Optimal control of networks: energy scaling and open challenges

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Control of Networks (1)

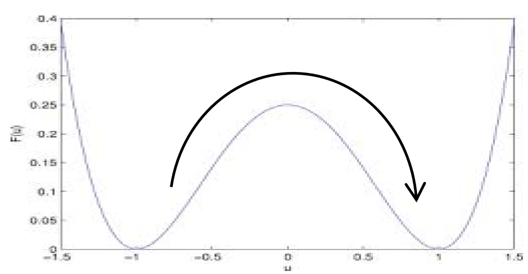
• Power Grid Dynamics: maintaining frequency of generators in the presence of perturbations

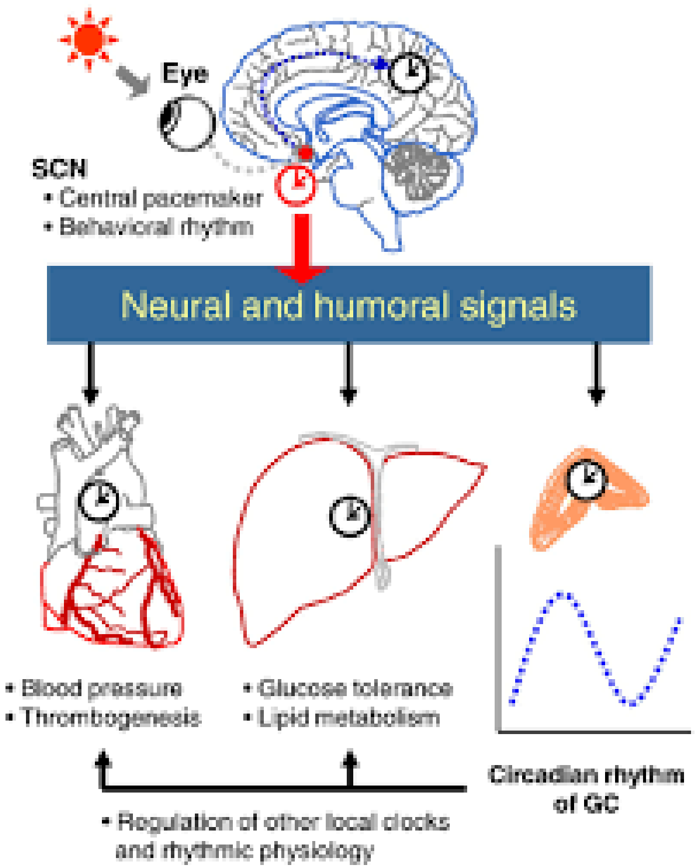
Control of Networks (2)

- Control of Mammalian Circadian
- Rhythm

- The dynamics is multistable (both fixed points and limit cycles)

- Problem: moving from one attractor to the basin of attraction of another attractor



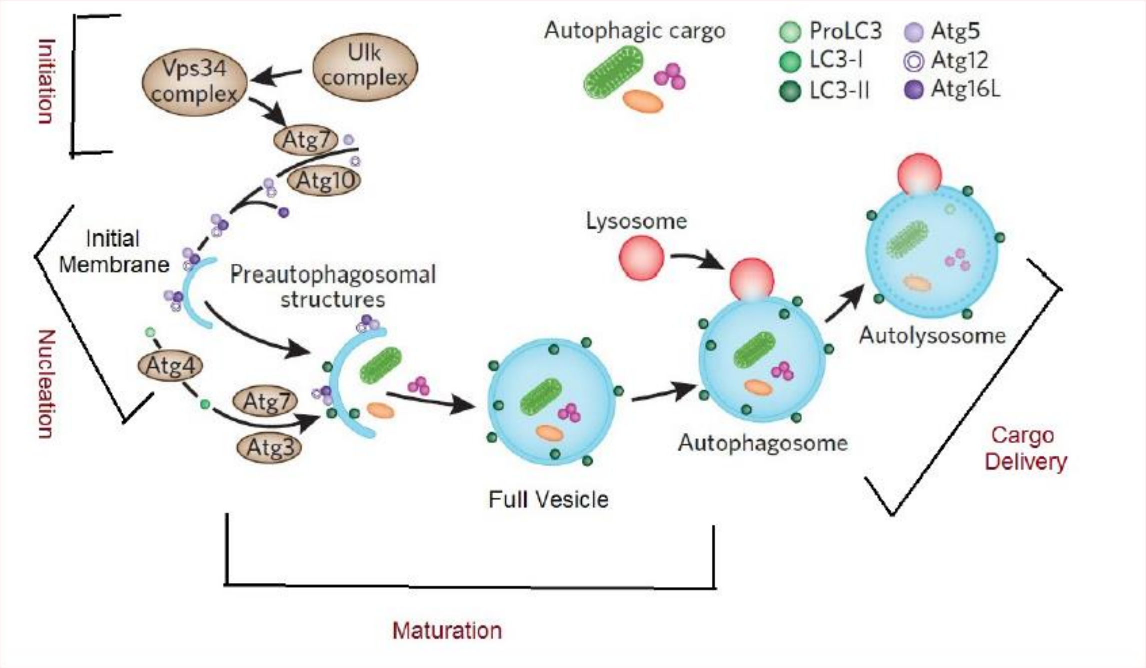


Chung, Son, Kim, Circadian rhythm of adrenal glucocorticoid: Its regulation and clinical implications

Control of Networks (3)

