice that the "action" is now in the exponents of the lag operators. It is equivalent to write

$$\sigma_\epsilon^2 \frac{1}{2\pi i} \oint z^r z^{-s} \frac{dz}{z} = \begin{cases} 0, & r \neq s \\ \sigma_\epsilon^2, & r = s \end{cases}$$

where the integration is around the unit circle. This is due to Cauchy's Theorem (Conway, 1978).

Whiteman (1985) demonstrated that a discounted conditional covariance involving complicated lags can be succinctly expressed as a convolution. I duplicate the demonstration here. Consider two serially correlated processes, a_t and b_t, where $a_t = \sum_k^{\infty} A_k\epsilon_{t-k}$ and $b_t = \sum_k^{\infty} B_k\epsilon_{t-k}$. The discounted conditional covariance as of time t, setting realized innovations to zero, is

$$E_t \sum_{s=1}^{\infty} \beta^s a_{t+s} b_{t+s} = E_t \sum_{s=1}^{\infty} \beta^s \left(\sum_{k=0}^{\infty} A_k \epsilon_{t+s-k} \right) \left(\sum_{k=0}^{\infty} B_k \epsilon_{t+s-k} \right).$$

Because cross-product terms drop out, coefficients of like lags of ϵ_t can be grouped. This yields

$$= \beta[A_0 B_0 + \beta A_1 B_1 + \beta^2 A_2 B_2 + \ldots] E_t \epsilon_{t+1}^2 + \beta^2 [A_0 B_0 + \beta A_1 B_1 + \beta^2 A_2 B_2 + \ldots] E_t \epsilon_{t+2}^2 + \ldots$$

$$= \frac{\beta \sigma_\epsilon^2}{1-\beta} \sum_{s=0}^{\infty} \beta^k A_k B_k = \frac{\beta \sigma_\epsilon^2}{1-\beta} \frac{1}{2\pi i} \oint A(z) B(\beta z^{-1}) \frac{dz}{z}.$$

This is a useful transformation because the integrand is a product. Since the optimal policy for individuals and for the contract is an analytic function, this representation of the objective makes the search for an optimal policy more direct.

Hansen and Sargent (1980, 1981) showed that the first-order conditions of stochastic optimization problems could be expressed in lag operator notation, z-transformed, and solved. Whiteman noticed that the z-transformation could be done on the objective function itself, skipping the step of finding the time-domain version of the Euler condition. The objective is then a functional, that is, a mapping of functions into the real line. He then applied the calculus of variations to find the optimum policy function, directly obtaining the z-transformed Euler equation, with the solution expressible in a succinct and general way using operator analysis.

To solve the z-transformed Euler equation of a stochastic linear-quadratic optimization problem, one first factors the equation so that the nonanalytic parts can be separated from the analytic parts. The annihilator then eliminates the nonanalytic parts, leaving the analytic parts unaffected. The solution yields an unknown function that must be analytic because it operates on stochastic innovations and hence must be backward looking.

The most difficult part of an operator problem is the factorization. For example, the factorization in (29) involves the sum of two unknown spectral densities. Although the existence of an invertible and analytic κ has been shown, and although there are algorithms for factoring $\kappa\kappa_*$, there is no known general expression for κ explicitly in terms of δ and ϕ. It would be desirable to have such an analytic expression because the optimization is over δ and ϕ. The Szegö factorization is as close as we can get. It is a generalization of the identity

$$y = e^{1/2 \log(y^2)}.$$

The Szegö factorization thus permits an explicit connection between the factored function κ and its constituents δ and ϕ.

More information about z-transform methods can be found in Sargent (1979), Whiteman (1983) and the appendix of Whiteman (1985).

Appendix B

■ The derivations and proofs of the equivalence result in Section 7 follow.

Proof of Proposition 1. I will first show that the unstable part of the general solution is ruled out by the adjustment cost. Since $\lambda_2 > \beta^{-1} > \beta^{-1/2}$, if $c_2 \neq 0$, debt will grow without bound faster than it can be discounted by $\beta^{1/2}$. Since debt grows geometrically, so does the change in debt and the resulting adjustment cost, and since $\lambda_2 > \beta^{-1}$, faster than it is discounted. The result of such growth of debt might be to allow consumption to approach the bliss point asymptotically. However, the contribution of such bliss is bounded by zero, while the adjustment cost is unbounded. This cannot be optimal, and therefore $c_2 = 0$ is a necessary condition.

