Estimation Uncertainty and the Equity Premium

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October 9, 2001

Abstract

This paper studies a dynamic equilibrium model of asset prices in a partially observable exchange economy. It shows that the precautionary savings motive in response to estimation uncertainty can dominate the risk aversion effect, resulting in the reduction of the equity premium over short horizons. This exacerbates the equity premium puzzle. Over longer holding horizons, however, estimation uncertainty does induce higher risk premiums on equity over risk-free coupon bonds of matching maturities, as long-term bond yields are lowered due to the precautionary savings effect.

JEL classification: E1, E43, E44, G1, G12

Keywords: Equity premium; Estimation risk; Precautionary savings; Learning; Long-horizon expected return

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1 Introduction

Uncertainty about economic prospects involves not only pervasive fluctuations or "noise" in the economy, but also unobservable "structural parameters" such as the rate of productivity and the expected growth rate of earnings. Uncertainty about the economic fundamentals and the learning process Bayesian agents undertake to infer these parameters introduces an additional dimension of uncertainty, often referred to as estimation uncertainty. It is tempting to think that this additional uncertainty may contribute to the risk premium that equity commands over government bonds and hence may help resolve the equity premium puzzle, first discussed by Mehra and Prescott (1985). Indeed, Barsky and De Long (1993) and Timmerman (1996) have investigated how learning about the unobservable fundamentals may generate excess volatility in stock returns. Recently, Brennan and Xia (2001) have developed a dynamic equilibrium model with an element of learning to match the stock price volatility and the equity premium in the U.S. market.

In this paper, however, I demonstrate through a dynamic equilibrium model of asset prices in a partially observable exchange economy that the estimation uncertainty does *not* necessarily give rise to higher excess volatility and a higher equity risk premium when agents have constant relative risk aversion (CRRA).[1] In my model, the estimation error in the learning process has a direct effect on the intertemporal marginal rate of substitution (IMRS) of the representative agent. This effect is due to the precautionary savings motive[2] and drives down long-term interest rates in response to increased uncertainty about future consumption. Meanwhile, with additional uncertainty in expected future dividend stream the risk aversion effect would decrease the demand for the stock. In the equilibrium, however, the precautionary savings

[1] The utility function of constant relative risk aversion is used in the previously cited papers, including Mehra and Prescott (1985) and Brennan and Xia (2001).

[2] The precautionary savings motive is first studied by Leland (1968), Sandmo (1970), and Drèze and Modigliani (1972). More recent development can be found in Kimball (1990).

effect dominates the risk aversion effect. Consequently, the stock price is higher with greater uncertainty and the equity premium over short horizons is actually reduced. This exacerbates the equity premium puzzle. In contrast, interest rates are constant either by assumption, as in Barsky and De Long (1993) and Timmerman (1996), or by the specific feature of the assumed aggregate consumption process in Brennan and Xia (2001), and hence the precautionary savings effect of the estimation risk does not affect the IMRS. Therefore the results in these papers are driven mainly by the risk aversion effect.

This result is consistent with the findings of Veronesi (2000), who independently shows in a model of unobservable regime shifts that imprecise signals about the current state of the economy tend to increase the stock price and reduce the equity premium. Even though my model and Veronesi's have different structures, their shared emphasis on estimation uncertainty and their agreement about its effect on the equity premium illustrate the robustness of the result. Moreover, in this paper I connect explicitly the intuition behind this result with the precautionary savings motive and breakdown the impact of the estimation uncertainty into the two opposing effects described above. This makes the "hedging demand" explanation in Veronesi (2000) more concrete.

Another unique contribution of this paper is to examine the impact of estimation uncertainty on the expected returns and equity premiums over different holding horizons. While most empirical studies focus on stock returns over short holding periods, the time-varying nature of the expected return results in variations of the expected holding-period return over different horizons. Therefore, studying long-horizon returns is important both for testing asset pricing models and for understanding the asset allocation decision in the presence of the return predictability. Consistent with the empirical evidence, the model implies low expected returns around the peaks of economic expansion and high expected returns in the depths of economic contraction.

Furthermore, higher estimation uncertainty about growth rates reduces the expected return across all horizons, although over longer horizons, as the estimation uncertainty is resolved over time, the expected returns converge with those in a full-information economy.

While the term structure of expected returns exhibits widely varied shapes across economic cycles, I find that the equity premium for a given holding horizon is remarkably stable. This pattern holds for both full-information and partially observable economies, although the equity premium is reduced over short horizons with increasing estimation uncertainty. Over longer holding horizons, however, the additional risk from uncertain growth prospects does entail higher risk premiums compared with those in a corresponding full-information economy, as the long-term interest rates are lowered due to the precautionary savings effect and the long-horizon expected equity returns converge with those in the full-information economy.

In the model presented in this paper, agents with constant relative risk aversion (CRRA) first infer the unobservable consumption growth rate from its past realizations and then optimize their time-additive expected utility functions to determine their optimal intertemporal consumption and investment policies. The framework adopted here is developed by Detemple (1986), Dothan and Feldman (1986) and Gennotte (1986), and is used by a number of authors, such as Brennan and Xia (2001), David (1997), and Veronesi (2000), to study asset price dynamics in various models of exchange economies. This paper emphasizes the importance of the precautionary savings motive associated with the learning process in determining asset prices, and its implication for the equity premium puzzle.

The rest of the paper is organized as follows: Section 2 lays out the partially observable economy, describes the agents' learning process, and establishes the equilibrium. Section 3 discusses the effect of the estimation uncertainty on the term structure of interest rates and on the expected return of the stock and the equity premium. Results of expected returns and equity premiums over different holding horizons are also presented. Concluding remarks are offered in Section 4. Appendix A discusses the parameters used in numerical calculations, and Appendix B provides a lemma that is needed for proofs which are given in Appendix C.

2 The Model

2.1 The Economy

I consider a continuous-time version of the Lucas (1978)-type single-good pure exchange economy.[3] In this economy, there is one locally risk-free bond and one risky stock, and trading takes place continuously. The price of the risk-free bond, $B(t)$, is described by

$$dB(t) = r(t)B(t)dt, \tag{1}$$

where $r(t)$ is the stochastic (real) interest rate that is to be determined endogenously. The risk-free bond is in zero net supply.

The risky stock is a claim to the exogenous flow of endowment, $D(t)$, which I will call dividend. This "dividend" may be thought of as the real aggregate output in the economy, in units of the non-storable single consumption good. The evolution of the economy is described by the dynamics of $D(t)$, which is assumed to satisfy the stochastic differential equation

$$\frac{dD(t)}{D(t)} = \alpha(t)dt + \nu dZ_1(t). \tag{2}$$

Here the expected growth rate $\alpha(t)$ may not be observable to the agents, although it

[3]This type of model, in various representations, can be found in Rubinstein (1976), Mehra and Prescott (1985), Abel (1988), Kandel and Stambaugh (1990), and Gennotte and Marsh (1993), among others.

is assumed to follow a mean-reverting process:

$$d\alpha(t) = \kappa(\overline{\alpha} - \alpha(t))dt + \rho dZ_1(t) + \eta dZ_2(t), \tag{3}$$

with known constant parameters $\kappa, \overline{\alpha}, \rho$ and η. $Z_1(t)$ and $Z_2(t)$ are independent Brownian motions with $Z_2(t)$ capturing innovations to the growth rate that are not correlated with the dividend process.[4] The assumption of mean reversion in the expected growth rate of dividends is consistent with the real business cycles (see, e.g., Kandel and Stambaugh, 1990). In addition, the volatility in the dividend process, ν, is assumed to be constant in this model, as our focus here is on the uncertainty in the expected growth rate, $\alpha(t)$.

There is a continuum of identical agents living in a period $[0, \mathcal{T}]$ in the economy. Each agent has a constant relative risk aversion (CRRA) utility function and an initial endowment. This assumption ensures the existence of a representative agent in this economy. The representative agent maximizes her expected lifetime time-additive utility of consumption

$$\sup_{\{c(t),\pi(t)\}} E\left\{\int_0^{\mathcal{T}} e^{-\delta t}\frac{c(t)^{1-\gamma}-1}{1-\gamma}dt\right\} \tag{P1}$$

with the budget constraint

$$dW(t) = \pi(t)\frac{dG(t)}{S(t)} + \left(r(t)(W(t) - \pi(t)) - c(t)\right)dt,$$

where δ is the constant time-preference parameter, $c(t)$ is the agent's time-t consumption, $W(t)$ is the wealth process, and $\pi(t)$ is the amount of wealth invested in the stock. The gain process $dG(t)$ for the stock includes both capital gains and dividend

[4]In this economy, $Z = (Z_1, Z_2)$ are standard Brownian motions defined on a complete probability space $(\Omega, P, \mathcal{F})$. Implicitly, we assume that all necessary and sufficient technical conditions are satisfied so that the stochastic differential equations all have a solution. Agents' information structure is summarized by the filtration $\mathcal{F}^D$ generated by $D(t)$, and we have $\mathcal{F}_t^D \subset \mathcal{F}_t$. Processes, such as α and Z, are adapted to $\mathcal{F}$, but not to $\mathcal{F}^D$, and thus they are not observed by the agents. Therefore, agents face a dynamically incomplete market in the sense of Harrison and Kreps (1979).

flows, and it is expressed as

$$dG(t) = dS(t) + D(t)dt,$$

where $S(t)$ is the price of the stock at time t. The dynamics of the stock price will be endogenously determined. In this economy, the only income for the agent is from the dividend, and she will not have a bequest motive. As $\mathcal{T} \to \infty$, the economy converges to an infinite-horizon one.