- Autophagy regulation

◆ Consider the Autophagic system in a single cell,



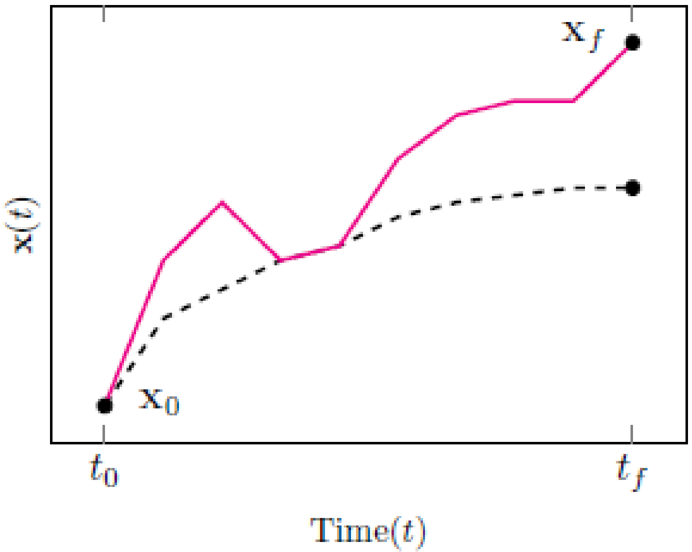
Wang et al., The Molecular Mechanism of Autophagy (2003)

One option: Optimal Control

Controllability

Consider the continuous time system,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t) + B\mathbf{u}(t)$$
$$\mathbf{x}(t_0) = \mathbf{x}_0$$
$$\mathbf{x}(t_f) = \mathbf{x}_f$$



One option: Optimal Control

Control Energy

Control Energy,

$$J = \int_{t_0}^{t_f} \mathbf{u}(t)^T \mathbf{u}(t) dt$$

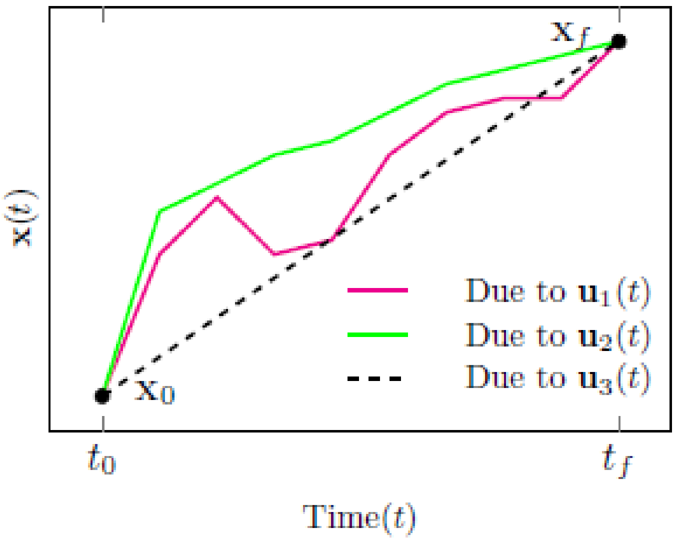
Optimal Control Input

Optimal Control Input = $\mathbf{u}^*(t)$.

Optimal Control Energy

Optimal Control Energy,

$$J^* = \int_{t_0}^{t_f} \mathbf{u}^*(t)^T \mathbf{u}^*(t) dt$$



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Issues

- The dynamics of complex networks is nonlinear

- Control of nonlinear systems is difficult!

- Optimal control strategies for nonlinear systems are typically obtained numerically

- Numerical optimal control solutions for large high-dimensional nonlinear systems are computationally expensive

Target Control of Networks

Isaac Klickstein, Afroza Shirin,Francesco Sorrentino

A network is described by two sets:

1. A set of nodes, $\mathcal{V}$ (often these coincide with the states), and
2. A set of edges, $\mathcal{E}$ (these are the linearized dynamical relations between nodes)

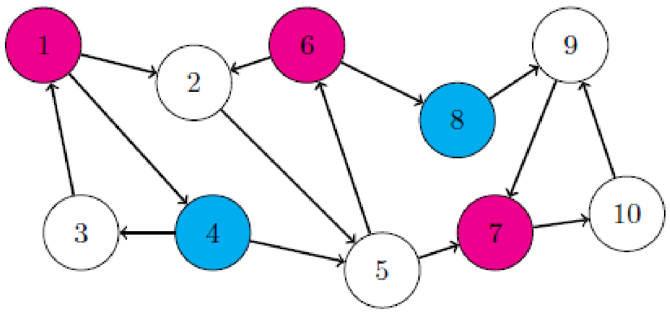

Figure: A 10 Node Network

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_j + \sum_{k=1}^{m} b_{ik} u_k$$

There are three types of nodes:

1. Driver Nodes: These can be directly influenced by our control inputs, u_k, $k = 1, \ldots, m$.
2. Target Nodes: These are nodes with a desired final condition.
3. Neither: These are nodes that are neither driven nor targeted.

Target Control of Networks

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A state, $x_i(t)$, $i = 1, \dots, n$ corresponds to a node $v_i \in \mathcal{V}$.
We define our state vector as,

$$\mathbf{x}(t) = \left\{ \begin{array}{c} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{array} \right\} \qquad (4)$$

The **adjacency matrix**, $A = \{a_{ij}\}$, contains the **edges** $\in \mathcal{E}$ where if $a_{ij} \neq 0$, the state of v_j affects v_i.

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$$

Target Control of Networks

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To start, we consider all nodes as target nodes.
We define the **control energy** as,

$$E = \int_{t_0}^{t_f} ||\mathbf{u}(t)||^2 dt \tag{5}$$

The optimization problem is:

$$\min_{\mathbf{u}(t)} \quad J = \frac{1}{2}E = \frac{1}{2}\int_{t_0}^{t_f} ||\mathbf{u}(t)||^2 dt$$
$$\text{such that} \quad \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{6}$$
$$\mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x}(t_f) = \mathbf{x}_f$$

$J(\mathbf{x}(t), \mathbf{u}(t))$ is the **cost function**, or penalty function.