With $y_t = \bar{y}$, a constant, the difference equation (8b) then has the simple first-order form

$$B_{t+1} = \lambda_1 B_t + K,$$

with

$$K = -\lambda_1 \lambda_0^{-1} (1 - \lambda_2^{-1} L^{-1})^{-1} (1 - \beta(1+\rho) L^{-1}) y_{t+1}.$$

Since $y_t = \bar{y} < 0$ is constant, the terms $(1 - \beta(1+\rho)L) y_t$ become $(1 - \beta(1+\rho)L)\bar{y}$, with the sign determined by the magnitude of ρ. The pattern of debt accumulation will be stable around the steady state of $K/(1-\lambda_1)$. If $\rho < (1-\beta)/\beta$, $K > 0$ and net borrowing occurs in the long run. This results in net saving by the intermediary at an interest rate below the intermediary's opportunity cost, and cannot be an equilibrium. Conversely, if $\rho > (1-\beta)/\beta$, $K > 0$, net saving occurs in the long run and the intermediary borrows at a rate above the opportunity cost it faces; this cannot be an equilibrium either. Finally, if $\rho = (1-\beta)/\beta$, $K = 0$ and net borrowing is zero in the long run. With an initial condition of $B_0 = 0$, net profit is zero for the intermediary. Equation (9) reduces to $B_{t+1} = B_t$, that is, no change in deterministic debt over time. Since in the initial period aggregate debt is necessarily zero, in all subsequent periods deterministic and hence aggregate debt is therefore also zero, which is an equilibrium. *Q.E.D.*

Derivation of (34). The stochastic part of the policy function, (32), can be substituted into the objective (31) to yield

$$-\frac{1}{2\pi i}\oint\left\{(A-\delta\kappa^{-1}[\delta_*\kappa_*^{-1}A]_+)(A_*-\delta_*\kappa_*^{-1}[\delta_*\kappa_*^{-1}A]_{+*})+\gamma\phi\kappa^{-1}[\delta_*\kappa_*^{-1}A]_+\phi_*\kappa_*^{-1}[\delta_*\kappa_*^{-1}A]_{+*}\frac{\delta z}{z}\right\},$$

where I have dropped the leading constant $\beta\sigma_\epsilon^2/(1-\beta)$ for convenience. Algebraic manipulation yields

$$-\frac{1}{2\pi i}\oint\{AA_*-2\delta\kappa^{-1}A_*[\delta_*\kappa_*^{-1}A]_++(\delta\delta_*+\gamma\phi\phi_*)\kappa^{-1}\kappa_*^{-1}[\delta_*\kappa_*^{-1}A]_+[\delta_*\kappa_*^{-1}A]_{+*}\}\frac{dz}{z}.$$

Recalling the definition of κ, this yields

$$-\frac{1}{2\pi i}\oint\{AA_*-2\delta\kappa^{-1}A_*[\delta_*\kappa_*^{-1}A]_++[\delta_*\kappa_*^{-1}A]_+[\delta_*\kappa_*^{-1}A]_{+*}\}\frac{dz}{z}.$$

Complete the square to express this as

$$\frac{1}{2\pi i}\oint\{AA_*+(\delta_*\kappa_*^{-1}A-[\delta_*\kappa_*^{-1}A]_+)(\delta\kappa^{-1}A-[\delta_*\kappa_*^{-1}A]_{+*})-\delta_*\kappa_*^{-1}A\delta\kappa^{-1}A_*\}\frac{dz}{z}.$$

Using Lemma C.2 from Taub (1986) yields (34). *Q.E.D.*

Proof of Proposition 2. The proposition follows if consumption fluctuates, resulting in a nonzero stochastic component of utility, that is, if

$$\|A(z)+\delta(z)B(z)\|^2>0.$$

From (37), the proof reduces to asking, is $(1-\delta(z)\hat{\delta}(z)^{-1})=\hat{\delta}(z)^{-1}(\hat{\delta}(z)-\delta(z))$ identically zero on the boundary of the disk, $D(\beta)$? Since $\delta(\beta)=0$, that is, since there is a zero of $\delta(\cdot)$ inside the disk $D(\beta)$, and $\hat{\delta}(\beta)\neq 0$, $\hat{\delta}(\cdot)$ is not identically equal to $\delta(\cdot)$ inside the disk. Since both $\hat{\delta}$ and δ are analytic, their real parts are harmonic and their behavior inside the disk is determined by the behavior of the real parts on the boundary. If the two functions were identical on the boundary of the disk, they would necessarily be identical inside the disk since one could obtain the interior values using the formula

$$(\hat{\delta}(\beta)-\delta(\beta))=\frac{1}{2\pi i}\oint_{|\zeta|=1}\frac{\zeta+\beta}{\zeta-\beta}\mathrm{Re}(\hat{\delta}(\zeta)-\delta(\zeta))\frac{d\zeta}{\zeta}\neq 0$$

