LEARNING AND MODEL VALIDATION

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Abstract. This paper studies the following problem: An agent takes actions based on a possibly misspecified model. The agent is 'large', in the sense that his actions influence the model he is learning about. The agent is aware of potential model misspecification and tries to detect it, in real-time, using an econometric specification test. If his model fails the test, he selects a new better-fitting model. If his model passes the test, he uses it to formulate and implement a policy based on the provisional assumption that the current model is correctly specified, and will not change in the future.

We claim this testing and *model validation* process is an accurate description of many macroeconomic policy problems. We study the dynamics of this process. Several results emerge from the analysis: (1) We describe the sense in which model validation justifies the use of constant gain learning algorithms, (2) We describe conditions under which model validation supports the persistence of misspecified models, (3) We study how model validation dynamics influence the large deviation properties of Sargent's (1999) *Conquest* model, and explain how endogenous model selection influences beliefs about the effectiveness of monetary policy, and (4) We offer conjectures about how model validaton could potentially deliver an endogenous reference model in robust control and filtering models.

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1. Introduction

In his Ryde Memorial lectures, Sargent (1993) discussed several potential benefits from studying models in which agents 'act like' econometricians, using traditional econometric methods to formulate and revise their expectations. First and foremost, they permit an analysis of transition dynamics. By definition, Rational Expectations models have nothing to say about transition dynamics. They presume convergence has already occurred.[1]. Econometric learning models can also provide a basis for equilibrium selection when there are multiple Rational Expectations equilibria.[2] They can also provide tools for computing Rational Expectations equilibria.

Despite these benefits, Sargent (1993) expressed reservations about the usefulness of these models for explaining actual time-series data. Given their focus on convergence and stability, adaptive learning models naturally produce econometric specifications with time-varying parameters, which depend on initial conditions. This creates econometric

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[1]See Bray and Kreps (1987)

[2]See Evans and Honkapohja (2001) for a detailed discussion.

difficulties, unless one has prior information about these initial conditions (as might be the case when studying transitional dynamics). Sargent also highlighted the need for substantial 'prompting' in these models. Typically, agents are presumed to know the model up to a small number of unknown parameters. One could argue, however, that endowing agents with the correct model specification 'throws the baby out with the bath water' when it comes to justifying the Rational Expectations Hypothesis. The macroeconomic learning literature was originally motivated by doubts about the strong informational requirements of Rational Expectations. From this perspective, the real issue is *model* uncertainty, not parameter uncertainty. Once agents know the model, you might as well assume Rational Expectations! So at least in stationary environments, learning models only seemed to be of interest to economic theorists, useful perhaps for justifying or computing a given Rational Expectations equilibrium, but of little relevance beyond that.

Fortunately, the macroeconomic learning literature has made a lot of progress since Sargent's Ryde lectures. An important step was taken by Sargent (1999) himself, in his monograph *The Conquest of American Inflation*, which built on the work of Sims (1988) and Chung (1990). This introduced two key innovations. First, it introduced the idea of 'constant gain' learning, which effectively discounts old data. Discounting past data makes sense when agents suspect the underlying data-generating process might be time-varying. Somewhat ironically, these doubts about the stability of the data-generating process produce a (stochastic) stationary equilibrium, since discounting washes out initial conditions and prevents learning from dissipating over time. Stationary equilibria are much more amenable to standard econometric estimation methods. The second innovation was to introduce the idea of a Self-Confirming Equilibrium (SCE).[3] This relaxes the assumption that agents have a correctly specified model, and hence, reduces the 'prompting' agents receive. In a self-confirming equilibrium agents' beliefs are correct, or 'rational', along the equilibrium path, but they can be incorrect, or misspecified, about off-equilibrium path events.[4] Sargent (1999) showed that the combination of constant gain learning and model misspecification could produce recurrent cycles, which resemble the kind of Markov-Switching processes that are commonly fit to macroeconomic data. Cho, Williams, and Sargent (2002) (CWS) went on to show that the stochastic properties of these cycles could be characterized using the tools of large deviations theory.

Although Self-Confirming Equilibria and Restricted Perceptions Equilibria can be attractive alternatives to Rational Expectations Equilibria, they raise obvious, and as yet unaddressed, questions: Wouldn't agents be able to discover the misspecifications of their models? What if we allowed agents to *test* their models, as well as estimate them? Our primary goal is to begin to address these questions. In doing this, we push Sargent's analogy between agents and econometricians one step further. Econometrics textbooks generally consist of two halves: the first half discusses estimators and their properties, and the second half discusses inference and specification analysis. The learning literature assumes that agents have only read the first half of the book, focusing on issues like decreasing versus constant gain learning, recursive least-squares versus stochastic gradient

[3]Self-Confirming Equilibria were originally developed in the game theory literature by Fudenberg and Levine (1993). Sargent (1999) argued that this equilibrium concept could be useful for macroeconomists.

[4]Evans and Honkapohja (2001) discuss the related concept of Restricted Perceptions Equilibria (RPE), in which agents optimially (in a least-squares sense) fit misspecified econometric models.

learning, etc. Our analysis lets agents use the second half of the book.

Econometricians have made substantial progress during the past two decades on the problem of testing and comparing misspecified models.[5] A unifying theme in this work is to view models as inducing probability distributions over the data, and to then compare models using information-theoretic measures of the difference between two probability distributions. Most recent work on comparing misspecified models is based on the Kullback-Leibler Information Criterion (KLIC), which measures the relative entropy between two probability distributions. Models with smaller estimated KLICs are preferred. A key result in this literature is that consistent estimates of the difference between two KLICs can be obtained *without* prior knowledge of the true data-generating process. As a result, models can be compared without having to assume that any of them are correctly specified. Basing model selection on relative entropy also has the virture that it naturally encapsulates a measure of model 'complexity', thereby striking a balance between bias and variance, and allowing models with different dimensions to be sensibly compared.

The decisionmaker in our model exploits these recent developments. We contend that this makes his rationality a little less bounded. However, it is still bounded, for the same reason it is bounded when estimating his model. Traditional econometric methods (for both estimation and inference) presume the data-generating process is exogenous. In our model this isn't the case. Our agent actually uses his model to make decisions, and as a result, the data-generating process *responds to* the agent's own estimation and testing efforts. That is, the data are *endogenous*. When it comes to inference, this means there will be a difference between the nominal and actual sizes of our agent's test statistics. We assume this discrepancy goes undetected.

The model validation process we study can now be described as follows: (1) An agent begins with a fixed, finite, collection of potentially misspecified parametric models, (2) Each period the agent compares his current reference model to all other candidate models. He does this by recursively estimating their KLICs. In our gaussian linear regression examples this reduces to recursive computation of Akaike Information Criteria (AIC). Note that all models are always being estimated, even if they are not currently being used for policy. (3) If the difference between the current reference model's estimated KLIC and the current minimum estimated KLIC is less than some (fixed) threshold, the current reference model is deemed to be *validated*, and is then used as the basis for current policy. The threshold is set to control Type I errors. (4) If the difference exceeds the threshold, the current reference model is rejected, and a new reference model is selected, based on the model with the minimum estimated KLIC statistic. This new reference model is then used to formulate policy, and the whole process repeats itself next period.[6]

This recursive procedure defines a completely general model validation process. Although it might be possible to characterize the induced dynamics of this process in a

[5]A highly selected sample includes: White (1982), Vuong (1989), Hansen and Sargent (1993), Sin and White (1996), Kitamura (2001), Rivers and Vuong (2002), and Marcellino (1999). White (1994) and Burnham and Anderson (2002) contain textbook treatments.

[6]Note, this process does not permit the agent to engage in model averaging which, from the standpoint of Bayesian decision theory, would generally be a preferred strategy. Keep in mind, however, that our agent uses his model to solve a control problem. We assume the computational advantages of model selection outweigh the statistical advantages of model averaging.

general setting, we have chosen to narrow our focus to two well known examples from the macroeconomic learning literature. Our first example is taken from Sargent (1999). We first consider the case where model complexity is ignored, so that validation becomes a simple question of fixed versus time-varying parameters. Sargent's recursive learning model is based on the implicit prior of slow, random walk, parameter drift. Parameters do appear to drift, but only because of the agent's own model misspecification. One might question the plausibility of this prior, however, in light of the time-varying, heteroskedastic nature of the model's induced time paths. Wouldn't a good econometrician look at these time paths and revise his specification of parameter drift?[7] Model validation allows for this. Although the agent *monitors* his model continuously, he only updates it if it goes sufficiently far off-track. A time invariant testing threshold automatically generates a greater rate of model revision during turbulent periods.[8].

We show that as the testing threshold converges to zero, model validation converges in a very strong sense to recursive learning. In particular, both strategies induce probability distributions over sample paths. Our convergence result implies that these two distributions coincide not only in the center of the distribution, *but also in the tails.* From this, we can conclude that the two processes share the same large deviations properties (e.g., escape routes and escape times). This aymptotic equivalence lends some support to constant gain recursive learning models.

The next step is to incorporate a complexity cost. We assume the agent fits both static and dynamic Phillips Curve models. The dynamic model fits the data better, but is more complex, so it will not be used unless its fit is sufficiently superior. As noted by Cogley, Colacito, and Sargent (2005), this example is of historical interest, as it allows for a distinction between short-run and long-run trade-offs between inflation and unemployment. This distinction was actively debated in the policy arena during the 1960s and 1970s. Using simulations, we show that model validation dynamics represent a sort of convex combination of the recursive learning dynamics of simple and complex models, with a weight determined by the complexity cost. It turns out that model validation dynamics consist of a subtle interplay between the escape dynamics of simple models and the mean dynamics of more complex models.

This first example features a fairly subtle form of model misspecification. Our second example studies a more mundane, and perhaps more common, form of model misspecification; namely, the exclusion of relevant variables from a (recursively updated) linear regression model. In particular, we study an example from Evans and Honkapohja (2001). They focus on how E-stability conditions are affected by model misspecification. We pose a different stability question. We ask whether our model validation process would reveal the correct model if it is contained in the agent's original collection of models. This sort of question has been the focus of a huge literature on the consistency of model selection criteria. While the lessons of this literature are suggestive, they are not directly applicable due to the endogenous nature of the data-generating process in our case. Still, as might be expected from this literature, the crucial issue is how model complexity is penalized.