Target Control of Networks

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The solution is,

$$\mathbf{u}^*(t) = B^T e^{A^T(t_f - t)} W^{-1} \beta \qquad (7)$$

where,

$$W = \int_{t_0}^{t_f} e^{A(t_f-\tau)} B B^T e^{A^T(t_f-\tau)} d\tau, \qquad \beta = \left(\mathbf{x}_f - e^{A(t_f - t_0)} \mathbf{x}_0\right)$$

W is the controllability Gramian.
More importantly, the minimum energy is,

$$\begin{aligned} E_{\text{min}} &= \int_{t_0}^{t_f} ||\mathbf{u}^*(t)||^2 dt \\ &= \beta^T W^{-1} \beta \end{aligned} \qquad (8)$$

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The controllability Gramian tends to be poorly conditioned when,

1. The time interval, $t_f - t_0$ is 'small', or
2. The percentage of nodes which are drivers is small.

Why does the condition of W matter?
Min-Max Theorem

$$E_{\min}^{(\min)} \leq \frac{1}{||\beta||^2} \beta^T W^{-1} \beta \leq E_{\min}^{(\max)} \tag{9}$$

So,

$$E_{\min}^{(\max)} = \frac{1}{\lambda_{\min}(W)} \tag{10}$$

which can be prohibitively large.

Target Control of Networks

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We define an output,

$$\mathbf{y}(t) = C\mathbf{x}(t), \qquad \mathbf{y}(t) \in \mathbb{R}^{p\times 1}, \qquad p \leq n \tag{11}$$

which is a linear combination of the states.

The output can be used to **target** nodes by choosing C such that each row has only one nonzero element.

Problem Statement for MEOCS:

$$\begin{aligned} \min_{\mathbf{u}(t)} \quad & J = \frac{1}{2}E = \frac{1}{2}\int_{t_0}^{t_f} ||\mathbf{u}(t)||^2 dt \\ \text{such that} \quad & \dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \\ & \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{y}(t_f) = \mathbf{y}_f \end{aligned} \tag{12}$$

Target Control of Networks

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The optimal control input,

$$\mathbf{u}^*(t) = Be^{A^T(t_f - t)} C^T \left(CWC^T \right)^{-1} \beta \tag{13}$$

The minimum energy is,

$$E_{\min} = \beta^T \left(CWC^T \right)^{-1} \beta = \beta^T W_p^{-1} \beta \tag{14}$$

where W_p is a **minor** of W.

This method reduces the **control space** of the system.

Target Control of Networks

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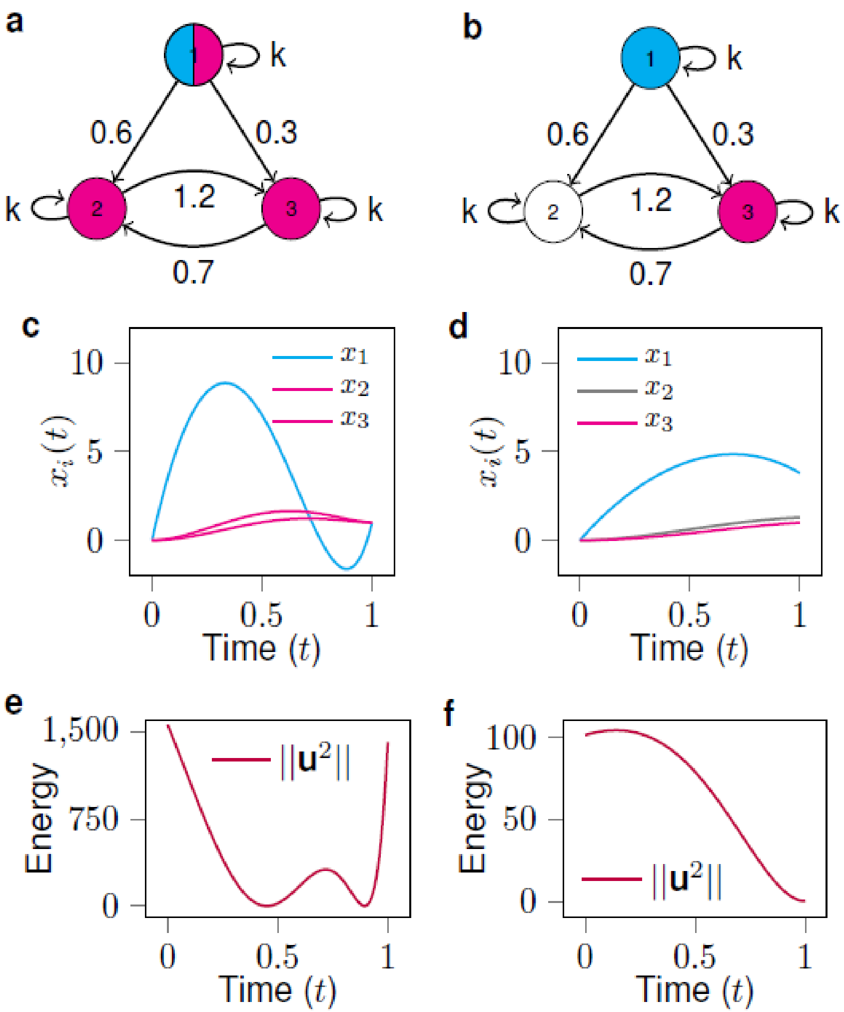

The system on the left uses the MECS formulation to place each node at a final condition. The integral of the energy magnitude curve is $E = 382$.

The system on the right assumes only node three needs to have a final condition, a MEOCS, and is the only node targeted. This time $E = 66.3$, only a sixth of the MECS formulation.

Target Control of Networks

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A four dimensional example:

$$W = \begin{bmatrix} W_{11} & W_{12} & W_{13} & W_{14} \\ W_{21} & W_{22} & W_{23} & W_{24} \\ W_{31} & W_{32} & W_{33} & W_{34} \\ W_{41} & W_{42} & W_{43} & W_{44} \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then,

$$W_p = CWC^T = \begin{bmatrix} W_{22} & W_{24} \\ W_{42} & W_{44} \end{bmatrix}$$

Cauchy Interlacing Theorem:
Proves that the minimum eigenvalue of the minor of a matrix is larger than the minimum eigenvalue of the original matrix.