When the agent does not directly observe the instantaneous growth rate, $\alpha(t)$, she seeks to extract information from the observed dividend process to estimate $\alpha(t)$. Using the assumed Gaussian-Markov structure,[5] Detemple (1986), Dothan and Feldman (1986) and Gennotte (1986) demonstrate that a separation principle holds - that the agent can first solve the inference problem to form her expectation, and then, based on this expectation, can solve her optimization problem. Here I adopt this methodology. In the following subsections, I first summarize the agent's process of learning about the expected growth rate, $\alpha(t)$, and then I examine the equilibrium in an equivalent fully observable economy.

2.2 The Agent's Learning Process

Initially (at time $t = 0$), the agent views the distribution of $\alpha(0)$ as Gaussian with a mean $m(0)$ and a variance $V(0)$. Then she updates the estimate of the state variable $\alpha(t)$ through observing the realized path of $D(t)$. As shown in Lipster and Shiryayev (1978), the conditional distribution of $\alpha(t)$ based on the observation of $\{D(s) : s \leq t\}$ is also Gaussian with a mean $m(t)$ and a variance $V(t)$.

To the agent, the dividend process becomes

[5] Detemple (1991) extends the Gaussian-Markov structure to include non-Gaussian prior distributions. In this paper, we maintain the assumption of the Gaussian-Markov structure because it enables us to solve asset prices in analytical forms and still yields important new insights into the effect of estimation uncertainty.

$$\frac{dD(t)}{D(t)} = m(t)dt + \nu d\varpi(t) \tag{4}$$

where the *estimated* expected growth rate $m(t)$follows

$$dm(t) = \kappa(\overline{m} - m(t))dt + \left(\rho + \frac{V(t)}{\nu}\right) d\varpi(t). \tag{5}$$

Here $\overline{m} \equiv \overline{\alpha}$. In the Gaussian-Markov setting adopted here, the measure of the estimation error, $V(t)$, is a deterministic function satisfying a Riccatti equation

$$\frac{dV(t)}{dt} = -2\kappa V(t) + \eta^2 + \rho^2 - \left(\rho + \frac{V(t)}{\nu}\right)^2 \tag{6}$$

with an initial condition $V(0) = V_0$.[6] This implies that the volatility of $m(t)$ will be time varying but non-stochastic. Furthermore, the innovation process

$$d\varpi(t) = \frac{1}{\nu}\left(\frac{dD(t)}{D(t)} - m(t)dt\right) = \frac{1}{\nu}\left(\alpha(t) - m(t)\right)dt + dZ_1(t)$$

is also a Brownian motion and now observable to agents. It should be noted that the information structure generated by $\{m(0), \varpi(s), 0 \leq s \leq t\}$ is the same as the one generated by $\{D(s), 0 \leq s \leq t\}$. Moreover, the market is now dynamically complete with respect to the *inferred* (or "derived") state variable, $m(t)$.

[6]The solution to this Riccati equation is

$$V(t) = \frac{V_+ - V_- \frac{V_0 - V_+}{V_0 - V_-} e^{-2\varphi t}}{1 - \frac{V_0 - V_+}{V_0 - V_-} e^{-2\varphi t}}$$

where

$$\begin{aligned} \varphi &= \sqrt{\left(\kappa + \frac{\rho}{\nu}\right)^2 + \eta^2} \\ V_+ &= \nu^2 \left[-\left(\kappa + \frac{\rho}{\nu}\right) + \varphi\right] \\ V_- &= \nu^2 \left[-\left(\kappa + \frac{\rho}{\nu}\right) - \varphi\right] \end{aligned}$$

As $t \to \infty$, $V(t) \to V_+$. In general, the estimation error does not necessarily go to zero as time progresses, and hence one may never be able to have an infinitely accurate estimation of $\alpha(t)$ if it is subject to a random shock uncorrelated to the signal, $D(t)$.

Later in this paper, I compare asset prices in the partially observable economy with those in the corresponding full-information economy in which $\alpha(t)$ is observable to the agents. For convenience, I will treat the latter as a special case of the former in which $V(t) \equiv 0$ and $\eta = 0$.[7]

2.3 The Equilibrium

In the partially observable economy, the information structure in (2) and (3) is equivalent to that in (4) and (5). In the continuous-time framework adopted here, it is demonstrated that the optimization problem (P1) faced by the representative agent is equivalent to that in a fully observable economy with the underlying processes described by (4) and (5) (Gennotte, 1986).

In this equivalent fully observable economy, the gain process for the stock, G, is

$$\frac{dG(t)}{S(t)} = \frac{dS(t)}{S(t)} + \frac{D(t)dt}{S(t)} = \mu(t)dt + \sigma(t)d\varpi(t) \tag{7}$$

where $\mu(t)$ and $\sigma(t)$ are functions of the derived state variable $m(t)$ and will be determined endogenously. Because the market is dynamically complete with respect to $m(t)$, one can apply the martingale approach, due to Cox and Huang (1989) and Karatzas, Lehoczky, and Shreve (1987), to transform the agent's intertemporal dynamic problem (P1) to the following static optimization problem,

$$\sup_{\{c(t)\}} E\left\{\int_0^T e^{-\delta t}\frac{c(t)^{1-\gamma}-1}{1-\gamma}dt\right\} \tag{P2}$$

such that

$$E^Q\left[\int_0^T e^{-\int_0^t r(s)ds}c(t)dt\right] \leq W_0 \tag{8}$$

[7] $\eta = 0$ in the full-information economy ensures a complete market. The full-information economic model corresponds to that in Goldstein and Zapatero (1996).

where E^Q represents the expectation with respect to the risk-neutral probability measure Q, defined by

$$E^Q[1_A] = E[\xi_T \cdot 1_A]$$

where

$$\xi_T = \exp\left\{-\int_0^T \theta(t)d\varpi(t) - \frac{1}{2}\int_0^T \theta(t)^2 dt\right\}$$

with

$$\theta(t) = \frac{\mu(t) - r(t)}{\sigma(t)}$$

as the market price of risk.

In the equilibrium, the security market clears so that the representative agent will hold zero net shares of the riskless bond and all shares of the equity. The goods market also clears, i.e., if $c^*(t)$ denotes the optimal consumption policy for the agent, then

$$c^*(t) = D(t), \qquad \forall t. \tag{9}$$

This leads to the following proposition.

Proposition 1 *In the equilibrium of the partially observable economy, the instantaneous interest rate is*

$$r(t) = \delta + \gamma m(t) - \frac{1}{2}\gamma(1+\gamma)\nu^2, \tag{10}$$

and the market price of risk is

$$\theta(t) = \gamma\nu. \tag{11}$$

This result is standard, except that $m(t)$ is the estimated expected growth rate. In a dynamic equilibrium model of unobservable regime shifts, Veronesi (2000) obtains the same result.[8] The dependence of the instantaneous interest rate $r(t)$ on $m(t)$

[8] In Veronesi (2000), agents' information set includes not only the realized dividend process, but also an additional noisy signal process. The existence of this signal process alters the market price of risk. But as the result from this paper shows, this additional signal process does not materially change the effect of estimation uncertainty on the equity premium.

indicates that the estimation process affects the intertemporal marginal rate of substitution (IMRS), in addition to affecting the expected future consumption stream. In contrast, in Brennan and Xia (2001), the aggregate consumption process is assumed to be a geometric Brownian motion with known constants, so the interest rate is constant and the IMRS is unchanged by the estimation process. Thus the asset prices in this model are determined only by the expected future dividend stream. As argued in the next section, both effects of the estimation uncertainty are important in assessing its impact on the equity premium on the aggregate level.

3 Equilibrium Asset Prices and Equity Premiums

3.1 Bond Prices and Term Structure of Interest Rates

From Proposition 1 and (5), the interest rate process can be written as

$$dr(t) = \kappa(\overline{r} - r(t))dt + \gamma\left(\rho + \frac{V(t)}{\nu}\right)d\varpi, \tag{12}$$

where

$$\overline{r} = \delta + \gamma\overline{m} - \frac{1}{2}\gamma(1+\gamma)\nu^2, \tag{13}$$

as implied from (10). This is an extended version of the Vasicek (1977) interest rate dynamics. Under the risk-neutral measure, the interest rate process becomes

$$dr(t) = \kappa\left(\overline{r} - \frac{\lambda(t)}{\kappa} - r(t)\right)dt + \gamma\left(\rho + \frac{V(t)}{\nu}\right)d\varpi^Q, \tag{14}$$

where $\lambda(t) = \gamma^2(\rho\nu + V(t))$ is the time-varying risk premium for the interest rate. In contrast, the risk premium for the interest rate is constant in the full-information economy.