[7]Sargent and Williams (2005) show that escape dynamics are sensitive to prior beliefs about parameter drift. If agents think parameters vary more rapidly than in Sargent (1999), then escape dynamics disappear.

[8]In this sense, model validation can mimic the kind of time-varying gain sequence used by Marcet and Nicolini (2003)

Standard model selection criteria embody a cost function that decreases with sample size. This reflects the fact that variance becomes less of an issue as sample size grows, so that agents can afford to consider progressively more complex models over time. Not surprisingly, we find that the RPE studied by Evans and Honkapohja is not stable with respect to standard model selection criteria. We argue that model validation leads to a new concept of Restricted Perceptions Equilibria, one based on minimum KLIC.

For simplicity, our paper focuses on parametric model uncertainty. In a sense then, we are subject to the same criticism we previously made of the existing macroeconomic learning literature. By only considering a finite collection models, we've essentially added a few unknown (discrete) parameters to a standard learning exercise. One route toward genuine model uncertainty would be to consider nonparametric alternatives, as in Chen and White (1998). This can be problematic, however, given the sample sizes usually available to macroeconomists. Nonparametric methods are also ill-suited to situations where the model is being used to solve a control problem. An attractive alternative, better suited to a control context, has been proposed by Hansen and Sargent (2006b). Their robust control methods adopt a (constrained) worst-case approach to model uncertainty. In Hansen and Sargent (2006a), they extend these methods to allow filtering of underlying hidden states. Using discrete, time invariant, hidden states to index unknown models allows agents to respond to model uncertainty in a way that is robust to general forms of distributional misspecification. Hansen and Sargent approach this problem from the perspective of model averaging. However, their methods could easily be adapted to model selection, which would deliver a form of 'robust model validation'. We argue that it would also deliver an endogenous reference model.

Finally, there has been some prior work attempting to link learning with model validation, which we should briefly mention. First, the early work of Bray and Savin (1986) touched on this issue. They ask whether agents could use standard diagnostics, like Chow tests and Durbin-Watson statistics, to detect the time variation in parameters that their own learning behavior generates. They found that when convergence is slow, agents are generally able to detect the misspecification of their models. In a repeated game context, Foster and Young (2003) allow players to construct, test, and revise simple models of their opponent's behavior. They show that hypothesis testing produces convergence to a Nash equilibrium in a relatively strong sense, although testing errors produce rare but recurrent experimentation phases. Perhaps closest in spirit to our analysis is recent work by Branch and Evans (2005). They also study a situation where agents not only update the coefficients of a given model, but also select among alternative parametric models based on their recent forecasting performance. Despite the similarities, our model validaton approach differs in one crucial respect from all these prior efforts - ours is based on formal statistical and information-theoretic model selection criteria. In particular, our paper is the first to exploit recent advances in the econometric analyis of misspecified models.

The remainder of the paper is organized as follows. Section 2 provides a general outline of our model validation process. Section 3 applies this framework to Sargent's (1999) *Conquest* model. Section 4 applies it to an RPE example from Evans and Honkapohja (2001). Section 5 briefly discusses how our model validation approach could be adapted to deliver an endogenous reference model in robust control problems. Section 6 provides a few concluding remarks, and an Appendix contains proofs of various technical results.

2. Model Validation

The concept of model validation is not new. For example, it has a long history in the engineering literature.[9]. Our goal is to apply and adapt some of the language and methods of this well developed literature to the macroeconomic learning literature. This is not a straightforward exercise in translation. One problem has already been discussed, namely, the fact that in our setting the data are endogenous to the validation process. Another problem concerns the real-time nature of the exercise. Most validation procedures are retrospective, or are based on hold-out samples, which are ill-suited to ongoing decision making problems. Yet another problem, again related to the decision making aspect of the problem, relates to the nature of the alternative hypothesis. Many existing real-time validation procedures, e.g., those based on CUSUM statistics, fail to specify an explicit alternative.[10]. Unfortunately, these methods are of little use to our agent, who, at the end of the day, must make a decision based on *some* model. Simply concluding a model is rejected is not very useful. Instead, our agent must adopt the perspective of "it takes a model to beat a model", which means that validation must be defined relative to an explicit alternative.

Suppose an agent begins with a fixed, finite set of parametric models, $\overline{\mathcal{M}}$. For expositional ease, suppose $\overline{\mathcal{M}}$ consists of just two elements. Now, consider a model parameterized by β, and denote its implied distribution over the relevant data by μ_β. How good is this model? Clearly, this depends on what we mean by 'good'. Suppose we define in it an absolute sense, relative to the true data-generating process, denoted by μ_0. The relative entropy between μ_β and μ_0 is then defined as

$$H(\mu_0||\mu_\beta) = \begin{cases} \int \log \frac{d\mu_0}{d\mu_\beta} d\mu_0 & \text{If } \mu_0 \ll \mu_\beta \\ \infty & \text{otherwise} \end{cases} \tag{2.1}$$

Note that relative entropy is just an expected log-likelihood ratio statistic. In the statistics literature, relative entropy is referred to as the Kullback-Leibler Information Criterion (KLIC), and we use these terms interchangeably. Although relative entropy is not a metric, since it is not symmetric, it does define a sensible measure of the distance between two probability distributions. For example, suppose our model is *really* good, in the sense that it *is* the true data-generating process. Then (2.1) implies that the relative entropy between our model and the data will be zero. On the other hand, if our model is misspecified, then one can show that the relative entropy will be strictly positive.

In practice, of course, we don't know the true model, and so we cannot compute the relative entropy between our model and the true DGP.[11] It turns out, however, that for purposes of model *comparison* the fact that we don't know the true model is irrelevant. To see this, let $f(\mu_0|\beta_1)$ be the log-likelihood of Model 1 and $f(\mu_0|\beta_2)$ be the log-likelihood of Model 2. They are indexed by μ_0 as a reminder that the value of both likelihoods depend

[9]Model validation is currently an active research front in the robust control literature. See, e.g., Gevers, Bombois, Codrons, Scorletti, and Anderson (2003)

[10]See, e.g., Chu, Stinchcombe, and White (1996) and Inoue and Rossi (2005)

[11]This is not quite correct. By using nonparametric methods, e.g., a sieve estimator, we *can* obtain a consistent estimate.

on the data, and hence on μ_0. Notice that if we substitute these into (2.1) and take their *difference*, the log-likelihood of the true model washes out, and we are left with

$$H(\mu_0|\beta_2) - H(\mu_0|\beta_1) = \int f(\mu_0|\beta_1)d\mu_0 - \int f(\mu_0|\beta_2)d\mu_0 \tag{2.2}$$

In the context of Gaussian linear regression, where models are 'good' and samples are 'large', we have (ignoring inessential constants)[12]

$$\int f(\mu_0|\beta_i)d\mu_0 \approx T\log(\hat{\sigma}_i^2) + 2K_i \tag{2.3}$$

where K_i is the number of parameters in model-i, and $\hat{\sigma}_i^2 = T^{-1}\sum \hat{\varepsilon}_t^2$, where $\hat{\varepsilon}_t$ is just the time-t model residual. The right-hand side of (2.3) is the well known Akaike Information Criterion (AIC). Substituting this into (2.2) yields the following asymptotically unbiased estimate of two models' relative (expected) KLICs

$$AIC_2 - AIC_1 = [\log(\hat{\sigma}_2^2) - \log(\hat{\sigma}_1^2)] + \frac{2}{T}(K_2 - K_1) \tag{2.4}$$

This expression provides a very natural, theoretically motivated, basis for model comparison and selection. In particular, positive values favor Model 1, while negative values favor Model 2. Notice that the first term on the right-hand side captures relative model fit, while the second captures relative model complexity. Hence, equation (2.4) resolves a trade-off between bias and variance that is present in all model selection problems. Of course, it is important to remember that there are other ways to resolve this trade-off. In fact, the AIC has been criticized as being an inconsistent model selection criterion, i.e., it has a non-vanishing probability of selecting over-parameterized models. This over-fitting tendency has led to the development of other model selection criteria, which impose higher model complexity costs. Perhaps the most commonly used rival is Schwarz's Bayesian Information Criterion (BIC). Sin and White (1996) discuss conditions on the complexity function that ensure consistency. In our analysis, we are fairly agnostic about this function, for a couple of reasons. First, the endogeneity of the data-generating process here invalidates these existing consistency theorems. Second, our agent selects models *recursively*. The repeated nature of the testing also calls into question the validity of these results.[13] As a reult, although we often couch our arguments in terms of AIC, in our proofs and simulations we also consider alternative specifications.

2.1. **Validation Dynamics.** We are now in a position to describe more precisely our recursive model validation dynamics. Again, the agent starts with an exogenously specified finite collection of parametric models, $\overline{\mathcal{M}}$. All of these models may be misspecified. Unlike most applications of model selection, our agent actually uses his model to solve a control problem. Let x_t be the time-t choice of his control variable. The next section provides an explicit example of the kind of control problem we have in mind. Note, as in the recursive learning literature, we assume that when the agent solves his control problem, he adopts

[12] Here 'good' means that White's (1982) Information Matrix Equality is approximately satisfied. When this is not the case, minimizing expected KLIC produces the so-called Takeuchi Information Criterion. See Burnham and Anderson (2002) for more details.

[13] In fact, Shen and Ye (2002) argue that, when applied recursively, model selection criteria should feature an adaptive, data-dependent, complexity cost.

the provisional assumption that his current model will apply in all future periods. This imparts a degree of 'bounded rationality' to the agent since, in fact, his model will evolve over time. On the other hand, we assume our agent *is* aware of sampling variability and Type I errors. This means he will retain his current model until the performance of the best model exceeds it by a given threshold, ρ. Effectively, the agent runs a sequence of Vuong (1989)-type quasi-likelihood ratio tests. Schematically then, the agent uses the following recursive testing and model validation process:

- Let $\mathcal{A}_t$ be the time-t value of the recursively estimated AIC statistic for the current reference model.
- Let $\mathcal{A}_t^*$ be the minimum AIC statistic across all models at time-t. (Note: All models are always being fit to the data, even if they are not currently being used for policy.)
- If $\mathcal{A}_t - \mathcal{A}_t^* < \rho$, then the current model is validated, and x_t is chosen based on it.
- If $\mathcal{A}_t - \mathcal{A}_t^* \geq \rho$, then the current reference model is replaced by the model corresponding to $\mathcal{A}_t^*$, and x_t is instead based on this new model.