Target Control of Networks

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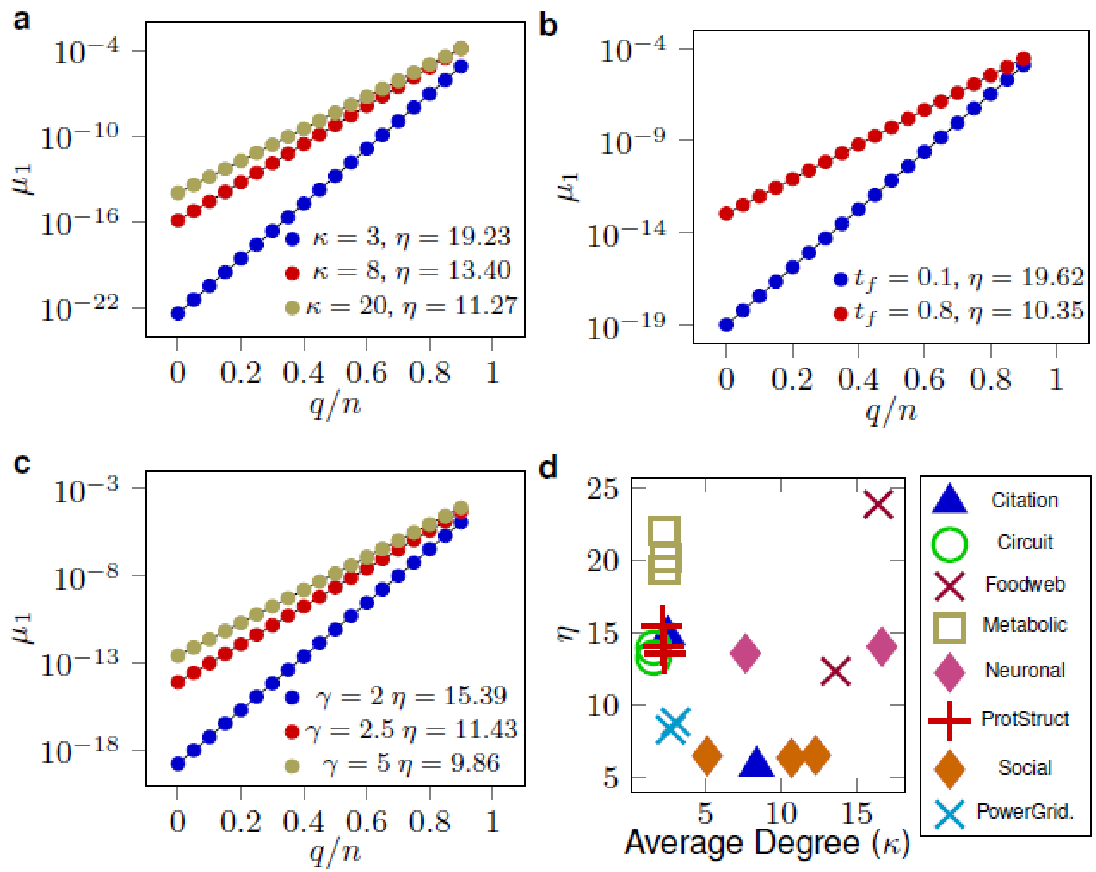

Note that,

$$E_{\text{min}}^{(\text{max})} = \frac{1}{\mu_1}$$

Four Cases:

a) Degree

b) Time Interval

c) Homogeneity

d) Real Datasets

Target Control of Networks

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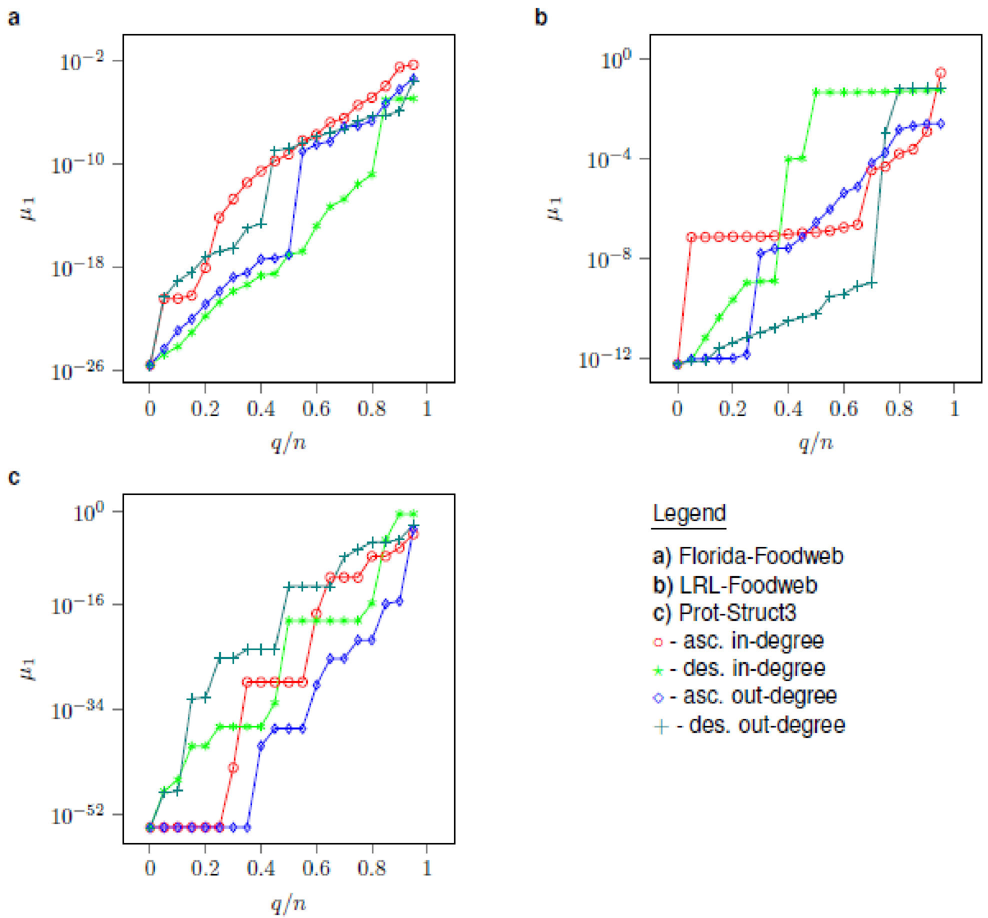

When nodes are chosen by degree, we see much less smooth behavior.