Although in this model the volatility of $r(t)$ is deterministic, as the estimation error $V(t)$ itself is deterministic, this will not affect my illustration of the qualitative effect of the estimation uncertainty on asset prices and the equity premium. In fact, the simple structure of my model facilitates an analysis of the term structure and leads to a clearer and more focused vantage point from which we can determine the causes of this effect. In comparison, while the estimation error $V(t)$ is stochastic in Veronesi (2000), there is no discussion of the term structure effect and the consequent multi-facet effect that the estimation uncertainty has on the asset prices.

The following proposition provides a formula for the price of a discount bond.[9]

Proposition 2 *In the partially observable economy, the price of the discount bond maturing at time T is given by*

$$\begin{aligned} P(t,T,r(t)) &= E_t^Q\left[e^{-\int_t^T r(u)du}\right] \\ &= e^{A(t,T)-B(t,T)r(t)} \end{aligned} \tag{15}$$

where

$$\begin{aligned} A(t,T) &= -\bar{r}\left[(T-t)-B(t,T)\right]+\gamma^2\nu\int_t^T \phi(\tau)B(\tau,T)d\tau \\ &\quad +\frac{\gamma^2}{2}\int_t^T \phi^2(\tau)B^2(\tau,T)d\tau \end{aligned} \tag{16}$$

$$B(t,T)=\frac{1-e^{-\kappa(T-t)}}{\kappa} \tag{17}$$

and

$$\phi(t)=\rho+\frac{V(t)}{\nu}. \tag{18}$$

The bond price given above solves the stochastic differential equation

$$\frac{dP(t,T,r(t))}{P(t,T,r(t))}=\left(r(t)-\lambda(t)B(t,T)\right)dt-\gamma\phi(t)B(t,T)d\varpi. \tag{19}$$

Note that now $P(t,T,r(t))\neq P(T-t,r(t))$, because $A(t,T)\neq A(T-t)$ due to the presence of the estimation error $V(t)$. The volatility of the discount bond price

[9] This proposition extends (and corrects an error in) Proposition 1 in Feldman (1989) which is obtained under log utility ($\gamma=1$) in a production economy.

is $\sigma_P(t,T) = \gamma\phi(t)B(t,T)$, which implies that the volatility of a constant-maturity bond price changes over time as uncertainty is resolved through learning.

Moreover, the term premium for a discount bond maturing at time T is $-\lambda(t)B(t,T)$. Given that $\lambda(t) = \gamma^2(\rho\nu + V(t))$, the immediate effect of the estimation uncertainty is a reduction in the term premium for the discount bond, because $V(t) > 0$. The amount of reduction is increasing in both risk aversion γ and the degree of estimation uncertainty $V(t)$. This is best illustrated by the last two terms in the following expression of the spot rate of a finite maturity discount bond:

$$\begin{aligned} R(t,T) &= -\log\left(P(t,T,r(t))\right)/(T-t) \\ &= \overline{r} + \frac{B(t,T)}{T-t}\left(r(t)-\overline{r}\right) \\ &\quad -\frac{1}{T-t}\int_t^T \lambda(\tau)B(\tau,T)d\tau \\ &\quad -\frac{1}{2(T-t)}\int_t^T \sigma_P^2(\tau,T)d\tau. \end{aligned} \tag{20}$$

Figure 1 shows a set of term structures of interest rates for different initial uncertainty in $m(t)$ as measured by V_0. The long-run mean growth rate is set to be $\overline{m} = 2\%$, which corresponds to a long-run mean interest rate $\overline{r} = 2.36\%$.[10] The panel of three graphs depicts three different growth regimes, and within each graph are the yield curves with different V_0. One can see that a higher V_0 depresses the slope of the term structure and results in lower long-term interest rates. In the most severe case presented in the graph, the term premium turns negative at the short end of the term structure, causing a dip in a generally upward-sloping yield curve.

An examination of the bond price formulation helps to explain this result. In the complete market setting considered here, the pricing kernel is proportional to the intertemporal marginal rate of substitution of the representative agent; hence, the

[10] These parameters are chosen for an illustrative purpose only. They are calibrated to be broadly consistent with empirical evidence, as discussed in Appendix A.

bond price is given by

$$P(t,T,r(t)) = E_t\left[e^{-\delta(T-t)}\frac{U'(c_T)}{U'(c_t)}\right] = \frac{e^{-\delta(T-t)}}{U'(c_t)}E_t\left[U'(c_T)\right],$$

where $U(c_t)$ is the period utility function at time t. For a *convex* marginal utility such as the power utility function, higher variances of the consumption process c_T induce higher bond prices. This is related to the precautionary savings motive first discussed in Leland (1968).[11] Noting the fact that, in equilibrium, consumption equals dividends, and applying the lemma in Appendix B to (4) and (5), one obtains the variance of $\log(c_T)$ at time t as

$$\begin{aligned} Var_t\left[\log(c_T)\right] &= \int_t^T \left(\nu + \phi(\tau)B(\tau,T)\right)^2 d\tau \\ &= \nu^2(T-t) + 2\nu\int_t^T \phi(\tau)B(\tau,T)d\tau + \int_t^T \phi^2(\tau)B^2(\tau,T)d\tau. \end{aligned}$$

The impact of the estimation risk comes through the volatility of the expected growth rate, $\phi(t) = \rho + \frac{V(t)}{\nu}$, and is on the order of $O\left((T-t)^2\right)$ as $T \to t$. For sensible parameter sets,[12] the variance of $\log(c_T)$ is increasing in $V(t)$, as is the variance of c_T.[13] Thus, although the estimation error does not affect the instantaneous rate $r(t)$ in this model, it does exert a significant influence on the term premium of a long-term bond.

The effect of $V(t)$ may also be understood using a hedging argument.[14] If the correlation between the estimated expected growth rate of consumption and the realized consumption is positive, a negative shock to dividends induces a lower estimate of

[11]Also in Sandmo (1970) and Drèze and Modigliani (1972). See Kimball (1990) for a more recent exposition on precautionary demand for savings.

[12]A sufficient condition is $1 + 2\phi(\tau)B(\tau,T) > 0$, for $t < \tau < T$. This condition is always satisfied if $\rho > 0$. If $\rho < 0$, which also seems consistent with data, this reduction only comes under some parameter restriction. I refrain from elaborating on this restriction as it is not a binding constraint in the argument.

[13]This is always true if the increasing spread in $\log(c_T)$ is mean-preserving, which is the case with the effect of the estimation error.

[14]See, *e.g.*, Campbell (1999) for a discussion of the hedging argument. Veronesi (2000) applies this argument to explain his result in an unobservable regime-switch model.

its expected growth rate in the future; hence, agents' demand for assets with future payoffs increases, raising the value of these assets. Note that this argument implicitly invokes the precautionary savings motive discussed earlier. The effect of the estimation error $V(t)$ is to promote this positive correlation, thereby increasing the hedging value of long bonds and reducing their yields.

3.2 Stock Price and Instantaneous Expected Return

The equilibrium price of the stock in the economy is

$$S(t) = E_t \left[\int_t^T e^{-\delta(s-t)} \left(\frac{D(s)}{D(t)} \right)^{-\gamma} D(s) ds \right]$$

where $D(t) = c(t)$ is expressed in terms of the estimated state variable $m(t)$ as in (4). The following proposition provides an expression for the stock price.

Proposition 3 *In the partially observable economy, the equilibrium stock price is given by*

$$S(t) = D(t) \int_t^T \exp\left(\psi\left(t, s; r(t)\right)\right) ds, \tag{21}$$

where

$$\begin{aligned} \psi\left(t, s; r(t)\right) &= \left[-\frac{\delta}{\gamma} + \frac{(1-\gamma)}{\gamma}\overline{r} + \frac{(1-\gamma)\gamma}{2}\nu^2 \right](s-t) \\ &\quad + \frac{(1-\gamma)^2}{2} \int_t^s \left(\nu + \phi(\tau)B(\tau, s)\right)^2 d\tau \\ &\quad + \frac{(1-\gamma)}{\gamma}\left(r(t) - \overline{r}\right) B(t, s). \end{aligned} \tag{22}$$

Applying Ito's Lemma to (21), we get

$$\frac{dG}{S} = \frac{dS + Ddt}{S} = \mu(r, t)dt + \sigma(r, t)d\varpi, \tag{23}$$

where

$$\mu(r, t) = r(t) + \gamma\nu^2 + (1-\gamma)\gamma\nu\phi(t)\Lambda(r, t), \tag{24}$$

$$\sigma(r,t) = \nu + (1-\gamma)\phi(t)\Lambda(r,t), \tag{25}$$

and

$$\Lambda(r,t) = \left(\frac{\int_t^T B(t,s)\exp\left(\psi\left(t,s;r(t)\right)\right)ds}{\int_t^T \exp\left(\psi\left(t,s;r(t)\right)\right)ds} \right) > 0$$

is a slow moving function of the state variable $r(t)$.