As an example, consider the case of two linear regression models. Notice that in this case there are effectively three models on the table at any given time: (1) The current reference model, (2) An updated version of the reference model, obtained by recursive updating of its coefficients, and (3) A recursively updated version of the alternative model.

Calibrating ρ to a given Type I error probability requires knowledge of the sampling distribution of the difference between two AIC statistics. Vuong (1989) did this for the case of i.i.d. data. It turns out that the sampling distribution depends on whether the models are nested or non-nested. These results were subsequently exended to the dynamic case by Rivers and Vuong (2002), for the case of non-nested models, and by Marcellino (1999), for the case of nested models. In principle, we could allow our agent to apply these tests. However, as we've noted before, this would still lead to incorrectly sized tests, due to both the endogeneity of the data and the recursive nature of the tests.

Instead, we assume the agent adopts the following quasi-Bayesian calibration strategy, based on the notion of 'Akaike weights' (see, e.g., Burnham and Anderson (2002)). Suppose that each period the agent computes the following value for each model:

$$\pi_{it} = \frac{\exp(-.5\Delta_{it})}{\sum_j \exp(-.5\Delta_{it})} \tag{2.5}$$

where $\Delta_{it} = AIC_{it} - AIC_{min,t}$ and the summation runs over all models in $\overline{\mathcal{M}}$. Notice that π_{it} can be interpreted as a frequentist estimate of the probability that model-i is currently the best KLIC model. We assume the agent retains the current reference model as long as $\pi_{max,t} < \rho \cdot \pi_{it}$, where $\rho > 1$. For example, if $\rho = 1.1$ then the agent retains the current reference model as long as the probability of the best model does not exceed its probability by more than 10%.

2.2. **Representation as a Stochastic Recursive Algorithm (SRA).** In the context of *estimation*, Marcet and Sargent (1989) showed that the dynamics of self-referential systems can be usefully approximated by writing them as a standard Stochastic Recursive Algorithm (SRA). The same applies to the problem of *inference.* The advantages from doing this stem from application of so-called 'singular perturbation' methods, which exploit the time-scale separation characterizing these systems under small gain updating. In particular, for the case of recursive least-squares, where the gain is $O(T^{-1})$, the behavior of the system can be approximated (asymptotically) by an ordinary differential equation. Under constant gain learning, the behavior of the system can be approximated by a diffusion process.

To adapt these methods to model validation, all we need to do is define a set of discrete indicator variables that index each model. For example, suppose there are just two models. Let $s_t \in \{0, 1\}$ indicate the period-t reference model. In particular, assume $s_t = 1$ if Model 1 is the current reference model. Suppose Model 1 is characterized by a parameter vector, β_1, and a state vector, X_1. Analogous definitions apply to Model 2. Further suppose that the agent's control problem is LQG, so that the policy function pertaining to each model is linear, and can be written as $g_i(\beta_i)X_i$. Defining the stacked vectors, $\beta = (\beta_1, \beta_2)'$ and $X = (X_1, X_2)'$, then delivers the following recursive representation of the full model validation process:

$$
\begin{aligned}
\beta_{t+1} &= \beta_t + \gamma h(\beta_t, X_t, s_t, \varepsilon_{t+1}) \\
X_{t+1} &= A(\beta_t, s_t)X_t + Bv_{t+1} \\
\psi^d_{t+1} &= (1-\gamma)^{-1} \cdot [\log(\hat{\sigma}^2_{2t}) - \log(\hat{\sigma}^2_{1t})] + 2(k_2 - k_1) \\
\hat{\sigma}^2_{it+1} &= \hat{\sigma}^2_{it} + \gamma(\hat{\varepsilon}^2_{it+1} - \hat{\sigma}^2_{it}) \\
s_{t+1} &= s_t + (1 - s_t) \cdot \mathcal{I}(\psi^d_t > \rho) - s_t \cdot \mathcal{I}(\psi^d_t < \rho)
\end{aligned}
$$

where $\mathcal{I}(\cdot)$ is an indicator function for the event in question, and $h(\cdot)$ is defined by a set of least-squares orthogonality conditions. Interestingly, one can show that if the transition rate of s_t is $O(\gamma)$, then this process converges to a so-called 'switching diffusion' process (see, e.g., Yin and Zhang (2005)). This transition rate is governed by the testing threshold, ρ. In principle, one could use approximations like those described in Siegmund (1986) to tune this transition rate. However, we leave this difficult problem for future research. Instead, we now turn to two simple examples of our model validation framework.

3. Example I: Sargent's (1999) *Conquest* Model

In this section we apply our model validation framework to Sargent's (1999) well know model of learning about the Phillips Curve. The analysis proceeds in two steps. We first ask what happens when $\rho \downarrow 0$. We then consider the case of strictly positive ρ.

3.1. **True Model.**

Assume the government selects a target inflation rate, x_t, to minimize the social cost of realized inflation, y_t, and unemployment, u_t:

$$\min_{x_t} \mathsf{E}(1-\delta)\sum_{t=1}^{\infty}(y_t^2 + u_t^2)\delta^{t-1} \tag{3.6}$$

for a given discount factor $\delta \in (0,1)$. The true law of motion governing $\{u_t, y_t\}$ is

$$\begin{aligned} u_t &= u^* - \theta(y_t - x_t^e) + v_{1t} \\ y_t &= x_t + v_{2t} \\ x_t^e &= x_t \end{aligned} \tag{3.7}$$

where x_t^e is the expectation of the private sector. The last condition is referred to as the 'Fed watcher' assumption.[14] It is motivated by the idea that the private sector knows the government's policy rule.

We assume that the perturbation terms $\{v_{1t}, v_{2t}\}$ are i.i.d. over time. To simplify the exposition, we assume that the distribution is Gaussian so that our model is embedded in the class of linear quadratic Gaussian (LQG) models.

Assumption 3.1. *$\forall i$, $v_{it} \sim N(0, \sigma_i^2)$ and i.i.d.*

We study this model under two alternative assumptions about the government's behavior. First, we follow Sargent (1999) and assume the government adopts a constant gain learning algorithm. Next we assume the government adheres to our previously outlined model validation strategy.

3.2. **Recursive Learning.** The government does not know the true law of motion, especially (3.7). Instead, the government's perception of the link between u_t and y_t is confined to the class of linear models $\overline{\mathcal{M}}$, having at most $\overline{q}$ variables other than the constant term and y_t. $\overline{\mathcal{M}}$ represents the limited knowledge of the government. While the government does not have a precise idea about the true law of motion, the models in $\overline{\mathcal{M}}$ are believed to be a reasonable approximation of the true law of motion. As we do *not* require apriori that the true law of motion is contained in $\overline{\mathcal{M}}$, the government's model can be misspecified. As a result, $\overline{\mathcal{M}}$ need not have the 'grain of truth' property, and so we cannot invoke Bayesian learning to conclude the government learns the true law of motion in the long run (Kalai and Lehrer (1993)).

For simplicity, we focus on the case where $\overline{\mathcal{M}}$ consists of models that contain at most $\overline{q}$ lags of inflation. Although adding more lagged variables helps the government fit the model to the data, at the same time it makes the model more complex and so makes out-of-sample forecasting less reliable. Our focus is on how the government strikes a balance between model fit (bias) and out-of-sample forecast reliability (variance).

A generic element of $\overline{\mathcal{M}}$ can be written as follows:

$$u_t = \hat{\gamma}_{0t} + \hat{\gamma}_{1t} y_t + \sum_{\ell=1}^{q} \hat{\lambda}_{tl} y_{t-\ell} + \xi_t \tag{3.8}$$

for some $q \in \{0, 1, \ldots, \overline{q}\}$, or more compactly,

$$u_t = \hat{\beta}_t' Z_t + \xi_t$$

[14] In Cho and Kasa (2006), we show that this assumption does not alter the model's large deviations properties.

where $Z_t = (1, y_t, y_{t-1}, \dots, y_{t-q})'$ and ξ_t is a regression residual. In addition to the functional form, the government assumes that

$$\xi_t \sim N(0, \sigma_\xi^2)$$

and independent over $t \geq 1$.[15] To simplify notation, we index the government's model by the coefficient vector $\hat{\beta}_t$.

Given these assumptions, it is optimal to estimate the coefficients by least squares. Again following Sargent (1999), we assume the government responds to detected parameter drift by discounting old data. Discounted least squares can be implemented recursively as follows:

$$\hat{\beta}_{t+1} = \hat{\beta}_t + aR_t^{-1} Z_t \hat{\epsilon}_t \tag{3.9}$$

$$R_{t+1} = R_t + a\left[Z_t Z_t' - R_t\right] \tag{3.10}$$

where

$$\hat{\epsilon}_t = u_t - \hat{\beta}_t' Z_t$$

and $a > 0$ is a constant gain parameter. Given $\hat{\beta}_t$, the government sets x_t by solving (3.6).

Let

$$\dot{\hat{\beta}} = \hat{\Psi}(\hat{\beta}) \tag{3.11}$$

be the associated ordinary differential equation (ODE), and $\hat{\beta}^s$ be the stationary solution of the ODE:

$$\hat{\Psi}(\hat{\beta}^s) = 0.$$

We call $\hat{\beta}^s$ a self-confirming equilibrium. Given $\hat{\beta}_t$ with q lagged inflations, the government sets the target

$$b^r(Z_t, \hat{\beta}_t) = g(\hat{\beta}_t) \cdot Z_t \tag{3.12}$$

where $g(\hat{\beta}_t)$ is computed using standard (discounted) LQR formulas. From Sargent (1999) and Cho, Williams, and Sargent (2002), we know the learning dynamics converge to the self-confirming equilibrium: $\forall \delta > 0$,

$$\lim_{a \to 0} \lim_{t \to \infty} \mathsf{P}\left(|\hat{\beta}_t - \hat{\beta}^s| > \delta\right) = 0.$$

3.3. **Simulations.** Before plunging into general analysis, it is helpful to see sample paths of the validation dynamics, and compare them to those generated from the learning dynamics. The government considers two classes of potential models: a static one,

$$u_t = \gamma_{10} + \gamma_{11} y_t + \xi_{1t}$$

which we refer to as Model 1, and a dynamic model in (3.8), which we repeat here for convenience

$$u_t = \gamma_{20} + \gamma_{21} y_t + \sum_{\ell=1}^{\overline{q}} \lambda_{tl} y_{t-\ell} + \xi_{2t}$$

We refer to the dynamic model as Model 2.