Energy vs. Distance in the Path Graph

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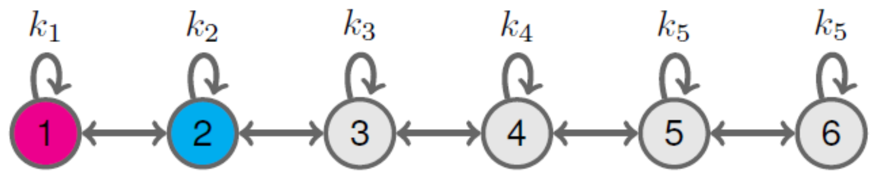

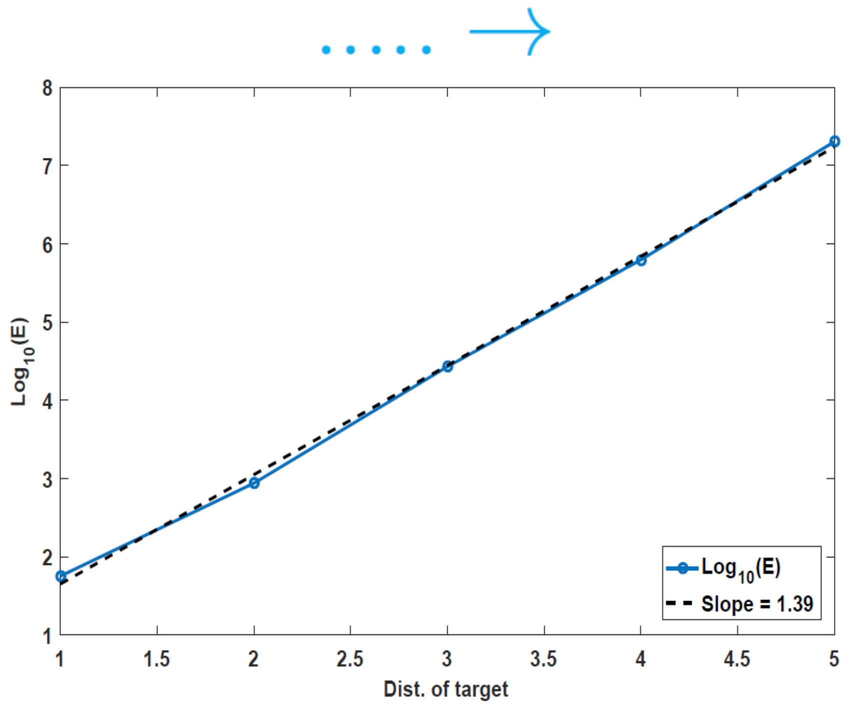

Control Energy increases exponentially with the distance

Single Target Control

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In the case of single target control, i.e., target node = *j*, we are interested in only the *j* element on the main diagonal of the Gramian matrix, $w_{j,j}$.

The single target minimum control energy is equal to

$$E = \frac{1}{w_{j,j}} \beta_j^2$$

The Lyapunov Equation

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The controllability Gramian matrix $W(t_f)$ satisfies the Lyapunov equation:

$$\dot{W}(t) = AW(t) + W(t)A + BB^T$$
$$W(0) = 0$$

One can find $W(t_f)$ by integrating the above equation forward in time from $t=0$ to $t = t_f$.

Infinite Path Graph

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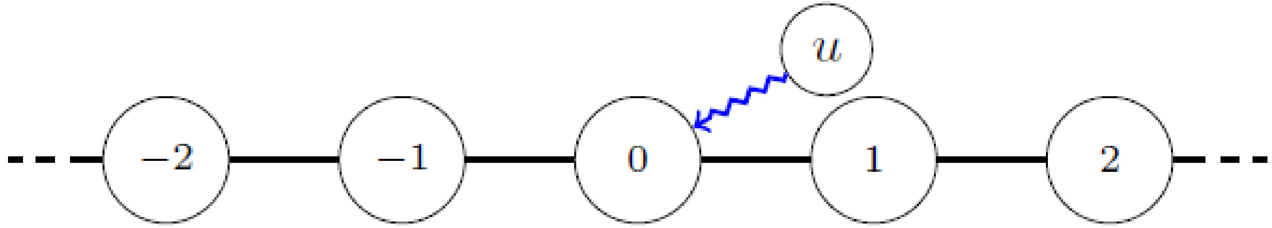

For the Path Graph, the Lyapunov equation becomes

$$\begin{aligned}\dot{w}_{i,j}(t) = -2pw_{i,j}(t) + fw_{i-1,j}(t) + fw_{i+1,j}(t) \\ + fw_{i,j-1}(t) + fw_{i,j+1}(t) + \delta_{i,0}\delta_{j,0}, \\ -\infty < i,j < \infty\end{aligned}$$

Initial condition is zero

Solution to the homogeneous equation

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$$\dot{w}_{i,j}(t) = -2pw_{i,j}(t) + fw_{i-1,j}(t) + fw_{i+1,j}(t) + fw_{i,j-1}(t) + fw_{i,j+1}(t)$$

$$-\infty < i,j < \infty$$

We apply a two-variable discrete-time Fourier transform:

$$w_{u,v}(t) = \sum_{i,j} e^{\mathcal{I}ui} e^{\mathcal{I}vj} w_{i,j}(t)$$

We obtain the following decoupled equation:

$$\dot{w}_{u,v}(t) = (-2p + 2f\cos u + 2f\cos v)w_{u,v}(t)$$

with solution

$$w_{u,v}(t) = e^{-2pt} e^{2ft\cos u} e^{2ft\cos v} w_{u,v}(0)$$

Solution (continues)

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Performing the inverse discrete-time Fourier transform:

$$w_{i,j}(t) = e^{-2pt} \sum_{\alpha,\beta} \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{-\mathcal{I}u(i-\alpha)} e^{-\mathcal{I}v(j-\beta)} \times e^{2ft\cos u} e^{2ft\cos v} du dv$$

$$= \sum_{\alpha,\beta} e^{-2pt} I_{i-\alpha}(2ft) I_{j-\beta}(2ft) w_{\alpha,\beta}(0)$$

Where the Modified Bessel Function of the First Kind,

$$I_\ell(z) = \frac{1}{2\pi} \int_0^{\pi} e^{z\cos\theta} \cos(\ell\theta) d\theta$$

Solution to the non-homogeneous Eq.