From (24), the instantaneous equity premium is

$$\begin{aligned} \mu(r,t) - r(t) &= \gamma\nu^2 + (1-\gamma)\gamma\nu\phi(t)\Lambda(r,t) \\ &= \gamma\nu^2 + (1-\gamma)\gamma\nu\rho\Lambda(r,t) + (1-\gamma)\gamma V(t)\Lambda(r,t), \end{aligned}$$

so the contribution to the equity premium from learning is mostly reflected by

$$(1-\gamma)\gamma V(t),$$

which is positive only when $\gamma < 1$. If one adopts the view consistent with most empirical estimations that $\gamma > 1$, then the estimation error actually reduces the instantaneous equity premium. This is consistent with the findings of Veronesi (2000), who shows that in a model of unknown regime shifts, imprecise signals tend to increase the stock price and reduce the equity premium.

The explanation for rising stock prices with increasing estimation risk stems from the precautionary savings motive discussed in the last subsection with regard to bond prices. To see this, one can rewrite the formula for the stock price into the following form:

$$S(t) = \int_t^T dsP\left(t,s,r(t)\right)E_t\left[D(s)\right]e^{-\gamma\int_t^s(\nu+\phi(\tau)B(\tau,s))^2 d\tau}, \tag{26}$$

where

$$E_t\left[D(s)\right] = D(t)\exp\left[\begin{array}{c} \left(\overline{m} - \frac{\nu^2}{2}\right)(s-t) + \left(m(t) - \overline{m}\right)B(t,s) \\ +\frac{1}{2}\int_t^s \left(\nu + \phi(\tau)B(\tau,s)\right)^2 d\tau \end{array} \right]$$

is the conditional expectation of future dividends, and $P\left(t,s,r(t)\right)$ is the bond price given in (15). An increase in the uncertainty of the expected growth rate, $V(t)$, will raise the bond prices as well as the expected future dividends. Although aversion to the additional risk reflected in the last term in (26) has the effect of reducing the stock price, it is more than offset by the rising bond prices and the expected future dividends. Therefore, the stock price increases. This demonstrates the important effect of the term structure of interest rates on stock valuation. Note that in a model with a constant interest rate, the effect of the term structure is absent and the risk aversion will drive down the stock price, hence increasing the expected return.[15]

There are many ways in which risk affects the stock price and its co-movement with the bond price. Abel (1988), Barsky (1989), and Gennotte and Marsh (1993) discuss the co-movement of stock and bond prices due to the change in the variance of dividend stream, ν, and in the expected dividend growth rate, $\alpha(t)$, in fully observable economies. They show that for the CRRA preferences, $\gamma = 1$ is a watershed point; that is, when $\gamma > 1$, the prices of stocks and bonds both increase with an increase in ν or a reduction in $\alpha(t)$, but when $\gamma < 1$ they move in opposite directions as the stock price drops with an increase in ν. But what we see here is that the uncertainty about the expected growth rate will move the prices of stocks and bonds in the same direction for all γs. This illustrates the point that, while the effect of changing ν and the effect of changing $V(t)$ are both attributable to the hedging motive arising from the precautionary demand for savings, their impacts on the prices of stocks and bonds and their co-movement are different.

The volatility of the instantaneous stock return is described by Equation (25). It shows that while a higher $V(t)$ drives up the stock price, it lowers the volatility at the same time. The explanation for this result is that, given the mean-reversion in the unobservable future dividend growth rate, agents' estimation of the growth rate will be less variable than the true one. This result implies that uncertainty about the ex-

[15]In Brennan and Xia (2001) and Timmerman (1996), the interest rate is either fixed or determined to be a constant, so their results are mainly driven by the risk aversion effect.

pected growth rate increases skewness in the return distribution as higher stock prices are associated with lower volatilities. This may have a bearing on understanding the asymmetric volatility structure of stock returns.

3.3 Expected Holding-Period Returns and Long-Horizon Equity Premiums

The previous subsection is concerned with the *instantaneous* expected return and risk premium. It is, however, often important to discuss returns over longer horizons. As the investment opportunity changes over time, the required risk premium will vary with the length of holding horizon. Yet we have had few studies that address the term structure of long-horizon risk premiums.[16] In this subsection, I examine the expected holding-period returns and equity premiums over different horizons.

The return from holding the stock over a time horizon (t, T) is usually represented by

$$RT(t,T) = \frac{S(T) + \int_t^T D(s)ds}{S(t)}. \tag{27}$$

The following corollary provides a formula for calculating the expected holding-period returns.[17]

Corollary 1 *The expected return from holding the stock over a time horizon* (t, T) *is*

$$\begin{aligned} E_t\left[RT(t,T)\right] &= \frac{1}{S(t)}\left[E_t\left[S(T)\right] + \int_t^T E_t\left[D(s)\right]ds\right] \\ &= \frac{1}{S(t)}\left[E_t\left[D(T)\right]\int_T^{\infty} ds e^{\Psi(T,s)+\Phi(t,T,s;r)} + \int_t^T E_t\left[D(s)\right]ds\right]. \end{aligned} \tag{28}$$

Figure 2 shows the expected return as a function of holding period for three different expectation scenarios: fast economic growth ($m = 5\%$), moderate growth

[16]Kandel and Stambaugh (1990) and Daniel and Marshall (1997) are among the exceptions.

[17]Full expressions of the functions in the formula are given along with a derivation in Appendix C.

($m = 2\%$), and expected downturn ($m = -1\%$). The long-run growth rate is set to be $\overline{m} = 2\%$. For each scenario, we compare cases in which $m(t)$ is either fully observable ($V_0 = 0$) or needs to be estimated (with error $V_0 \neq 0$). The parameters used here are for purposes of illustration, and their calibration is discussed in Appendix A. One should not be too concerned about the absolute value of the numbers, but instead should focus on the relative levels between the variables in the full-information economy and those in the partially observable economy. The expected returns we show in the graph are continuously compounded and calculated as

$$ER(t, T) = \ln\left(E_t\left[RT(t, T)\right]\right)/(T - t).$$

For all economic regimes with varied expected growth rates, the expected stock returns in a partially observable economy are lower than those in a full-information economy. This difference is glaringly large with short horizons, diminishing gradually over longer holding periods because of the temporal resolution of the uncertainty caused by the estimation error. The term structures of expected returns show different characteristic shapes under different growth conditions. For a given horizon, the expected return increases with the expected future growth rate of dividend. In general, the expected return declines with the holding horizon. However, when growth is expected to slow or a recession is anticipated, the term structure exhibits a humped shape. In particular, when an expected future recession is coupled with a high degree of estimation uncertainty, the expected return for short horizons can be negative, as the short-term (real) interest rates are also negative.

Empirically, it is difficult to measure *ex ante* expected returns (see, e.g., Merton, 1980). Nevertheless, studies by Chen (1991) and Fama and French (1989) suggest that expected returns are lower around peaks of the business cycles when the prospects of future real growth are poor, while expected returns are higher during recessions, especially late in the trough when future prospects are brighter. This is consistent with the general features in Figure 2.

I also attempt to gauge the model implications for the equity premiums for different holding horizons under various economic phases. The equity premium is customarily measured as the difference between the expected return on the equity and the short-term risk-free rate, which is often proxied by the one-month Treasury bill rate. This definition, however, is not satisfactory for measuring long-horizon equity premiums, as the short-term interest rate does not represent the risk-free return over the same term. Conceivably, a discount bond with a matching maturity might be a candidate for a risk-free instrument over a given horizon. But Wachter (2001) has shown that a coupon bond with a matching maturity is a more appropriate risk-free instrument for an agent with intermediate consumption and a fixed horizon. Therefore I measure the required compensation for the equity risk over a fixed horizon as the difference between the expected stock return over this holding period and the known return on the par yield coupon bond[18] with a matching maturity.[19]

Despite the widely variable term structures of the expected returns under different growth conditions shown in Figure 2, I find a remarkably stable pattern for the equity risk premium over different holding periods, as shown in Figure 3. Although Figure 3 only depicts the case of a full-information economy, the pattern holds true for partially observable economies as well. The stable pattern suggests that the cyclical variations in the economy are largely reflected in the term structure of interest rates,

[18]Specifically, for a horizon T, the coupon rate of the par yield coupon bond maturing at T is obtained as

$$c = \frac{1 - \exp\left(-R(T)T\right)}{\int_0^T \exp\left(-R(s)s\right) ds}$$

where $R(t)$ is the spot rate maturing at time t. The continuously compounded rate of return on the coupon bond is $\ln\left(1 + cT\right)/T$.