[15] This is where the government's model is misspecified. The condition fails, because y_t is not independent of u_t.

In the simulation we set $\bar{q} = 2$, so that the dynamic model contains two lags of inflation. We also set $\rho = 1.1$, so that the reference model is maintained unless the the probability of the alternative is at least 10% higher.

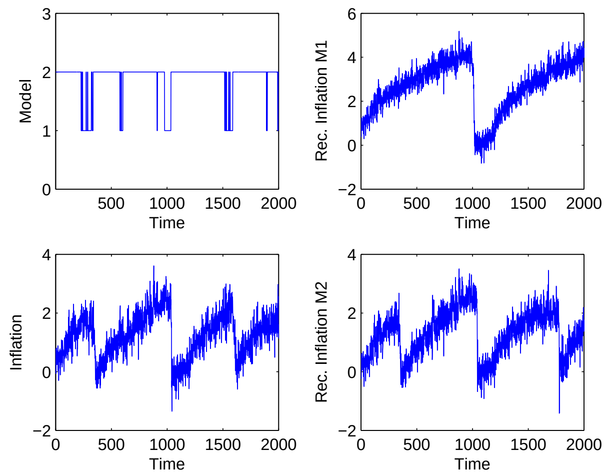

FIGURE 1. Validation and Recursive Learning Dynamics: $a = .02$, $\rho = 1.1$

The left-half of Figure 1 plots the validation dynamics. The upper half plots the sequence of selected models and the bottom half plots the resulting inflation path. The right-hand side of Figure 1 then reports the standard constant gain recursive learning dynamics. The upper half plots inflation when a simple static model is used, and the bottom half plots inflation when a more complex dynamic model is used. These two figures echo the results from Cho, Williams, and Sargent (2002). In particular, their simulations also revealed that bigger and more complex models appeared to escape more frequently. Later we shall prove why this is.

3.4. **Findings and Interpretations.** Three stylized facts emerge from these plots. First, the government uses the more complex model (ie., Model 2) most of the time. Second, although it is a little hard to tell from Figure 1, it turns out that the government switches to the simple model (Model 1) as inflation approaches the self-confirming equilibrium (which turns out to be 5%). This is illustrated more clearly in Figure 2, which provides a close-up of the escape taking place around observation 1050 in Figure 1. Evidently, a switch to Model 1 takes place about 50 periods before the escape, just as the model reaches the self-confirming equilibrium. Then, once an escape is ignited, the government switches back to Model 2. Finally, the third feature revealed in these plots is that the model validation dynamics closely resemble the recursive learning dynamics of the more complex model. In particular, the *apparent* frequency of escapes are quite similar.

The similarity between the model validation dynamics and the recursive learning dynamics of Model 2 is a little more subtle and warrants more discussion. As noted by CWS (2002), a remarkable feature of the learning dynamics is that the dynamic system has the

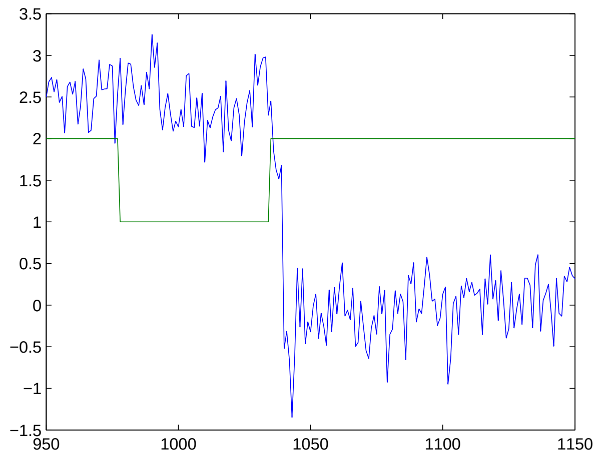

FIGURE 2. Escape Dynamics and Model Selection

self-confirming equilibrium as the globally stable point, but the constant gain learning algorithm produces repeated, long excursions away from the self-confirming equilibrium, as shown in Figure 1. The dynamics toward the self-confirming equilibrium (mean dynamics) are fundamentally different from the dynamics away from the self-confirming equilibrium (escape dynamics). The mean dynamics are characterized by an associated ordinary differential equation (ODE), which essentially characterizes the trajectory of the center of the distribution (i.e., mean) of the sample paths. On the other hand, because episodes of escape from the stable equilibrium are rare events when the gain sequence $a > 0$ is close to zero, the escape dynamics are characterized by the large deviation properties (LDP) around the self-confirming equilibrium.

When the government runs a fairly tight specification test, the learning dynamics differ little from the validation dynamics along the convergent path toward the self-confirming equilibrium, as shown in Figure 1. In fact, it is fairly straightforward to prove that the associated ODE of the learning dynamics is close to the associated ODE of the validation dynamics if the government imposes no complexity cost ($\mathcal{C}(q,t) = 0$).

However, the frequency of escapes from the self-confirming equilibrium under the recursive learning dynamics (right-hand panels in Figure 1) are evidently different from the escape frequency under the validation dynamics (lower-left panel in the same figure). Later we show that the frequency from the self-confirming equilibrium in fact decreases as the number of explanatory variables decreases. Hence, escapes under the validation dynamics occur less frequently than under the learning dynamics (except for the smallest model). As a result, the inflation exhibits more inertia around the self-confirming equilibrium under the validation dynamics than under the recursive learning dynamics.

Interestingly, it turns out that the escape dynamics of Model 2 (those with lagged variables) are completely irrelevant when it comes to explaining the model validation dynamics. Instead, Model 2 contributes only the dynamics toward the self-confirming equilibrium. Around the self-confirming equilibrium, the government switches to Model

1 (those without lagged variables, and therefore simpler models). Away from the self-confirming equilibrium, however, there are autocorrelated forecast errors. Adding lags to the model helps to soak up these forecast errors and improves the fit of the model. As long as the complexity cost is not too high, the decision maker will use Model 2 outside a neighborhood of the SCE to reduce the forecasting errors. The size of this neighborhood where the government opts for a simpler model is determined by the magnitude of the complexity cost. The switch to Model 1 in the neighborhood of the SCE is explained by the fact that the true model contains no dynamics and that around the self-confirming equilibrium, inflation is moving around a constant, which can be explained by the static model well. Hence, given the complexity cost, the government will always prefer the simpler model in the vicinity of the self-confirming equilibrium.

Finally, one issue that is not clearly revealed by this simulation is how the validation sample paths depend on the complexity cost function. Remember that AIC implicitly embodies the cost function

$$\mathcal{C}(q,t) = \frac{2\cdot(3+q)}{t}$$

where q is the number of lags in the model. The key feature of this function is that it decreases with sample size. This means that, eventually, if we run the simulations for a large enough number of periods, the validation dynamics will converge to the recursive learning dynamics associated with the biggest model. This partially explains the similarity between the left- and right-hand sides of the lower half of Figure 1. To reveal the role of complexity cost more clearly, we also ran simulations with the following ad hoc cost function:

$$\mathcal{C}(q,t) = \frac{5\cdot(3+q)}{\sqrt{t}}$$

which imposes a higher cost on model complexity. It turns out that the 5 in the numerator does not really matter. What matters is that the complexity cost falls much more slowly with sample size. Figure 3 compares the validation inflation paths for these two cost functions.

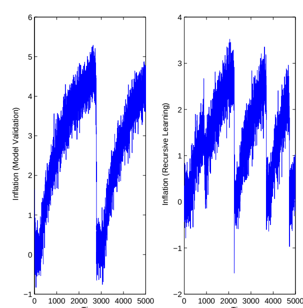

Figure 3. Validation vs. Learning with Greater Complexity Cost

The left-hand plot pertains to the higher cost function. Evidently, referring back to Figure 1, now the validation dynamics more closely resemble the learning dynamics of the *smaller* model, i.e. Model 1.

3.5. **Formal Analysis.** While the behavioral assumptions of model validation are more plausible than the assumptions of recursive learning, the dynamics pose a number of challenges. Because the government updates its model intermittently, the sample path of target inflation resembles a step function. In particular, a model's survival time is random. As a result, when the threshold level $\rho > 0$ is large, the sample path is discontinuous, which hinders formal analysis. The complexity cost aggravates this already difficult situation. As the government balances the benefit of a richer model against its additional complexity, the number of non-zero coefficients in the model changes, which produces another form of discontinuity in the actions of the government.

In order to highlight the impact of the complexity cost on the decision making process of the government, we first examine the case where the government ignores model complexity (subject to an a priori upper bound on the dimension of its model). Although the sample path induced by the validation dynamics is intrinsically discontinuous, we show that as the government runs a more stringent validation test, by reducing $\rho > 0$, the resulting sample paths converge to the sample paths of the recursive learning dynamics. Therefore, we can analyze the asymptotic properties of the validation dynamics in reference to the recursive learning dynamics, including the stability and large deviation properties. In the second step, we introduce the complexity cost to see how the validation dynamics are affected. More explanatory variables help the validation dynamics converge to the self-confirming equilibrium, but also make it easier to escape from the self-confirming equilibrium. This observation is substantiated by showing that the rate function of the richer model is smaller than the rate function associated with a more parsimonious model.

Because we are comparing the sample paths generated by two different dynamics, it is helpful to lay out the notational convention to differentiate the two before the analysis. We add '^' to the key variables that are induced by the recursive learning dynamics. The coefficient with " corresponds to the validation dynamics. For example, by $\hat{\beta}_t$, we mean the coefficient vector of the model at t updated by the recursive learning algorithm and Z_t as the coefficient vector induced by β_t in period t.