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The driver nodes are defined by the set of integers (the non-homogeneous term):

$$g = \sum_{k \in \mathcal{D}} \delta_{i,k} \delta_{j,k}$$

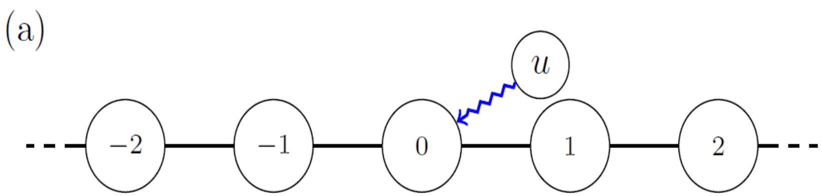
Now considering only one driver node at the origin of the infinite chain, i.e., $\mathcal{D} = \{0\}$,

$$w_{i,j}(t) = \int_0^t e^{-2p\tau} I_i(2f\tau) I_j(2f\tau) d\tau,$$
$$-\infty < i, j < \infty$$

For the single target problem (ℓ is now the *control distance*)

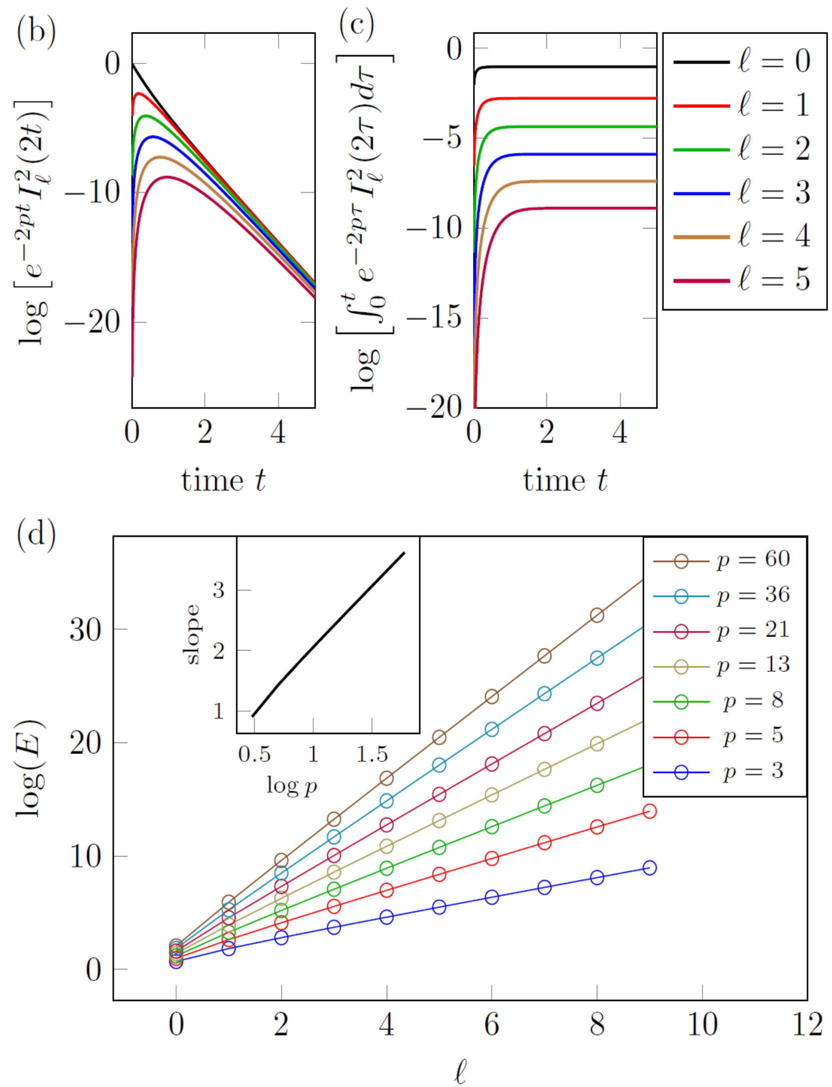
$$w_{\ell,\ell}(t) = \int_0^t e^{-2p\tau} I_\ell^2(2f\tau) d\tau.$$