[19]This measure for the horizon equity risk premium is different from the decomposition of equity premium into risk premium and term premium discussed in Abel (1999), where equity premium is still measured over a horizon of one period. The concern there is the difference in duration between equity and short-term risk-free assets.

a prediction that is consistent with empirical findings in Estrella and Hardouvelis (1991), Chapman (1997), and Roma and Torous (1997). Figure 3 indicates that the equity premium for the full-information model economy decreases as the holding horizon increases, although the equity remains a riskier asset than bonds with a positive risk premium at all horizons considered (up to 30 years). Moreover, the long-horizon equity premium is higher when the future prospects are brighter, i.e., m is larger, as this usually occurs after a series of downturns.

Figure 4 plots the equity premium with different levels of the estimation uncertainty across holding horizons. For short horizons, as discussed previously, the equity premium declines as the estimation uncertainty increases, while for long holding horizons the additional uncertainty due to the estimation error does entail higher equity premiums than those in the corresponding full-information economy. This is due to the effect that reduces the long bond yields while the expected returns on the stock over longer holding periods converge with those in a corresponding full-information economy, as illustrated in Figures 1 and 2.[20]

The results we obtain here may bear on our understanding of the previous spectacular performance of the stock market and its recent downturn. As technological advances converge to create a “new” economic paradigm, uncertainty about this “new” economy is high as we do not have much past experience to draw from. This uncertainty tends to depress the term structure of interest rates, as we observed in recent years, and lower the expected return as well as the risk premium of the stock. In addition, with growing participation in 401K plans for retirement, the time horizons of average investors may be significantly lengthened, hence further reducing the required equity risk premium. With the dynamics of this “new” economic paradigm now becoming more clear and the associated uncertainty being gradually resolved, the

[20]While this feature should be robust, the specific mechanism described here may be special to this model.

expected return and the equity premium increase, and therefore stock prices drop. Of course, this is just one possible explanation that is consistent with the model described in this paper.

4 Concluding Remarks

The results presented in this paper imply that, in the simple setting of a model with additive power utilities, the estimation risk associated with learning is *not* helping to resolve the puzzles relating to equity premium and excess volatility. This analysis highlights the effect of the precautionary savings motive associated with the learning process on asset prices, and its role in exacerbating the equity premium puzzle. Specifically, it demonstrates that the estimation uncertainty has an impact both on the intertemporal marginal rate of substitution - hence the pricing kernel - and on the expectation of future consumption. Thus, ignoring the effect of learning on the pricing kernel, and in turn on the dynamics of interest rates, can lead to an incomplete conclusion.

The precautionary savings effect of the estimation uncertainty is the explanation behind the often-used hedging demand argument, as the learning process promotes the positive correlation between the estimated expected growth rate and the innovation to the consumption. The temporal resolution of the uncertainty in the expected growth rate makes it informative to examine the expected returns and equity premiums over different holding horizons as well. The interaction of this temporal resolution of risk and the precautionary savings effect reduces the short-term equity premiums while increasing the long-term equity premiums in the presence of the estimation uncertainty.

Obviously, the model discussed in this paper is simplistic. A more complicated model - that is one that is situated somewhere between this model and that of Brennan and Xia (2001) and allows an aggregate consumption process to be affected by the estimation uncertainty, even with non–dividend income - may temper the effect of the estimation error on the equity premium. Nevertheless the point made in this paper - that the estimation uncertainty is not helpful for resolving the equity premium puzzle - is likely to be robust, especially considering the inherent unobservability of the contemporaneous inflation rate and other macroeconomic factors. On the other hand, the time-additive CRRA preference used in the model, and in Brennan and Xia (2001) and Veronesi (2000), has the problem confounding the effects of risk aversion and intertemporal substitution. Therefore, allowing the separation of these two effects using a non-time additive recursive utility model or a habit-formation model could prove to be revealing.[21] This represents a promising direction for future research.

Appendix

A Parameter Calibration

Since, in this paper, I make no direct quantitative comparison between theoretical results and empirical evidence, I have not attempted to fit the model to empirical data. I choose to calibrate to dividend data instead of aggregate consumption data, because Hagiwara and Herce (1997) demonstrate that, by using dividend data instead of aggregate consumption data, the intertemporal marginal rate of substitution (IMRS) is consistent with the Hansen-Jaganathan bound with a relative risk aversion of 3. Moreover, the historical participation rate in the stock market has been low. Mankiw and Zeldes (1991) show that the consumption of stockholders, who correspond more to the agents in the model, is much more volatile than that of non-stockholders, yet the aggregate consumption data do not reflect this distinction. Basak and Cuoco (1998)

[21] Duffie, Schroder, and Skiadas (1997) have studied a term structure model with incomplete information. Their purpose is to examine the effect of the timing of resolution of uncertainty with a form of non-additive recursive utility, and their approach may be a good starting point.

find that, taking limited market participation into account, the equity premium is mainly determined by the market participants and can be reconciled with a relative risk aversion around 3. So the practice of calibrating to dividend data instead of aggregate consumption data in the model economy seems to be reasonable.

The choice of parameters ν, ρ, and κ is guided by the existing empirical studies on historical volatilities of real interest rate and dividend (see, e.g., Campbell, Lo, and MacKinlay, 1997). κ is taken to be 0.4, roughly matching the value in some empirical results for the interest rate dynamics. In the model economy, $\rho = -0.02$, the negative sign is in accord with the mean-reverting trend in the growth rate, and $\nu = 0.12$ for the dividend. In addition, the relative risk aversion parameter, γ, is set to be 3, conforming to the results in Hagiwara and Herce (1997) and Basak and Cuoco (1998), and $\overline{m}$ is chosen to be between 2% and 2.5%, corresponding to the long-run real growth rate of the U.S. economy. In addition, δ is set to be 0.05, i.e., the time discount factor is 5% per year. For the measure of the estimation error, V_0, a standard deviation of 1% from the true growth rate corresponds to $V_0 = 0.0001$. The range of values for V_0 used in the text magnifies the effect to illustrate the idea.

B A Lemma

The following lemma will be useful for the proofs in Appendix C.

Lemma 1 *Suppose $Y(t)$ and $m(t)$ are jointly Markovian and satisfy the following stochastic differential equations:*

$$dY(t) = m(t)dt + \nu d\omega(t)$$

$$dm(t) = \kappa(\overline{m} - m(t))dt + \phi(t)d\omega(t).$$

Here $\omega(t)$ is a Brownian motion, $\phi(t)$ a deterministic function of time, and ν, κ, and $\overline{m}$ are constants. Then the joint conditional probability density function, $P[Y(s), m(s)|\mathcal{F}_t]$, is bivariate normal, and the corresponding moments are:

$$E\left[m(s)|\mathcal{F}_t\right] = \overline{m} + (m(t) - \overline{m})e^{-\kappa(s-t)},$$

$$Var\left[m(s)|\mathcal{F}_t\right] = e^{-2\kappa s}\int_t^s e^{2\kappa\tau}\phi^2(\tau)d\tau,$$

$$E\left[Y(s)|\mathcal{F}_t\right] = Y(t) + \overline{m}(s-t) + \frac{(m(t)-\overline{m})}{\kappa}\left(1 - e^{-\kappa(s-t)}\right),$$

$$\begin{aligned} Var\left[Y(s)|\mathcal{F}_t\right] &= \nu^2(s-t) + \frac{2\nu}{\kappa}\int_t^s \phi(\tau)\left(1 - e^{-\kappa(s-\tau)}\right)d\tau \\ &\quad + \frac{1}{\kappa^2}\int_t^s \phi^2(\tau)\left(1 - e^{-\kappa(s-\tau)}\right)^2 d\tau, \end{aligned}$$

$$Cov\left[Y(s), m(s)|\mathcal{F}_t\right] = \frac{1}{\kappa}\int_t^s e^{-\kappa(s-\tau)}\left(1 - e^{-\kappa(s-\tau)}\right)\phi^2(\tau)d\tau + \int_t^s \nu e^{-\kappa(s-\tau)}\phi(\tau)d\tau.$$

Proof. From these SDEs, we get

$$m(s) = \overline{m} + e^{-\kappa(s-t)}(m(t) - \overline{m}) + e^{-\kappa s}\int_t^s e^{\kappa\tau}\phi(\tau)d\varpi_\tau, \quad \text{for } s > t$$

and

$$Y(s) = Y(t) + \int_t^s m(\tau)d\tau + \int_t^s \nu d\varpi_\tau.$$

Then straightforward calculations provide the above expressions for the moments. ■

C Proofs

In the following proofs, all $\mathcal{F}_t$ corresponds to $\mathcal{F}_t^D$, the filtration generated by the realized dividend process, $\{D(s) : s \leq t\}$.