The validation dynamics require additional notation, because the target inflation is set by a model selected earlier. Define $\{t_k\}_{k=1}^{\infty}$ to keep track of how long a model survives the validation test. Let $t_0 = 1$. Define t_1 as the last period when the model selected in period 1 is used. Given $t_1, \ldots, t_{k-1}$, the government has updated the model k times. Let t_k be the last time when the k-th model was used before it is invalidated at the end of period t_k. Thus, the k-th model last $t_k - t_{k-1} + 1$ periods, which is a random variable parametrized by (a, ρ). To emphasize this link, we sometimes write $t_k(a, \rho)$ in place of t_k. By subscript $_{kt}$ we mean a variable at t generated by the k-th model. For example, let β_{t_k} be the k model selected by the government. The target inflation at period t is $b^r(\beta_{t_k})$ and the realized inflation is $b^r(\beta_{t_k}) + v_{2t}$. We write y_{kt} in place of y_t to illuminate the role of β_{t_k} and Z_{tk} for the vector of explanatory variables induced by β_{t_k}.

3.6. **Case 1: $\mathcal{C}(q,t) = 0$.** It is straightforward to see that, as $\rho \to 0$, the sample paths of validation and recursive learning have the same associated ODE. Recall that $\dot{\hat{\beta}} = \hat{\Psi}(\hat{\beta})$ is the ODE for the recursive learning dynamics. Similarly, we can define the ODE for the validation dynamics as

$$\dot{\beta} = \Psi^{\rho}(\beta)$$

parameterized by threshold $\rho > 0$. Since updating occurs intermittently, Ψ^ρ does not have a continuous trajectory. Instead, its trajectory looks more like a step function, with the average length of each step determined by $\rho > 0$. The substance of our first result is that if β is not a self-confirming equilibrium, then the step function converges to the trajectory induced by Ψ as $\rho \to 0$.

Proposition 3.2. *$\forall\beta$ where $\hat{\Psi}(\beta) \neq 0$.*

$$\lim_{\rho\to 0} \hat{\Psi}(\beta) - \Psi^\rho(\beta) = 0.$$

Because the proof is straightforward, we only sketch the main idea. Fix β, which is not a self-confirming equilibrium. If the short term Phillips curve is a reasonable approximation of the actual law of motion, then the government expects (u_t, y_t) to be distributed along the short term Phillips curve. However, because β is *not* a self-confirming equilibrium, the new data (u_t, y_t) are instead generated according to a distribution centered around $(u^*, b^r(\hat{\beta}))$ which is not located along the short term Phillips curve. As the government observes more data, relative entropy will exceed the threshold in a finite number of periods with probability 1. For small $a > 0$, the interval that a reference model survives converges to a deterministic time. This time interval shrinks as the decision maker runs a tighter specification test: $\rho \to 0$. Thus, the reference model is updated frequently as in the recursive learning model.

Proposition 3.2 remains silent about the tail portion of the distributions of the sample paths. In particular, it does not imply that the large deviation properties (LDP) of the recursive learning dynamics converge to LDP of the validation dynamics, which is crucial for the frequency of escapes from the self-confirming equilibrium. We need a stronger result than Proposition 3.2, ensuring the two dynamics not only share the same properties along the mean, but also in the tails of the distribution over sample paths. To this end, we shall prove that the models induced by the validation dynamics and the recursive learning dynamics are exponentially equivalent, from which Proposition 3.2 follows as a corollary.

Since the relative entropy test can detect deviations in any moments, $\forall\rho > 0\ \exists\rho' > 0$ such that $H(\beta|\epsilon^t) > \rho$ whenever $|\sum_{s=1}^t u_s - Z_{ks}\beta|/t > \rho$. Moreover, as $\rho \to 0$, we can choose small $\rho' > 0$.

In order to emphasize the underlying model that generates the data, we add subscript k to the variables. For example, by Z_{kt}, the data in period t is generated by the k-th model. By the definition of t_k, we can write

$$\beta_{t_{k+1}} = \beta_{t_k} + a(t_{k+1} - t_k)R^{-1}_{kt'}Z_{kt'}\left(\frac{1}{t_{k+1} - t_k}\left[\sum_{t'=t_k+1}^{t_{k+1}} y_{kt'} - Z_{kt'}\beta_{t_k}\right]\right)$$

Define $a_k = a(t_{k+1} - t_k)$ and construct the fictitious time scale according to $\{a_k\}$. Define $\beta(\tau)$ as the continuous time process obtained by interpolating the outcome of β_{t_k} with respect to the fictitious time scale. For $K > 0$, define $\beta^K(\tau) = \beta(K + \tau)$ as the left shifted process of $\beta(\cdot)$. We can repeat exactly the same manipulation for $\hat{\beta}_t$ with respect to the same fictitious time scale constructed for the validation dynamics. Let $\hat{\beta}^K(\tau)$ be the resulting left-shifted process of $\hat{\beta}$, which is the continuous time process obtained by interpolating $\hat{\beta}_t$.

The next result shows that following any history, the sample paths induced by the two different dynamics must remain very close as the government runs an increasingly stringent validation test.

Proposition 3.3. *$\forall K$, $\forall \mu > 0$, $\exists \tau' > 0$ such that $\forall \tau \in (0, \tau')$, $\exists a(\tau)$ such that $\forall a \in (0, a(\tau))$,*

$$\mathsf{P}\left(|\beta^K(\tau) - \hat{\beta}^K(\tau)| > \mu \mid \beta^K(0) = \hat{\beta}^K(0)\right) = 0.$$

Proof. See Appendix. □

Remark 3.4. *Notice that the probability that the two sample paths differs must be zero rather than merely converging to zero. Thus, if the gain function is smaller than $a(\tau)$, then the two sample paths are indistinguishable in any reasonable sense.*

The two different dynamics share the same LDPs as $\rho \to 0$, because they generate practically identical sample paths. This equivalence result provides a nice behavioral foundation for recursive learning models, which have been criticized for the implicit assumption that agents do not question the validity of their models. We can now view recursive learning as a reduced form of model validation, in the special case where agents do not face a complexity cost. Even if agents do not suspect the functional form of the model, they can still test the validity of the parameter values, which provides them an opportunity to revise their models. This equivalence is remarkable in the sense that two very different behavioral assumptions induce not only the same sample paths toward the self-confirming equilibrium, but also induce the same pattern of escapes from the self-confirming equilibrium.

3.7. **Case 2: $\mathcal{C}(q, t) > 0$.** A natural question at this point is how the complexity cost influences the validation dynamics. Subject to a small complexity cost and a finite data record, a policy maker has to balance a better forecast from a larger model against the greater precision of the forecast from a smaller model. From a purely descriptive viewpoint, the complexity cost captures an important aspect of the model building process. While a policy maker explores different models by adding and subtracting different variables, the selection of a new explanatory variable is subject to a careful test before it is included into a new model such as how much the new variable contributes to reducing the mean forecasting error and how much complexity of computation the new variable incurs.

3.7.1. *Preliminaries.* Let us for a moment suppress the endogenous model selection. Instead, we shall compare the various properties of the learning algorithms with different numbers of explanatory variables. Given the benchmark model

$$u_t = \beta' Z_t + \xi_t$$

where $Z_t = (1, y_t, y_{t-1}, \ldots, y_{t-\overline{q}})'$ and ξ_t is the regression residual, the coefficient vector β is estimated recursively according to (3.9).

Consider a matrix

$$\mathbf{B} = [\mathbb{I} \quad \mathbb{O}]$$

where $\mathbb{I}$ is the $(q+2) \times (q+2)$ identity matrix, and $\mathbb{O}$ is a $(q+2) \times (\overline{q} - q)$ zero matrix. Note that $\mathbf{B}Z_t$ extracts the first $q + 2$ elements of the explanatory variables.

Let us consider a model with fewer than $\overline{q}$ lag variables:

$$u_t = \beta^{q\prime}\mathbf{B}Z_t + \xi_t^q$$

where β^q contains $q+2$ coefficients and ξ_t^q is the regression residual. Let $\hat{\beta}_t^q$ be the estimator for β^q based on the information up to period t.

$$\hat{\beta}_{t+1}^q = \hat{\beta}_t^q + a(R_t^q)^{-1}\mathbf{B}Z_t\hat{\epsilon}_t^q \tag{3.13}$$

$$R_{t+1}^q = R_t^q + a\left[(\mathbf{B}Z_t)(\mathbf{B}Z_t)' - R_t^q\right] \tag{3.14}$$

where

$$\hat{\epsilon}_t^q = u_t - \hat{\beta}_t^q(\mathbf{B}Z_t).$$

Note that $\hat{\beta}_t^{\overline{q}} = \hat{\beta}_t$. We examine how the large deviation properties of $\hat{\beta}_t^q$ is affected by q. To this end, we need more notation.

Define the H-functional as

$$\Lambda(\alpha, \beta^q, t) = \limsup_{\tau\to 0}\limsup_{a\to 0}\frac{a}{\tau}\log \mathsf{E}\left[\exp\langle\alpha, \sum_{k=1}^{\lceil\tau/a\rceil}(R_t^q)^{-1}\mathbf{B}Z_t\hat{\epsilon}_t^q\rangle \mid \hat{\beta}_t^q = \beta^q, \mathcal{H}_t\right], \tag{3.15}$$

where $\mathcal{H}_t$ is the sigma algebra generated by information at t. The Legendre transform of the H-functional is defined as

$$\Lambda^*(\beta^q, \zeta, t) = \sup_\alpha\left[\langle\alpha, \zeta\rangle - \Lambda(\alpha, \beta, t)\right] \tag{3.16}$$

and the action functional is then defined as

$$S^q(\beta^q, T, \phi) = \int_0^T \Lambda^*(\dot{\phi}, \phi, t)dt \tag{3.17}$$

where $\phi(0) = \beta^q$ and ϕ is absolutely continuous; otherwise, $S(\beta, T, \phi) = \infty$. Given $\delta > 0$, define

$$D^q = \{\phi : |\phi(0) - \beta^q| < \delta, \quad \exists t < \infty, \phi(t) \notin G(\beta^q)\}$$

where $G(\beta^q) \subset \mathbb{R}^{q+2}$ is an open ball around β^q with radius $\delta > 0$. The key large deviation parameter is obtained by solving

$$S^*(\beta^q, T) = \inf_{\phi\in D^q} S^q(\beta, T, \phi).$$

Dupuis and Kushner (1989) proved that

$$S^*(\beta^q, T) > 0$$

and the sample path in $\phi \in D^q$ that solves the deterministic minimization problem in the most likely escape path from β^q to the boundary of $G(\beta^q)$. The frequency of escape from β^q vanishes at the rate of $e^{-aS^*(\beta^q,T)}$.