Infinite chain: scaling with control distance

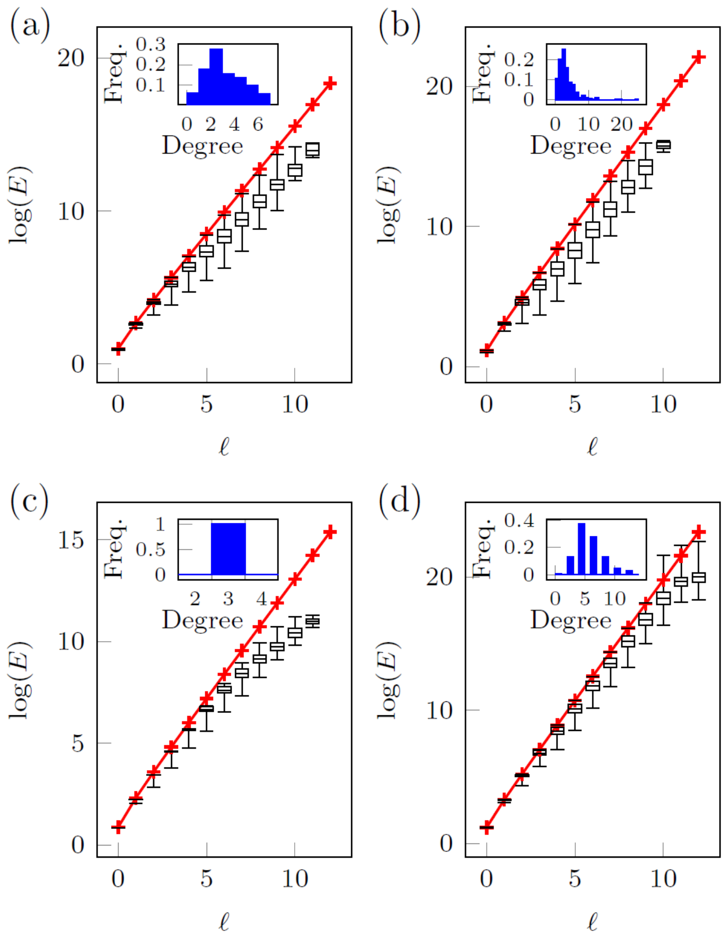


$$w_{\ell,\ell}(t) = \int_0^t e^{-2p\tau} I_\ell^2(2f\tau)d\tau.$$

Assumption:
For an arbitrary network and assigned control distance, an upper bound to the control energy is provided by the path graph



Upper bound for arbitrary networks



(a) ER Network, 300 nodes

(b) Scale Free network, 300 nodes

(c) A 3-regular graph, 300 nodes

(d) Northern European power grid

Single driver – single target

Linearization and Optimal Control

For nonlinear systems the optimal control solution of the minimum energy problem is typically *nonloncal*

This means that the optimal path from *a* to *b* goes through regions of state space that are far away from a and b

Result: the optimal control solution cannot be applied to nonlinear linearized systems as the optimal solution would leave the area of validity of the linearization, the "valid linearization region"

Sun, Nishikawa, Motter, *PRL,* 2014

Moreover, current approaches to optimally control nonlinear systems are mostly computational, e.g., GPOPS by Aniel Rao

These approaches are computationally expensive when the dimension of the system (network) is large

Locally Optimal Control Strategy ('LOCS')

Isaac Klickstein, Afroza Shirin,Francesco Sorrentino

Our first result determines *the set of final conditions* that guarantee the locality of the minimum energy controlled state trajectory of a linear dynamical system.

All the minimum energy control trajectories are included in an hyper-ellipsoid that can be parametrized by time and space

The hyper-ellipsoid represents the set of states reachable with an energy amount E(t).

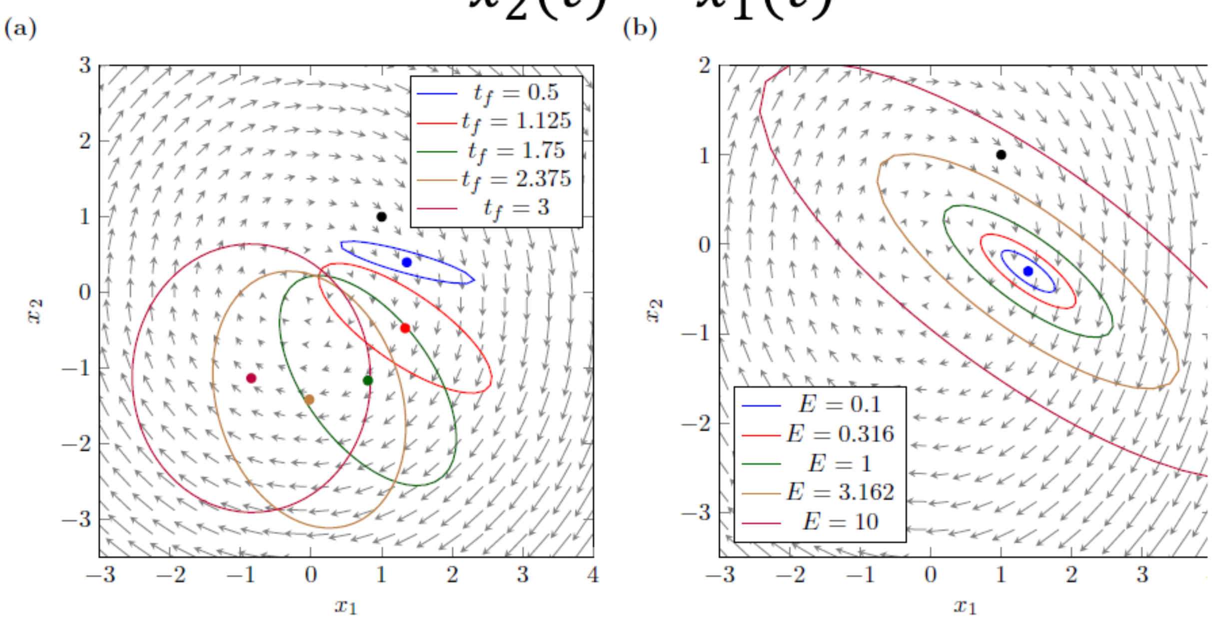
By restricting the amount of energy available, *E(tf),* we can determine a set of final conditions, *S(tf)* such that the state trajectory remains ***local***.

Locally Optimal Control Strategy ('LOCS')

Isaac Klickstein, Afroza Shirin, Francesco Sorrentino

Visualizing the ellipsoid by varying t_f and E. The dynamics are linear, two dimensional with equations:

$$\dot{x}_1(t) = x_2(t) + u(t)$$
$$\dot{x}_2(t) = x_1(t)$$



Locally Optimal Control Strategy ('LOCS')