Proof of Proposition 1:

Proof. The optimal consumption policy is easily shown to be

$$c^*(t) = \lambda^{-1/\gamma} \exp\left\{\frac{1}{\gamma}\int_0^t \left(r(s) - \delta + \frac{1}{2}\theta(s)^2\right) ds + \frac{1}{\gamma}\int_0^t \theta(s) d\varpi(s)\right\}$$

where λ is the Lagrangian multiplier that can be solved by substituting $c^*(t)$ into (8) with equality. The process for c^* is then

$$\frac{dc^*(t)}{c^*(t)} = \frac{1}{\gamma}\left(r(t) - \delta + \frac{(\gamma+1)}{2\gamma}\theta(t)^2\right) dt + \frac{1}{\gamma}\theta(t) d\varpi(t).$$

In the equilibrium, consumption equals the dividend paid, i.e.,

$$c^*(t) = D(t), \qquad \forall t.$$

Equations (10) and (11) are then established by comparing processes for $D(t)$ and $c^*(t)$. ■

Proof of Proposition 2:

Proof.

$$\begin{aligned} P(t,T;r(t)) &= E^Q\left[e^{-\int_t^T r(u)du} | \mathcal{F}_t\right] \\ &= E\left[e^{-\int_t^T \frac{1}{2}\theta^2 du - \int_t^T \theta d\varpi_u} e^{-\int_t^T r(u)du} | \mathcal{F}_t\right] \\ &= e^{-\frac{1}{2}\theta^2(T-t)} E\left[e^{-\int_t^T r(u)du - \int_t^T \theta d\varpi_u} | \mathcal{F}_t\right]. \end{aligned}$$

If we denote

$$Y(T) = Y(t) + \int_t^T r(u)du + \int_t^T \theta d\varpi_u,$$

we then have

$$P(t,T;r(t)) = e^{-\frac{1}{2}\theta^2(T-t)} e^{-E_t[Y(T)-Y(t)] + \frac{1}{2}Var_t[Y(T)]}.$$

By the lemma above and (12) and recall that $\theta = \gamma\nu$, we get

$$E\left[Y(T) - Y(t)|\mathcal{F}_t\right] = \overline{r}(T-t) + \frac{(r(t) - \overline{r})}{\kappa}\left(1 - e^{-\kappa(T-t)}\right)$$

and

$$
\begin{aligned}
Var\left[Y(T)|\mathcal{F}_t\right] &= \gamma^2\nu^2(T-t) + \frac{2\gamma^2\nu}{\kappa}\int_t^T \left(\rho + \frac{V(\tau)}{\nu}\right)\left(1-e^{-\kappa(T-\tau)}\right)d\tau \\
&\quad + \frac{\gamma^2}{\kappa^2}\int_t^T \left(\rho + \frac{V(\tau)}{\nu}\right)^2\left(1-e^{-\kappa(T-\tau)}\right)^2 d\tau.
\end{aligned}
$$

Hence, we arrive at (15). ■

Proof of Proposition 3:

Proof. Using Fubini's Theorem and Girsanov's Theorem, we have

$$
\begin{aligned}
S(t) &= E^Q\left[\int_t^T e^{-\int_t^s r(u)du} D(s)ds|\mathcal{F}_t\right] \\
&= \int_t^T E^Q\left[e^{-\int_t^s r(u)du} D(s)|\mathcal{F}_t\right] ds \\
&= \int_t^T E\left[\exp\left\{\int_t^s -\frac{1}{2}\theta^2 du - \int_t^s \theta d\varpi_u\right\}\exp\left\{-\int_t^s r(u)du\right\} D(s)|\mathcal{F}_t\right] ds,
\end{aligned}
$$

but

$$
D(s) = D(t)\exp\left\{\int_t^s \left(m(u) - \frac{1}{2}\nu^2\right)du + \int_t^s \nu d\varpi_u\right\}
$$

and

$$
r(s) = \delta - \frac{1}{2}\gamma(1+\gamma)\nu^2 + \gamma m(s),
$$

thus

$$
\begin{aligned}
S(t) &= D(t)E\left[\int_t^T e^{\int_t^s\left(-\frac{(1-\gamma)}{2}\nu^2-\delta\right)du} e^{(1-\gamma)\left[\int_t^s m(u)du + \int_t^s \nu d\varpi_u\right]} ds|\mathcal{F}_t\right] \\
&= D(t)\int_t^T \exp\left\{\left(-\frac{(1-\gamma)}{2}\nu^2 - \delta\right)(s-t)\right\} E\left[\exp\left\{(1-\gamma)\left(Y(s)-Y(t)\right)\right\} ds|\mathcal{F}_t\right]
\end{aligned}
$$

where

$$
dY(t) = m(t)dt + \nu d\varpi_t.
$$

Then by the lemma in Appendix B, we have

$$
\begin{aligned}
S(t) &= D(t)\int_t^T \exp\left\{\left(-\frac{(1-\gamma)}{2}\nu^2 - \delta\right)(s-t)\right\} \\
&\quad \times \exp\left\{(1-\gamma)E\left[Y(s)-Y(t)|\mathcal{F}_t\right] + \frac{(1-\gamma)^2}{2}Var\left[Y(s)-Y(t)|\mathcal{F}_t\right]\right\} ds.
\end{aligned}
$$

Collecting terms, we arrive at (21) and (22). ■

Proof of Corollary 1:

Proof.

$$\begin{aligned}
E_t\left[S(T)\right] &= E_t\left[D(T)p(r(T),T)\right] \\
&= E_t\left[D(t)e^{\int_t^T\left(m(s)-\frac{\nu^2}{2}\right)ds+\int_t^T \nu d\varpi_s}\int_T^{\mathcal{T}} e^{\psi(T,s;r(T))}ds\right] \\
&= D(t)e^{-\frac{\nu^2(T-t)}{2}}\int_T^{\mathcal{T}} dse^{\Psi(T,s)}E_t\left[e^{Y(T)-Y(t)+(1-\gamma)(m(T)-\overline{m})B(T,s)}\right]
\end{aligned}$$

where

$$\begin{aligned}
\Psi(T,s) &= \left[-\frac{\delta}{\gamma}+\frac{(1-\gamma)}{\gamma}\overline{r}+\frac{(1-\gamma)\gamma}{2}\nu^2\right](s-T) \\
&\quad +\frac{(1-\gamma)^2}{2}\int_T^s\left(\nu+\phi(\tau)B(\tau,s)\right)^2 d\tau
\end{aligned}$$

and

$$E_t\left[e^{Y(T)-Y(t)+(1-\gamma)(m(T)-\overline{m})B(T,s)}\right]=\exp\left\{\begin{array}{c} E_t\left[Y(T)-Y(t)\right]+(1-\gamma)B(T,s)E_t\left[m(T)-\overline{m}\right] \\ +\frac{1}{2}Var_t\left[Y(T)\right]+\frac{(1-\gamma)^2}{2}B(T,s)^2Var_t\left[m(T)\right] \\ +(1-\gamma)B(T,s)Cov_t\left[Y(T),m(T)\right] \end{array}\right\}.$$

Here we have

$$Y(T)-Y(t)=\int_t^T m(s)ds+\int_t^T \nu d\varpi_s.$$

Using the lemma in Appendix B, we get

$$E_t\left[S(T)\right]=E_t\left[D(T)\right]\int_T^{\mathcal{T}} dse^{\Psi(T,s)+\Phi(t,T,s;r)},$$

where

$$E_t\left[D(T)\right]=\exp\left\{\begin{array}{c} \left(\overline{m}-\frac{\nu^2}{2}\right)(T-t)+(m(t)-\overline{m})B(t,T) \\ +\frac{1}{2}\int_t^T\left(\nu+\phi(\tau)B(\tau,T)\right)^2 d\tau \end{array}\right\}$$

and

$$\begin{aligned}
\Phi(t,T,s;r) &= (1-\gamma)B(T,s)(m(t)-\overline{m})e^{-\kappa(T-t)} \\
&\quad +\frac{(1-\gamma)^2}{2}B(T,s)^2e^{-2\kappa T}\int_t^T e^{2\kappa\tau}\phi^2(\tau)d\tau \\
&\quad +(1-\gamma)B(T,s)\left[\begin{array}{c} \int_t^T e^{-\kappa(T-\tau)}\frac{\left(1-e^{-\kappa(T-\tau)}\right)}{\kappa}\phi^2(\tau)d\tau \\ +\int_t^T \nu e^{-\kappa(T-\tau)}\phi(\tau)d\tau \end{array}\right].
\end{aligned}$$ ■

References

Abel, A.B., 1988. Stock Prices under Time-varying Dividend Risk: An Exact Solution in an Infinite-Horizon Equilibrium Model. *Journal of Monetary Economics* 22, 375-393.

Abel, A.B., 1999. Risk Premia and Term Premia in General Equilibrium. *Journal of Monetary Economics* 43, 3-33.

Barsky, R.B., 1989. Why Don't the Prices of Stocks and Bonds Move Together? *American Economic Review* 79, 1132-1145.