3.7.2. *Escape Probability.* Note that the number of arguments in $S^*(\cdot, T)$ changes as q changes, which makes the direct comparison of $S^*(\beta^q, T)$ to $S^*(\beta^{q+1}, T)$ difficult. We shall construct an artificial system which serves as an intermediate step to facilitate the comparison. $S^*(\beta^q, T)$ is built around the H-functional $\Lambda(\alpha, \beta^q, t)$. Thus, if two stochastic processes share the same H-functional, then they have the identical large deviation properties.

Recall the recursive learning algorithm for $\hat{\beta}_t^q$:

$$\begin{aligned} \hat{\beta}_{t+1} &= \hat{\beta}_t + aR_t^{-1}Z_t\hat{\epsilon}_t \\ R_{t+1} &= R_t + a\left[Z_tZ_t' - R_t\right] \end{aligned}$$

where

$$Z_t = \begin{bmatrix} 1 \\ x_t + v_{2t} \\ x_{t-1} + v_{2t-1} \\ \vdots \\ x_{t-\overline{q}} + v_{2t-\overline{q}} \end{bmatrix}.$$

Let us consider an artificial learning algorithm obtained by assuming

$$v_{2t-\overline{q}} = \ldots = v_{2t-q+1} = 0$$

or equivalently, by suppressing the perturbations from the lag variables beyond q periods. Let us denote $\tilde{\beta}_t$ as the resulting coefficient:

$$\tilde{\beta}_{t+1} = \tilde{\beta}_t + a\tilde{R}_t^{-1}Z_t\hat{\epsilon}_t \tag{3.18}$$

$$\tilde{R}_{t+1} = \tilde{R}_t + a\left[Z_tZ_t' - \tilde{R}_t\right]. \tag{3.19}$$

This algorithm is not quite well defined, because $\tilde{R}_t$ may not be invertible. By the same token, the associated H-functional for $\tilde{\beta}_t$ is not well defined. In this case, we have to calculate the generalized inverse of $\tilde{R}_t$ by focusing on the first $q+2$ variables. The resulting H-functional of $\tilde{\beta}_t$ is precisely the H-functional of $\hat{\beta}_t^q$.

Let $S^*(\tilde{\beta}_q, T)$ be the action functional associated with $\tilde{\beta}_q$. Because $\tilde{\beta}_q$ is driven by $q+2$ perturbations, the set of escape paths out of β must be embedded in $\mathbb{R}^{q+2}$, which itself is a subspace of $\mathbb{R}^{\overline{q}+2}$. Recall that

$$S^{\overline{q}}(\beta^{\overline{q}}, T, \phi) = \int_0^T \Lambda^*(\dot{\phi}, \phi, t)dt \tag{3.20}$$

where $\phi(0) = \beta^{\overline{q}}$ and ϕ is absolutely continuous; otherwise, $S(\beta, T, \phi) = \infty$. Given $\delta > 0$, define

$$D^q = \{\phi : |\phi(0) - \beta^q| < \delta, \;\; \exists t < \infty, \;\; \phi(t) \notin G(\beta^q)\}.$$

Since any escape path outside of the subspace $\mathbb{R}^{q+2}$ makes the value of the action functional ∞, the minimum of the action functional must be achieved, if at all, within $\mathbb{R}^{q+2}$ instead of $\mathbb{R}^{\overline{q}+2}$. This feature has the same effect as restricting the domain of minimization from $\mathbb{R}^{\overline{q}+2}$ to $\mathbb{R}^{q+2}$ in (3.20). Since the domain of minimization is smaller under $\tilde{\beta}_t^q$,

$$S^*(\tilde{\beta}^q, T) \geq S^*(\beta^{\overline{q}}, T).$$

By repeating the same logic combined with

$$S^*(\beta^q, T) = S^*(\tilde{\beta}^q, T),$$

we obtain the monotonicity of the action functional with respect to q, which implies that the model with fewer explanatory variables have exponentially smaller probability of escape from the self-confirming equilibrium than the model with more explanatory variables.

Proposition 3.5. *$\forall q \in \{0, \ldots, \overline{q}\}$, $\forall T > 0$,*

$$S^*(\beta^q, T) \geq S^*(\beta^{q+1}, T).$$

This result explains the difference between the recursive learning sample paths in Figure 1.

3.8. **Model Reduction.** It remains to show that around a small neighborhood of the self-confirming equilibrium, the validation dynamics selects a model with fewer explanatory variables than the recursive learning models. Combined with Proposition 3.5, we can then conclude that validation dynamics exhibit less frequent escapes from the self-confirming equilibrium.

We can decompose the model selection process into two steps. First, among models with q lagged inflation, the government solves

$$\min_{\beta^q \in \mathbb{R}^{q+2}} H(\epsilon^t | \beta)$$

to find β_t^q for a given sequence ϵ^t of forecasting errors. Then, the government solves

$$\min_{0 \leq q \leq \overline{q}} H(\epsilon^t | \beta_t^q) + \mathcal{C}(q, t)$$

to select the reference model in period t, in case that the status quo model is invalidated.

Let us examine more carefully the first step of the model selection. Under the Gaussian assumption on v_{it} $\forall i, \forall t$,

$$\min_{\beta^q \in \mathbb{R}^{q+2}} H(\epsilon^t | \beta)$$

is equivalent to

$$\min_{\beta^q \in \mathbb{R}^{q+2}} \mathsf{E} \frac{1}{2} \sum_{s=1}^{t} \delta^{t-1} (u_s - Z_s' \beta^q)^2.$$

Obviously,

$$\min_{\beta^{\overline{q}} \in \mathbb{R}^{\overline{q}+2}} \mathsf{E} \frac{1}{2} \sum_{s=1}^{t} \delta^{t-1} (u_s - Z_s' \beta^{\overline{q}})^2 \leq \min_{\beta \in \mathbb{R}^{\overline{q}+2}} \mathsf{E} \frac{1}{2} \sum_{s=1}^{t} \delta^{t-1} (u_s - Z_s' \beta)^2 \tag{3.21}$$

subject to the constraint that the last $\overline{q} - q$ components of β must be 0:

$$\beta_{q+3} = \cdots = \beta_{\overline{q}+2} = 0.$$

Given an optimal solution $\beta_t^{\overline{q}} = (\gamma_0, \gamma_1, \lambda_1, \ldots, \lambda_{\overline{q}})$ of the minimization problem, we can write the sum of the forecasting errors as

$$\sum_{s=1}^{t} \left(u_s - \gamma_0 - (\gamma_1 y_s + \sum_{\ell=1}^{\overline{q}} \lambda_\ell y_{s-\ell}) \right) + \mathbf{R} \tag{3.22}$$

where

$$\mathbf{R} = \sum_{\ell=1}^{\overline{q}} \sum_{\ell'=1}^{\ell} \lambda_{k\overline{q}-\ell'+1} \left(y_{t-\overline{q}+\ell} - y_{-\overline{q}+\ell} \right).$$

Except for a few elements at the beginning and at the end of the summation, we can represents the summation of the forecasting errors as those from the model without lagged variables. Moreover, because $\overline{q} < \infty$, we only have a finite number of terms left over, which are collected in the second summation.

Let

$$\tilde{\beta}_t = \left(\gamma_0, \gamma_1 + \sum_{\ell=1}^{\overline{q}} \lambda_\ell y_{t-\ell}, 0, \ldots, 0 \right) \in \mathbb{R}^{\overline{q}+2}$$

which is a feasible, if not optimal, solution. Hence,

$$H(\epsilon^t | \beta_t^{\overline{q}}) \leq H(\epsilon^t | \tilde{\beta}_t).$$

By (3.22), $\exists M > 0$ such that

$$H(\epsilon^t | \tilde{\beta}_t) \leq H(\epsilon^t | \beta_t^{\overline{q}}) + \frac{M\mathbf{R}^2}{t}.$$

Since $\tilde{\beta}_t$ is also a feasible solution for the constrained minimization problem, we have

$$H(\epsilon^t | \beta_t^0) \leq H(\epsilon^t | \beta_t^{\overline{q}}) + \frac{M\mathbf{R}^2}{t}.$$

Applying the same logic, we conclude that $\forall q \in \{0, \ldots, \overline{q} - 1\}$,

$$H(\epsilon^t | \beta_t^{q+1}) \leq H(\epsilon^t | \beta_t^q) \leq H(\epsilon^t | \beta_t^q) + \frac{M\mathbf{R}^2}{t}.$$

Thus, a smaller model is favored if

$$\frac{M\mathbf{R}^2}{t} < C(q+1, t) - C(q, t).$$

In particular, the static model without the lagged inflation is selected if

$$\frac{M\mathbf{R}^2}{t} < C(1, t) - C(0, t).$$

In case of AIC, this condition is equivalent to

$$\mathbf{R}^2 \leq \frac{2}{M}$$

and in case of BIC,

$$\mathbf{R}^2 \leq \frac{\log t}{M}.$$

If $C(\overline{q}, t)$ vanishes at a slower rate than $1/t$ as in BIC, we can show that as the gain sequence $a > 0$ converges to 0, the government selects the static model without a lagged inflation in the neighborhood of the self-confirming equilibrium.

Proposition 3.6. *Suppose that*

$$\frac{\mathcal{C}(\overline{q}, t)}{t} \to \infty.$$

Then, $\forall \delta > 0$*,*

$$\lim_{a \to 0} \mathsf{P}\left(\forall t \geq T, \quad \beta_t = \beta^0 \mid \beta_T \in G(\delta)\right) = 1$$

where $G(\delta)$ *is the* δ *neighborhood of the self-confirming equilibrium, and* β^0 *is the coefficient vector of the static model.*

Proof. See Appendix. □

It is interesting to note that this condition on the complexity cost function is exactly one of the conditions identified by Sin and White (1996) as being necessary for consistent model selection. If the complexity cost vanishes at the same rate as $1/t$ as in AIC, then the conclusion is ambiguous. However, if v_{it} has a small second moment,

$$\mathbf{R}^2 \leq 2\overline{q}$$

can be satisfied. Our simulation reported in subsection 3.3 uses the AIC type complexity cost which reveals that around the self-confirming equilibrium, the government switches to a simpler static model.