(Conway, 1978, p. 259, Corollary 2.9). Therefore they must differ on the boundary of the disk as well. Thus

$$\|A(z)+\delta(z)B(z)\|^2=\|\,|\delta|^{-2}|\hat{\delta}-\delta|^2\|^2\neq 0. \qquad Q.E.D.$$

Proof of Proposition 3. I will use the Szegö factorization: see Rudin (1974, Th. 17.16(b)) and Appendix A. The factorization first requires that the functions with domain $D(\beta)$ be mapped into the corresponding functions on the unit disk. Denote

$$\tilde{\delta}(z)\equiv\sum_{s=0}^{\infty}\delta_s(\beta^{1/2})^s z^s.$$

Then

$$\frac{1}{2\pi i}\oint|\tilde{\delta}|^2\frac{dz}{z}=\frac{1}{2\pi i}\oint\delta\delta_*\frac{dz}{z}$$

with the integration around the unit circle in both cases. The minimization problem is equivalent to

$$\min_{\phi(\cdot)}\tilde{\kappa}(\tilde{\alpha})^2, \tag{A1}$$

subject to

$$\phi(0)=1, \tag{A2}$$

where $\tilde{\alpha}\equiv\beta^{1/2}\alpha$. Now use the Szegö factorization, since an explicit expression for $\tilde{\kappa}(\tilde{\alpha})$ in terms of its components is not available. Recalling from Rozanov's theorem that the coefficients of $\tilde{\kappa}(\cdot)$ can be chosen to be real, so that $\tilde{\kappa}(\tilde{\alpha})^2=|\tilde{\kappa}(\tilde{\alpha})|^2$,

$$\min \tilde{\kappa}(\tilde{\alpha})^2 = \min \exp\left\{2 \cdot \frac{1}{2}\frac{1}{2\pi i}\oint \frac{\zeta+\tilde{\alpha}}{\zeta-\tilde{\alpha}} \log\left(\tilde{\delta}(\zeta)\tilde{\delta}(\zeta^{-1}) + \gamma\tilde{\phi}(\zeta)\tilde{\phi}(\zeta^{-1})\right)\frac{d\zeta}{\zeta}\right\}$$

$$= \min \exp\left\{\frac{1}{2\pi i}\oint \frac{\zeta+\tilde{\alpha}}{\zeta-\tilde{\alpha}} \log\left(|\tilde{\delta}(\zeta)|^2 + \gamma|\tilde{\phi}(\zeta)|^2\right)\frac{d\zeta}{\zeta}\right\}.$$

Given the irrelevance of the choice of $\delta(\cdot)$ once constraint (33) is satisfied, the additional condition $\delta(0) = 1$ can be imposed, and the minimand can be re-expressed

$$\tilde{\delta}(\tilde{\alpha})^2 \exp\left\{\frac{1}{2\pi i}\oint \frac{\zeta+\tilde{\alpha}}{\zeta-\tilde{\alpha}} \log\left(1 + \gamma|\tilde{\psi}(\zeta)|^2\right)\frac{d\zeta}{\zeta}\right\},$$

where $\tilde{\psi} \equiv \phi/\delta$. We have the following lemma.

Lemma 1. No gain can be had by making the penalty and redistribution filters different, that is, under the restriction $\tilde{\psi}(0) = 1$,

$$\arg\min_{\tilde{\psi}(\cdot)} \exp\left\{\frac{1}{2\pi i}\oint \frac{\zeta+\tilde{\alpha}}{\zeta-\tilde{\alpha}} \log\left(1 + \gamma|\tilde{\psi}(\zeta)|^2\right)\frac{d\zeta}{\zeta}\right\} = 1.$$