Barsky, R.B., and J.B. De Long, 1993. Why Does the Stock Market Fluctuate? *Quarterly Journal of Economics* 108, 291-311.

Basak, S., and D. Cuoco, 1998. An Equilibrium Model with Restricted Stock Market Participation. *Review of Financial Studies* 11, 309-341.

Brennan, M.J., and Y. Xia, 2001. Stock Price Volatility and Equity Premium. *Journal of Monetary Economics 47, 249-283.*

Campbell, J.Y., 1999. Asset Prices, Consumption, and the Business Cycle, in John B. Taylor and Michael Woodford, eds., *Handbook of Macroeconomics*, Vol. 1 (North-Holland, Amsterdam) 1231-1303.

Campbell, J.Y., A.W. Lo, and A.C. MacKinlay, 1997. The Econometrics of Financial Markets (Princeton University Press, Princeton, New Jersey).

Chapman, D.A., 1997. The Cyclical Properties of Consumption Growth and the Real Term Structure. *Journal of Monetary Economics* 39, 145-172.

Chen, N., 1991. Financial Investment Opportunities and the Macroeconomy. *Journal of Finance* 46, 529-554.

Cox, J.C., and C. Huang, 1989. Optimal Consumption and Portfolio Policies When Asset Prices Follow a Diffusion Process. *Journal of Economic Theory* 49, 33-83.

Daniel, K., and D. Marshall, 1997. Equity-Premium and Risk-Free-Rate Puzzles at Long Horizons. *Macroeconomic Dynamics* 1, 452-484.

David, A., 1997. Fluctuating Confidence in Stock Markets: Implications for Return and Volatility. *Journal of Financial and Quantitative Analysis* 32, 427-462.

Detemple, J.B., 1986. Asset Pricing in a Production Economy with Incomplete Information. *Journal of Finance* 41, 383-392.

Detemple, J.B., 1989. Further Results on Asset Pricing with Incomplete Information. *Journal of Economic Dynamics and Control* 15, 425-453.

Dothan, M., and D. Feldman, 1986. Equilibrium Interest Rates and Multiperiod Bonds in a Partially Observable Economy. *Journal of Finance* 41, 369-382.

Drèze, J., and F. Modigliani, 1972. Consumption Decisions under Uncertainty. *Journal of Economic Theory* 5, 308-335.

Duffie, D., M. Schroder, and C. Skiadas, 1997. A Term Structure Model with Preferences for the Timing of Resolution of Uncertainty. *Economic Theory* 9, 3-22.

Estrella, A., and G.A. Hardouvelis, 1991. The Term Structure as a Predictor of Real Economic Activity. *Journal of Finance* 46, 555-576.

Fama, F.F., and K.R. French, 1989. Business Conditions and Expected Returns on Stocks and Bonds. *Journal of Financial Economics* 23-49.

Feldman, D., 1989. The Term Structure of Interest Rates in a Partially Observable Economy. *Journal of Finance* 44, 789-812.

Gennotte, G., 1986. Optimal Portfolio Choice under Incomplete Information. *Journal of Finance* 41, 733-746.

Gennotte, G., and T. Marsh, 1993. Variations in Economic Uncertainty and Risk Premium on Capital Assets. *European Economic Review* 37, 1021-1041.

Goldstein, R., and F. Zapatero, 1996. General Equilibrium with Constant Relative Risk Aversion and Vasicek Interest Rates. *Mathematical Finance* 6, 331-340.

Hagiwara, M., and M.A. Herce, 1997. Risk Aversion and Stock Price Sensitivity to Dividends. *American Economic Review* 87, 738-745.

Harrison, M., and D. Kreps, 1979. Martingale and Multiperiod Securities Markets. *Journal of Economic Theory* 20, 381-408.

Kandel, S., and R.F. Stambaugh, 1990. Expectations and Volatility of Consumption and Asset Returns. *Review of Financial Studies* 3, 207-232.

Karazas, I., J. Lehoczky, and S. Shreve, 1987. Optimal Portfolio and Consumption Decisions for a "Small Investor" on a Finite Horizon. *SIAM Journal of Control and Optimization* 25, 1157-1186.

Kimball, M.S., 1990. Precautionary Savings in the Small and in the Large. *Econometrica* 58, 53-73.

Leland, H., 1968. Savings and Uncertainty: The Precautionary Demand for Savings. *Quarterly Journal of Economics* 82, 465-473.

Lipster, R.S., and A.N. Shiryayev, 1978. Statistics of Random Processes I &II (Springer-Verlag, New York).

Lucas, R., 1978. Asset Prices in an Exchange Economy. *Econometrica* 46, 1429-1445.

Mankiw, N.G., and S.P. Zeldes, 1991. The Consumption of Stockholders and Nonstockholders. *Journal of Financial Economics* 29, 97-112.

Mehra, R., and E.C. Prescott, 1985. The Equity Premium: A Puzzle. *Journal of Monetary Economics* 15, 145-161.

Merton, R.C., 1980. On Estimating the Expected Return on the Market: An Exploratory Investigation. *Journal of Financial Economics* 8, 323-362.

Roma, A., and W. Torous, 1997. The Cyclical Behavior of Interest Rates. *Journal of Finance* 52, 1519-1542.

Rubinstein, M., 1976. The Valuation of Uncertain Income Streams and the Pricing of Options. *Bell Journal of Economics and Management Science* 7, 407-425.

Sandmo, A., 1970. The Effect of Uncertainty on Saving Decisions. *Review of Economic Studies* 37, 353-360.

Timmerman, A.G., 1996. Excess Volatility and Predictability of Stock Prices in Autoregressive Dividend Models with Learning. *Review of Economic Studies* 63, 523-557.

Vasicek, O., 1977. An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics* 5, 177-188.

Veronesi, P., 2000. How Does Information Quality Affect Stock Returns? *Journal of Finance* 55, 807-837.

Wachter, J.A., 2001. Risk Aversion and Allocation to Long-Term Bonds. Working Paper, New York University.

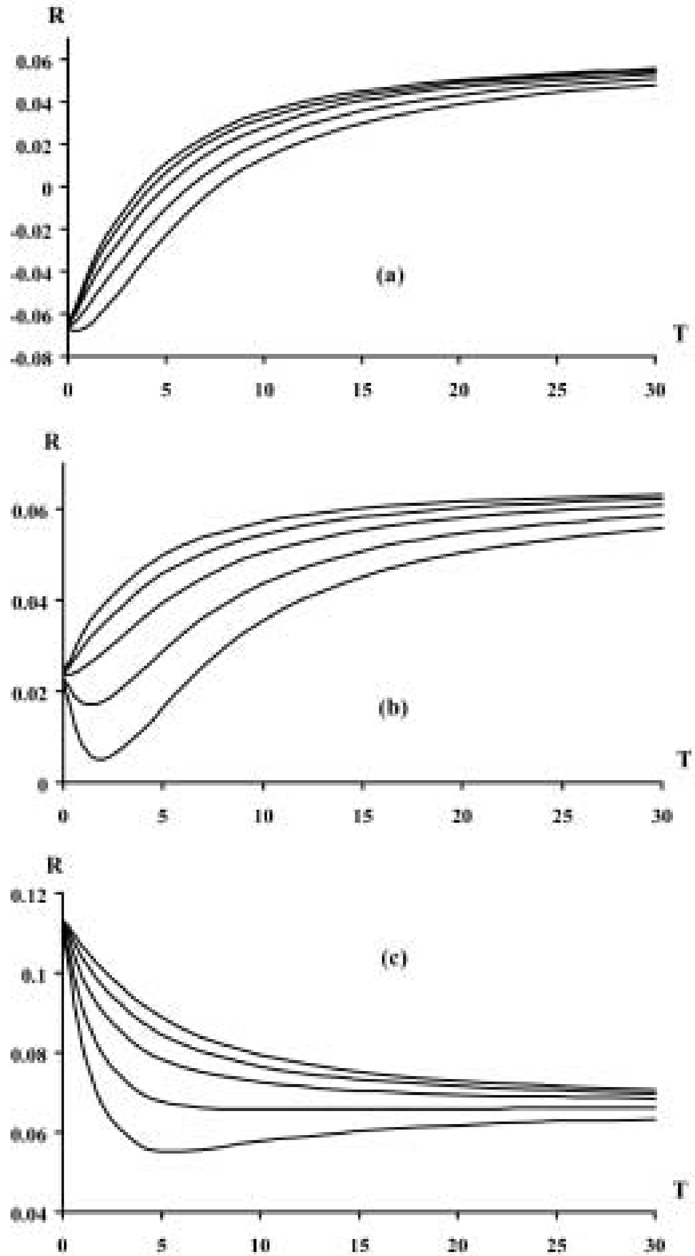

Figure 1: Term structure of (real) interest rates for (a) $m = -1\%$ (recession), (b) $m = 2\%$ (moderate growth), and (c) $m = 5\%$ (fast growth). In each graph the curves, from top to bottom, correspond to $V_0 = 0$ (fully observable), 0.001, 0.0025, 0.005, and 0.008, respectively. The parameters are $\delta = 0.05$, $\nu = 0.12$, $\kappa = 0.4$, $\eta = 0$, and $\rho = -0.02$. In addition, $\gamma = 3$ and $\overline{r} = 2.36\%$.