3.9. **Cogley and Sargent Puzzle.** Cogley and Sargent (2004) suggest that the Fed actually began to suspect the short-term Phillips curve was misspecified by the mid-1970s, several years *before* the Volker disinflation. Cogley and Sargent attribute this policy inertia to Bayesian model uncertainty. The Fed stuck to a high inflation policy because it risk-dominated the policy implied by the better-fitting model. Our result provides an alternative interpretation of policy inertia. Even if a model is rejected, it is typically the case that a similar model is the best fitting model. That is, even though a tail-sensitive test, like our relative entropy test, may detect a change, estimators like least squares or maximum likelihood, which focus on fitting the center of the distribution, will typically dictate a modest model revision. Drastic policy changes only take place at self-confirming equilibria, where model rejections are surprising, and therefore, informative. In particular, if the monetary authority constructs a model by judiciously selecting the explanatory variables, then the escape from the self-confirming equilibrium would be slower than otherwise. In our model, the inertia is a consequence of bounded rationality of the decision maker who is trying to economize his limited computational capability in building and forecasting the short term Phillips curve.

4. Example II: Stability of Restricted Perceptions Equilibria

The previous example featured a subtle form of model misspecification, based on the government misinterpreting the parameter drift in its model. Discovering this misspecification would require the government to rethink the way it identifies its model, using cross-equation restrictions.

Our second example features a cruder, but perhaps more general, kind of model misspecificaton, i.e, the exlusion of relevant variables. This case has been studied by Evans and Honkapohja (2001). They argue that even in this case we can think of agents as revising the coefficients of their misspecified models so as to obtain a (constrained) statistically optimal fit. They call these constrained-optimal models *Restricted Perceptions Equilibria* (RPE).

There are many reasons why agents might exclude relevant variables. For example, the necessary data might not be available, or suitable proxies might be difficult to construct. In this case, model validation will not make much difference. However, Evans and Honkapohja also motivate RPE as a response to insufficient degrees of freedom. Here validation could make a difference, since the resolution of bias-variance trade-offs is an endogenous outcome of the model validation process.

In this section, we study an RPE example from Chapter 13 (pgs. 320-322) in Evans and Honkapohja (2001). They show that misspecification can alter a model's E-stability conditions. However, they did not allow agents to test for misspecification. This raises the question of whether RPE can persist under model validation. We show that two key factors determine the stability of RPE: (1) Whether the correct model is in the agent's original set of models, $\overline{\mathcal{M}}$, and (2) The form of the complexity cost function. If $\overline{\mathcal{M}}$ contains the true model, and the complexity cost is $o(T^{-1})$, then RPE are necessarily unstable; the true model will be discovered with probability one. For an RPE to be stable, the complexity cost must continue to dominate the improved fit obtainable by using the true model. On the other hand, if $\overline{\mathcal{M}}$ does not contain the true model, then (by assumption) RPE can persist, even with standard complexity costs, but only those models that achieve minimum (expected) KLIC are viable candidates for a stable RPE.

4.1. **A Macroeconomic Workhorse.** The following expectational difference equation shows up all the time in macroeconomics, e.g., in linear-quadratic adjustment cost models.

$$y_t = \alpha + \beta E_t^* y_{t+1} + \delta y_{t-1} + v_t \tag{4.23}$$

where for simplicity v_t is presumed to be i.i.d./mean zero. The E^* notation reminds us that we do not necessarily assume Rational Expectations. However, this is a useful place to start. Under Rational Expectations there are two minimum-state variable (MSV) solutions of the form,

$$y_t = a + b y_{t-1} + c v_t \tag{4.24}$$

where b is a (real) root of the characeristic equation

$$b^2 - \beta^{-1} b + \beta^{-1}\delta = 0$$

with roots $b = [1 \pm \sqrt{1 - 4\beta\delta}]/2\beta$. Denote these two roots as b_+ and b_-. The remaining coefficients satisfy

$$a = \frac{\alpha}{1 - \beta - \beta b} \qquad c = \frac{1}{1 - \beta b}$$

4.2. **E-Stability Under Correct Specification.** Now assume agents are not magically endowed with knowledge of the Rational Expectations equilibrium. Instead, they must learn it over time via an adaptive error-correction process. Suppose they get lucky and conjecture the correct functional form, and must only learn the parameters of this function. That is, they begin with the a Perceived Law of Motion (PLM) given by equation (4.24), with cv_t replaced by a regression residual. Using this PLM in (4.23) gives the Actual Law of Motion (ALM):

$$y_t = T_a(a, b) + T_b(b) y_{t-1} + v_t$$

where $T_a(a,b) = (\alpha + \beta a)/(1 - \beta b)$ and $T_b(b) = \delta/(1 - \beta b)$. A Rational Expectations Equilibrium is defined by the fixed-point conditions, $a = T_a(a,b)$ and $b = T_b(b)$.

Define the vectors $\pi = (a,b)'$ and $T(\pi) = (T_a(a,b), T_b(b))'$. Then the equilibrium is E-stable if and only if the eigenvalues of the Jacobian of $T(\pi) - \pi$ have negative real parts. This will be the case if $\beta/(1 - \beta b) < 1$ and $\delta\beta/(1 - \beta b)^2 < 1$. When these conditions are satisfied, recursive least-squares estimation of (4.24) will be locally stable in the neighborhood of the Rational Expectations Equilibrium. Evans and Honkapohja show that if $|\beta + \delta| < 1$ then b_- is uniquely stationary and E-stable. However, there are regions of the parameter space, satisfying $4\beta\delta < 1$ and $|\beta + \delta| > 1$, where both b_+ and b_- are stationary, but neither is E-stable, or where one of the two is E-stable.

4.3. **E-Stability Under Misspecification.** Now suppose agents omit y_{t-1} from their PLM, so instead they recursively estimate the misspecified model,

$$y_t = a + \varepsilon_t \tag{4.25}$$

Again using this in (4.23) delivers the following ALM,

$$y_t = \alpha + \beta a_t + \delta y_{t-1} + v_t \tag{4.26}$$

where a_t is governed by the recursive least-squares algorithm,

$$a_t = a_{t-1} + t^{-1}(y_{t-1} - a_{t-1})$$

with y_{t-1} given by the ALM in (4.26). One can readily verify that the mean ODE in this case is,

$$\dot{a} = \frac{\alpha + \beta a}{1 - \delta} - a$$

which has a unique stationary point at $a = \alpha/(1 - \beta - \delta)$. This point defines an RPE. It is (globally) E-stable if and only if $\beta/(1-\delta) < 1$. Given the stationarity condition $|\delta| < 1$, this is equivalent to the condition $\beta + \delta < 1$. Evidentally, the E-stability conditions are now quite different. It is easy to construct examples where the REE is E-stable and the RPE isn't, or vice versa.

4.4. **Stability Under Model Validation.** An important feature of the previous RPE is that the agent thinks his regression model should have serially uncorrelated disturbances, when in fact one can easily see they follow an AR(1) process: $\varepsilon_t = \delta\varepsilon_{t-1} + w_t$. Wouldn't a modestly competent econometrician detect this, perhaps by looking at a Durbin-Watson statistic? With a large enough sample the answer seems clear enough. However, remember that in small samples, degrees of freedom *are* an issue.

5. Robust Model Validation

Earlier we alluded to a debate between Sims (2001) and Cogley and Sargent (2001) about the presence of regime changes in U.S. inflation data. Sims points out that if agents fit models with homoskedastic error terms when in fact the data are heteroskedastic, they may be fooled into inappropriately inferring that there have been breaks in the data, and as a result, inappropriately reject their models. Sims' point is relevant for us too. Until now we have assumed the decision maker knows the model's error distribution, even if he doesn't know its parameters. What if his beliefs about this distribution are wrong? If he ignores this possibility then he exposes himself to the kind of error that Sims highlighted. Our goal here is not just to model policymakers as econometricians, but to model them as *good* econometricians. Good econometricians worry about *robustness*.

If the only thing he had to worry about was parameter estimation and distributional uncertainty, there would be a straightforward response - use GMM rather than maximum likelihood. However, the agent we model is not just an econometrician, he is a decision maker who *uses* his model to devise a control policy. Moreover, this control policy influences the data-generating process.[16] This means that he must make enough assumptions about his environment that he can solve his control problem, and this necessarily exposes him to greater specification risk than if he just needed to estimate parameters.

Our approach to this problem is to blend the recent literature on robust control and filtering (Hansen and Sargent (2006b)) with the recent literature on robust inference in moment condition models (Kitamura and Stutzer (1997) and Kitamura and Otsu (2005)). We do this by building on Pandit (2004) and Pandit and Meyn (2004). As in the econometric literature on information-theoretic GMM and empirical likelihood, we define models by parameterized *moment conditions*. However, for us, these moment conditions do not come from economic theory; they define a permitted class of *model perturbations*, within which Hansen and Sargent's 'evil agent' can select a model to subvert the agent's model validation and control efforts. The more moment conditions there are, the less freedom the evil agent has, and hence, the less robust will be the outcome.

An important by-product of a robust model validation approach is that it endogenously delivers a 'robustified' reference model. Existing work on robust control is silent about where the robustness-seeking agent's reference model comes from. It only considers perturbations to a *given* reference model. In addition, due to the links between KLIC and Type I and Type II error rates, our approach endogenously generates *detection errors*. Current work on robustness specifies these errors exogenously, as a device to calibrate 'reasonable' amounts of robustness.[17] Instead, our approach exogenously specifies a set of moment conditions.

Let us now drop the assumption that the decision maker knows the exact distribution of ξ_t so that the decision maker faces some form of model uncertainty. In this case, it is natural for the decision maker to pursue some form of robustness in the validation and decision making process. We formulate the robust validation process following the framework of Pandit (2004) and Pandit and Meyn (2004). Let ϕ be the marginal distribution of X. If the decision maker knows the distribution of ξ_t, he can calculate the probability distribution

$$\sum_{\ell=0}^{\overline{\ell}} Z_{t-\ell}\beta_{t-\ell}.$$

However, we now assume the decision maker has only partial information about the distribution. Instead of the exact distribution, the decision maker knows only a few moments.

[16]Remember, however, that in keeping with our assumption of bounded rationality, we assume the agent ignores this feedback.

[17]See Anderson, Hansen, and Sargent (2003).