Proof. I will construct an upper bound for the objective and then show the bound is minimized by $\tilde{\psi}(0) = 1$. We have the successive inequalities

$$\exp\left\{\frac{1}{2\pi i}\oint \frac{\zeta+\tilde{\alpha}}{\zeta-\tilde{\alpha}} \log\left(1 + \gamma|\tilde{\psi}(\zeta)|^2\right)\frac{d\zeta}{\zeta}\right\} \le \exp\left|\frac{1}{2\pi i}\oint \frac{\zeta+\tilde{\alpha}}{\zeta-\tilde{\alpha}} \log\left(1 + \gamma|\tilde{\psi}(\zeta)|^2\right)\frac{d\zeta}{\zeta}\right|$$

$$\le \exp\left\{\frac{1}{2\pi i}\oint \left|\frac{\zeta+\tilde{\alpha}}{\zeta-\tilde{\alpha}} \log\left(1 + \gamma|\tilde{\psi}(\zeta)|^2\right)\right|\frac{d\zeta}{\zeta}\right\} \le \exp\left\{\left\|\frac{\zeta+\tilde{\alpha}}{\zeta-\tilde{\alpha}}\right\| \left\|\log\left(1 + \gamma|\tilde{\psi}(\zeta)|^2\right)\right\|\right\}$$

with the last inequality following from the Schwartz inequality (see Rudin, 1974). Since $\tilde{\alpha} < 1$, by Cauchy's theorem,

$$\left\|\frac{\zeta+\tilde{\alpha}}{\zeta-\tilde{\alpha}}\right\| = 1.$$

Finally,

$$\left\|\log\left(1 + \gamma|\tilde{\psi}(\zeta)|^2\right)\right\| = \left\|\log\left(1 + \gamma|\tilde{\psi}|^2\right)^{1/2}\right\|,$$

and

$$\log(1+\gamma) = \left\|\log\left(1 + \gamma|\tilde{\psi}(0)|^2\right)\right\| \le \left\|\log\left(1 + \gamma|\tilde{\psi}|^2\right)^{1/2}\right\|,$$

the last inequality following from a theorem of Rudin (1974). Imposing this minimum upper bound yields equality at each step. *Q.E.D.*

The finding that $\psi \equiv 1$ is equivalent to $\phi(z) = \hat{\delta}(z)$. This renders the factorization problem trivial, making the search for the optimal $\delta(\cdot)$ easier. Substituting $\psi(z) = 1$ into (38) yields the maximization problem,

$$\max_{\delta(\cdot)} (1+\gamma)^{-1}\delta(\beta\alpha)^2\hat{\delta}(\beta\alpha)^{-2} \tag{A3}$$

subject to $\delta(0) = 1$.

Lemma 2.

$$1 - \beta^{-1}z \in \{\arg\max_{\delta(\cdot)} (1+\gamma)^{-1}\delta(\beta\alpha)^2\hat{\delta}(\beta\alpha)^{-2}\}$$

and

$$\hat{\delta}(z) = 1 - z.$$

Proof. Recalling the feasibility restriction in (36), $\delta(\beta) = 0$, and so $\delta(z)$ can be expressed as $\delta(z) = (1 - \beta^{-1}z)f(z)$. Since the filter must operate only on current and past signals, it can have no poles in $D(\beta)$. Thus $f(z)$ is analytic in $D(\beta)$. Then (A3) becomes

$$\max_{f(\cdot)} \left((1-\alpha)(1-\beta\alpha)^{-1}f(\beta\alpha)\hat{f}(\beta\alpha)^{-1}\right)^2$$

or equivalently

$$\max_{f(\cdot)} \tilde{f}(\tilde{\alpha})\hat{\tilde{f}}(\tilde{\alpha})^{-2}.$$

Using the Szegö factorization again, this is equivalent to

$$\max \exp\left\{\frac{1}{2\pi i}\oint \frac{\zeta+\tilde{\alpha}}{\zeta-\tilde{\alpha}}\log\left(\frac{\tilde{f}(\zeta)\tilde{f}(\zeta^{-1})}{\hat{\tilde{f}}(\zeta)\hat{\tilde{f}}(\zeta^{-1})}\right)\frac{d\zeta}{\zeta}\right\}$$

But $\tilde{f}(\zeta)\tilde{f}(\zeta^{-1}) = \hat{\tilde{f}}(\zeta)\hat{\tilde{f}}(\zeta^{1})$ by definition, and so

$$\log\left(\frac{\tilde{f}(\zeta)\tilde{f}(\zeta^{-1})}{\hat{\tilde{f}}(\zeta)\hat{\tilde{f}}(\zeta^{-1})}\right) = 0$$

for any analytic f, including $f \equiv 1$. It is now straightforward to demonstrate that $\tilde{\delta}(z) = 1 - z$. *Q.E.D.*

Proposition 3 now follows directly from Lemma 2.

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