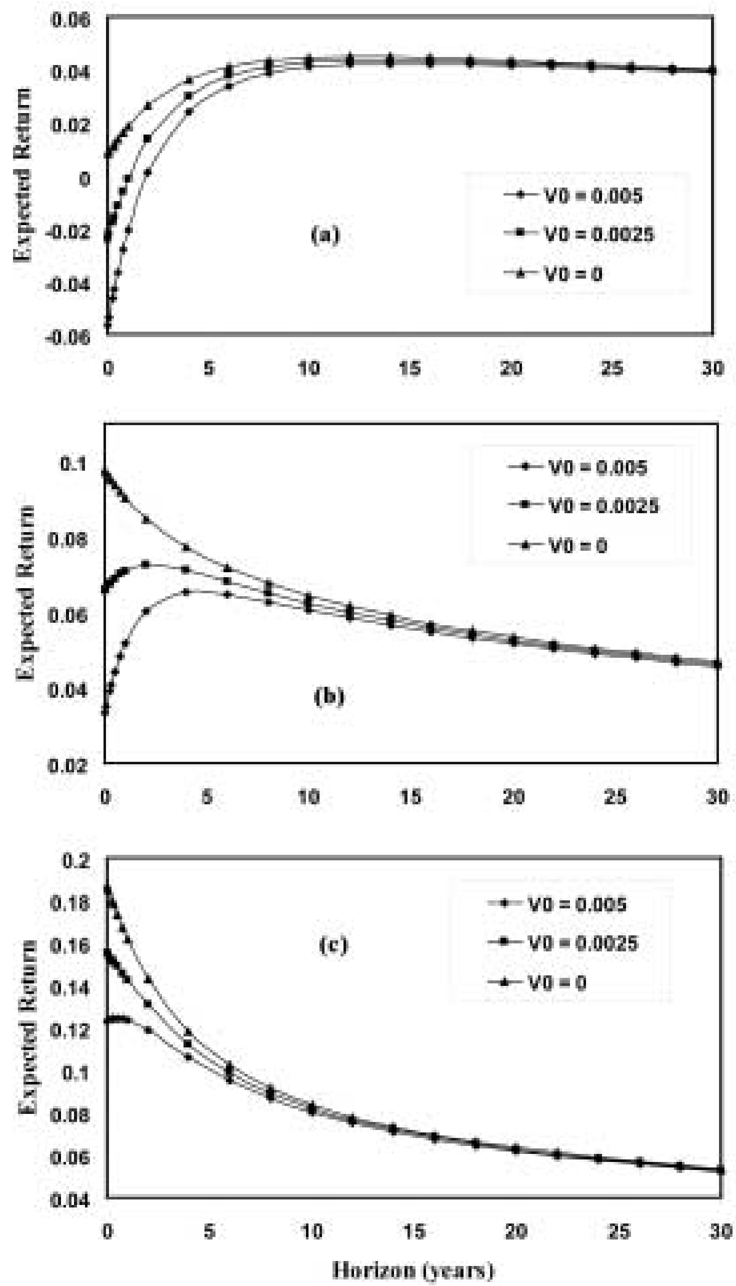

Figure 2: Expected return as a function of holding horizon with the expected growth rate (a) $m = -1\%$ (expected recession), (b) $m = 2\%$ (expected moderate growth), and (c) $m = 5\%$ (expected fast growth). Three different levels of uncertainty about future growth are represented: $V_0 = 0$ (full information), $V_0 = 0.0025$, and $V_0 = 0.005$. Other parameters are: $\delta = 0.05$, $\nu = 0.12$, $\kappa = 0.4$, $\rho = -0.02$, $\overline{m} = 0.02$, and $\gamma = 3$.

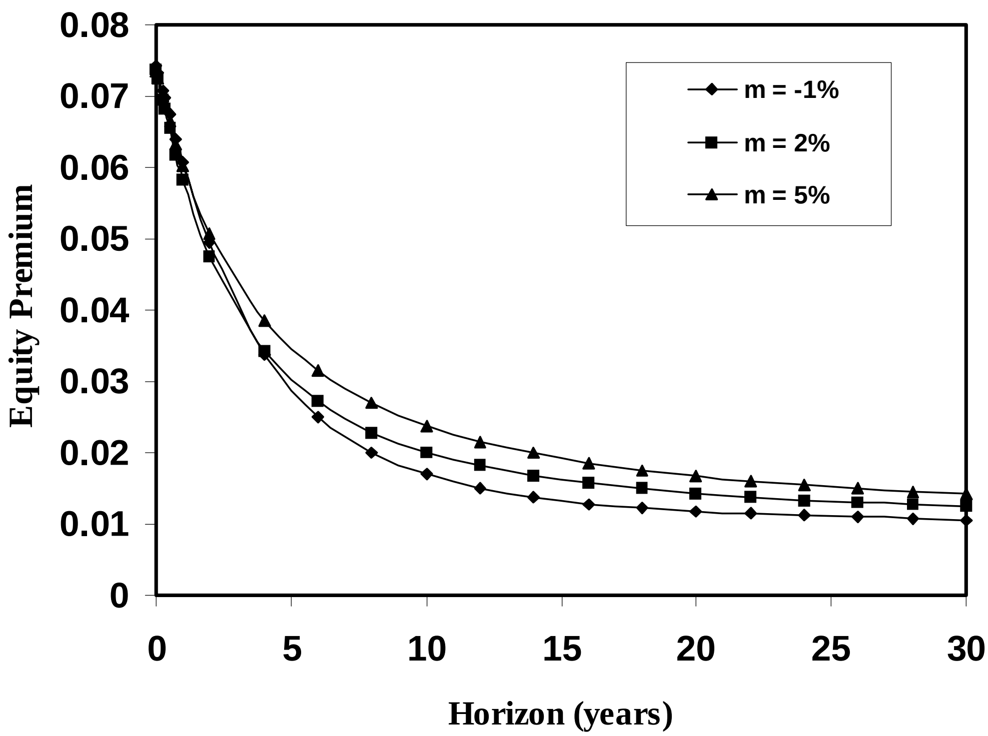

Figure 3: Equity premium as a function of holding horizon. Full information case with $V_0 = 0$. Other parameters are the same as those in Figure 2.

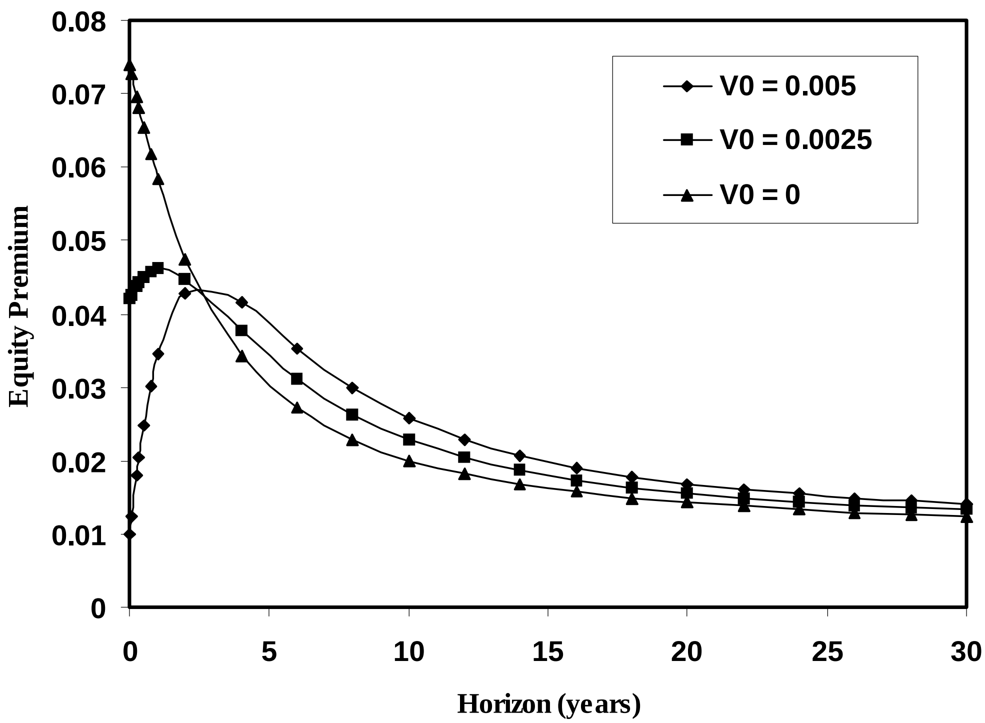

Figure 4: Equity premium as a function of holding horizon. The expected growth rate is $m = 2\%$. Other parameters are the same as those in Figure 2.