For example, consider the following moment-constrained set of parameterized models

$$\mathbb{P}(\beta) = \left\{ \phi \mid \mathsf{E}^{\phi} Z\beta = 0, \quad \text{and} \quad \mathsf{E}^{\phi}(Z\beta)^2 = \sigma_\xi^2 \right\}$$

where $\sigma_\xi^2 = \mathsf{E}\xi_t^2$. In this case, the decision maker does not know the exact distribution of the regression disturbance. He only knows the first two moments. The number of restrictions imposed on the moment class can be interpreted either as an expression of the decision maker's bounded rationality or as an expression of his preference for robustness. If he knows the correct distribution of ξ, he must know every moment of ξ, and therefore, the moment class is subject to an infinite number of constraints. The finiteness of the constraints can be interpreted as a bound on his capacity to process information, which exposes him to model uncertainty.

For $\rho_1 > 0$ and a probability distribution ϕ on $\mathbf{X}$, define

$$\mathcal{Q}_{\rho_1}(\phi) = \left\{ \phi' \mid H(\phi' \| \phi) < \rho_1 \right\}$$

and

$$\mathcal{Q}_{\rho_1}(\mathbb{P}) = \bigcup_{\phi \in \mathbb{P}} \mathcal{Q}_{\rho_1}(\phi).$$

For a small $\rho_1 > 0$, we can interpret $\mathcal{Q}_{\rho_1}(\phi)$ as the class of models which are difficult to differentiate from ϕ. Similarly, $\mathcal{Q}_{\rho_1}(\mathbb{P})$ is the class of models that are difficult to distinguish from models in $\mathbb{P}$.

The robust validation process can now be defined as follows. Let ϕ_k be the probability distribution over $\mathbf{X}$ induced by the present reference model parameterized by γ_k. Given a "smooth" empirical distribution $\hat{\mathcal{M}}_t$, define

$$\mathcal{L}(\hat{\mathcal{M}}_t, \gamma_k) = \inf_{\phi \in \mathbb{P}(\gamma_k)} H(\hat{\mathcal{M}}_t \| \phi)$$

as the "worst case" relative entropy over $\mathbb{P}(\gamma_k)$. If

$$H(\hat{\mathcal{M}}_t | \gamma_k) < \rho_2,$$

then the decision maker uses γ_k to solve

$$\sup_{\mathbf{u}_t} \inf_{\phi \in \mathbb{P}(\gamma_k)} \mathsf{E}^{\phi}(1-\delta) \sum_{k'=1}^{\infty} \delta^{k'-1} U(u_{t+k'}, Z_{t+k'}) \tag{5.27}$$

where $U(\cdot)$ is the one period payoff, and $\mathbf{u}_t = (u_t, u_{t+1}, \ldots)$ is the sequence of controls. If

$$H(\hat{\mathcal{M}}_t | \gamma_k) \geq \rho_2,$$

then γ_k is discarded, and a new reference model γ_{k+1} is constructed by solving

$$\sup_{\gamma} \inf_{\phi \in \mathbb{P}(\gamma)} H(\hat{\mathcal{M}}_t \| \phi).$$

With γ_{k+1} in place of γ_k, the decision maker solves (5.27).

6. Concluding Remarks

This paper has attempted to model macroeconomic policymakers as econometricians. We've done this by combining recent work in both macroeconomics and econometrics. From macroeconomics, we've borrowed from the work of Sargent (1993) and Sargent (1999) on boundedly rational learning dynamics. From econometrics, we've borrowed from recent work on robust hypothesis testing (Zeitouni and Gutman (1991), Pandit (2004) and Pandit and Meyn (2004)) and the analysis of misspecified models (Vuong (1989), Rivers and Vuong (2002), and Hansen and Sargent (1993)). As it turns out, this produces a rather difficult, and as yet unconsummated, marriage.

From a macroeconomic standpoint, it is difficult because we abandon the Rational Expectations Hypothesis, thereby putting ourselves into the 'wilderness of bounded rationality'. We do this not because we like to analyze difficult and ill-posed problems, but simply because of the casual observation that, as econometricians, macroeconomic policymakers do not spend their time refining estimates of a known model, but instead spend most of their time searching for new and better models. Of course, it is not *necessary* to abandon Rational Expectations and traditional Bayesian decision theory when confronting model uncertainty.[18] However, we think there are good reasons to explore alternative approaches.[19]

The marriage between macroeconomics and econometrics is difficult from an econometric standpoint because, presumably, policymakers have some influence over the data-generating processes they are attempting to learn about. The econometric analysis of misspecified models with endogenously generated data is truly uncharted territory.

We make progress on this problem by relating it to a problem that *is* relatively well understood, namely, the dynamics of constant gain recursive learning algorithms. We prove that as the government employs an increasingly stringent specification test, the dynamics generated by a process of testing and model revision, which we call *validation dynamics*, converge in a very strong way to the dynamics generated by recursive learning models. This is a useful connection to make, because it enables us to apply the results of Williams (2001) and Cho, Williams, and Sargent (2002) on escape dynamics to help us understand a wide range of Markov-switching macroeconomic dynamics. Looking at it from the other side, a second payoff from making this connection is that it provides a more secure behavioral foundation for recursive learning models.

Although we feel this paper takes a significant step forward in understanding the interplay between macroeconomics and econometrics, there are certainly many loose ends and unexplored avenues remaining. Perhaps the most promising one is to follow-up on the connections between robust validation, robust control, and robust inference in moment constrained models.

[18]See Brock, Durlauf, and West (2004) for Bayesian model averaging.

[19]See Hansen and Sargent (2006b), Kreps (1998), and Bray and Kreps (1987).

Appendix A. Proof of Proposition 3.3

Fix $\mu > 0$. By the definition of t_k, $\forall t \in \{t_k+1, \ldots, t_{k+1}\}$,

$$H(\epsilon^t | \beta_{t_k}) < \rho.$$

Under the Gaussian assumption, $H(\cdot)$ is determined by the sum of squared forecast errors. Thus, $\exists \rho'' > 0$ such that

$$\left| \frac{1}{t-t_k} \sum_{t'=t_k+1}^{t} R_{kt'}^{-1} Z_{kt'}(u_{t'} - Z_{kt'}\beta_{t_k}) \right| \leq \rho'' \qquad \forall t \in \{t_k+1, \ldots, t_{k+1}\} \tag{A.28}$$

where the subscript kt represents the variable at t generated by the k-th model, β_{t_k}. Moreover, we can let $\rho'' \to 0$ as $\rho \to 0$.

Fix an arbitrary sample path of $\{v_{it}\}$ where (A.28) holds for $t \in \{t_k+1, \ldots, t_{k+1}\}$. We claim that $M > 0$, $\rho'' > 0$, $\exists \tau' > 0$ such that $\forall \tau \in (0, \tau')$, $\forall t \in \{t_k+1, \ldots, t_{k+1}\}$,

$$\left| \frac{1}{t-t_k} \sum_{t'=t_k+1}^{t} R_{kt'}^{-1} Z_{kt'}(u_{t'} - Z_{kt'}\beta_{t_k}) - \frac{1}{t-t_k} \sum_{t'=t_k+1}^{t} R_{t'}^{-1} Z_{t'}(u_{t'} - Z_{t'}\hat{\beta}_{t'}) \right| \leq M\rho''. \tag{A.29}$$

We prove the claim by induction. If $t' = t_k + 1$, it is clear. Suppose that we have proved the claim for t. Since $\beta_{t_k} = \hat{\beta}_{t_k}$,

$$|\hat{\beta}_t - \beta_{t_k}| \leq a(t_{k+1} - t_k)(M+1)\rho'' \leq \tau(M+1)\rho''$$

since $\limsup_{a\to 0} a(t_{k+1} - t_k) \leq \tau$. Given the sequence of $\{v_{it}\}$, the difference in the updating terms arises from the difference of $\hat{\beta}_{t'}$ and β_{t_k}, which is at the order of ρ''. Thus, conditioned on the sequence $\{v_{it}\}$ satisfying (A.28),

$$\left| R_{t+1}^{-1} Z_{t+1}(u_{t+1} - Z_{t+1}\hat{\beta}_{t+1}) - R_{kt+1}^{-1} Z_{kt+1}(u_{t+1} - Z_{kt+1}\beta_{t_k}) \right| \leq L\tau(M+1)\rho''$$

where L depends upon the Lipschitz constant of b^r and the variable without t means the one generated by the recursive learning algorithm. Choose τ' sufficiently small so that $\forall \tau \in (0, \tau'')$, $L\tau(M+1) \leq M$, which proves the claim.

Since the rate of change of β is bounded by ρ'', which decreases as $\rho \to 0$, over any finite time interval $[0, \tau)$, the difference between the two sample path must vanish as $\rho \to 0$. Since this bound holds for an arbitrary sample path, we have the desired conclusion.

Appendix B. Proof of Proposition 3.6

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Fix $\delta > 0$ and the initial condition for the validation dynamics. Since the associated ODE for the validation has the self-confirming equilibrium as a unique stable point, $\exists T_0$ such that $\beta_{T_0} \in G(\delta)$. Since v_{2t} is i.i.d., for a fix initial condition, $\exists M > 0$ such that if

$$M\mathbf{R}^2 \leq t\mathcal{C}(\overline{q}, t),$$

then, the government will opt for the static model. Since $t\mathcal{C}(\overline{q}, t) \to t$, $\forall \mu > 0$, we can choose sufficiently large $T > T_0$ such that

$$\mathsf{P}\left(\forall t \geq T, \quad \mathbf{R}^2 \leq t\mathcal{C}(\overline{q}, t)\right) \geq 1 - \mu.$$

Since x^s is the stable point, the large deviation properties of the validation dynamics imply that

$$\mathsf{P}\left(\forall t \in \{T_0, \ldots, T\}, \beta_t \in G(\delta) \mid \beta_{T_0} \in G(\delta)\right) \geq 1 - e^{-S^*/a}$$

for some $S^* > 0$. Thus, before T period, the model stays within the small neighborhood of the self-confirming equilibrium. Because $T > T_0$ is sufficiently large, the simplicity of the static model outweighs the improvement in the forecasting errors. Thus, $\forall \mu > 0$ and $T' > 0$, $\exists a > 0$ sufficiently small so that

$$\mathsf{P}\left(\forall t \in \{T, \ldots, T'\}, \quad \beta_t = \beta^0 \mid \beta_T \in G(\delta)\right) \geq (1 - e^{-S^*/a})(1 - \mu)$$

from which the desired conclusion follows.

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