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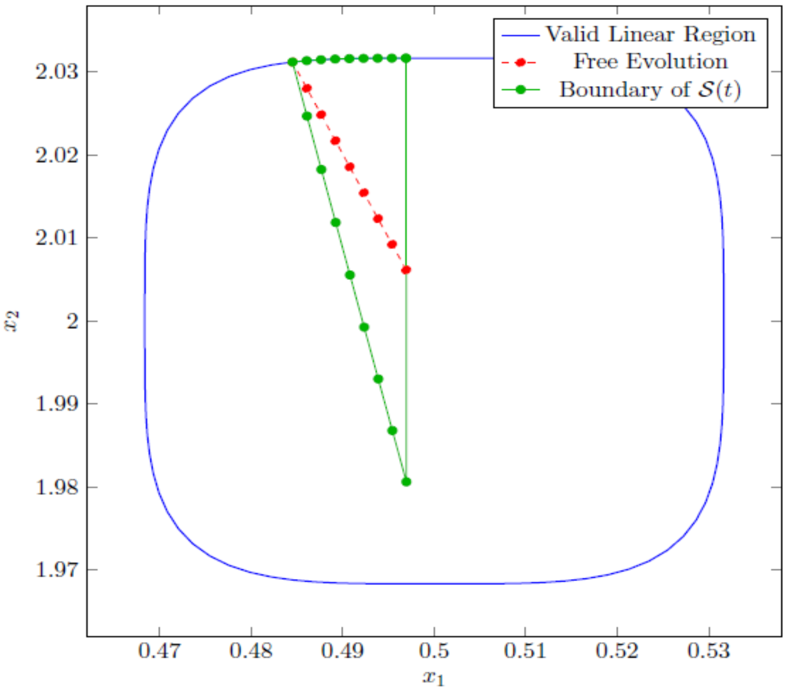
By intersecting the minimum energy ellipsoid with the valid linearization region, we can compute optimal control trajectories that are valid and remain local



The blue line depicts the region where the linearization is valid.

The red line plots the zero-input evolution of the system until it reaches the boundary of the valid linear region (VLR).

The green line depicts the region within the VLR of all states that, if assigned to be the final state for the minimum control action, the trajectory will remain within the VLR for all times.

Locally Optimal Control Strategy ('LOCS')

Isaac Klickstein, Afroza Shirin,Francesco Sorrentino

Optimal control of a nonlinear network (to some nonlocal point) can be achieved by performing a sequence of local optimal controls.

Our algorithm is based on the following steps:

1) Linearize equations
2) Define small error set
3) Find minimum energy solution that touches the small-error-set
4) Repeat(with properly chosen *intermediate targets*)

Iterating LOCS: a 2D system

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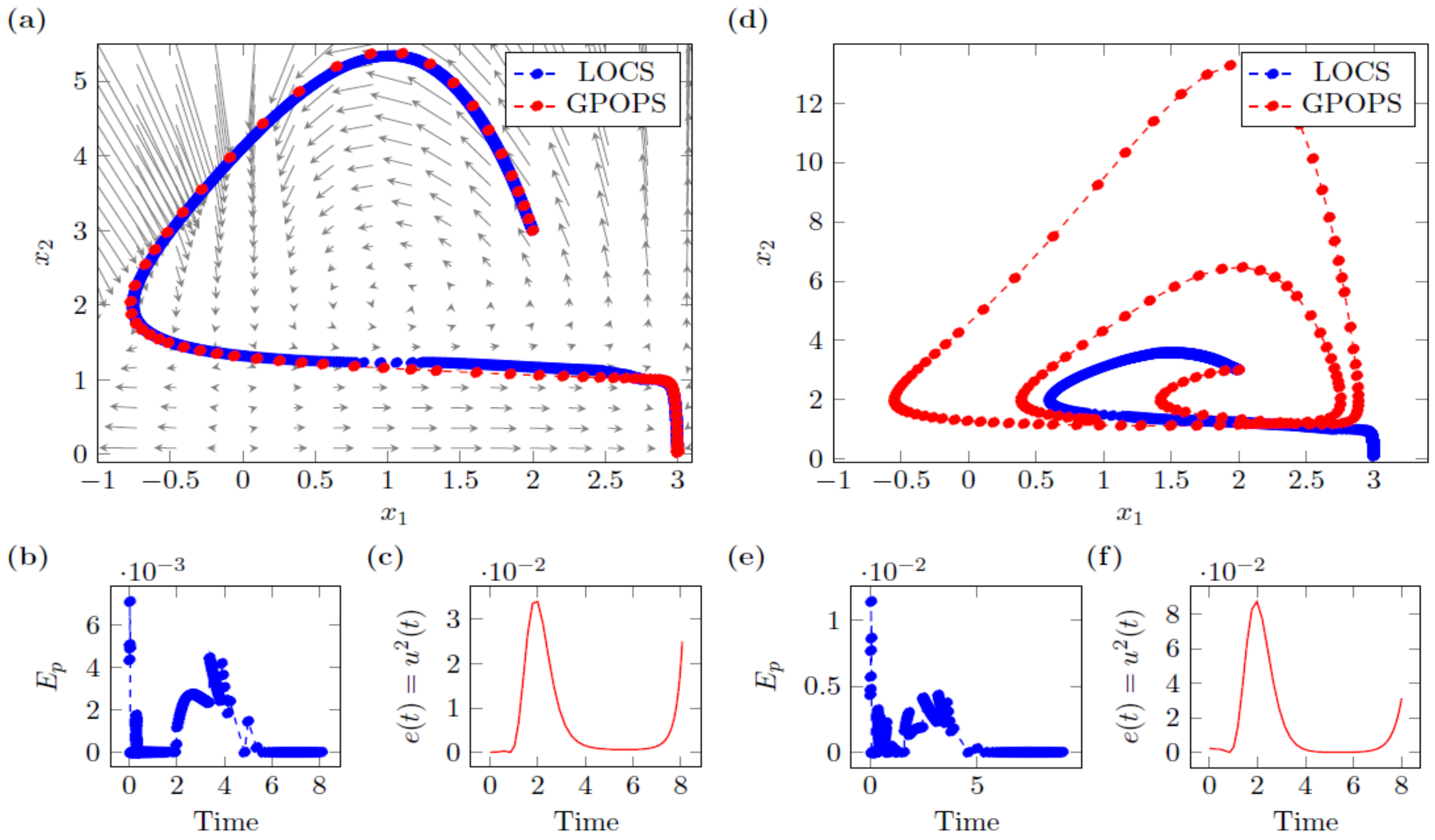

$$\dot{x}_1(t) = (x_1(t) - 3)(x_2(t) - 2)$$
$$\dot{x}_2(t) = x_2(t)(x_1(t) + 1)(x_2(t) + 1) + u(t)$$

Example: Lotka-Volterra dynamics

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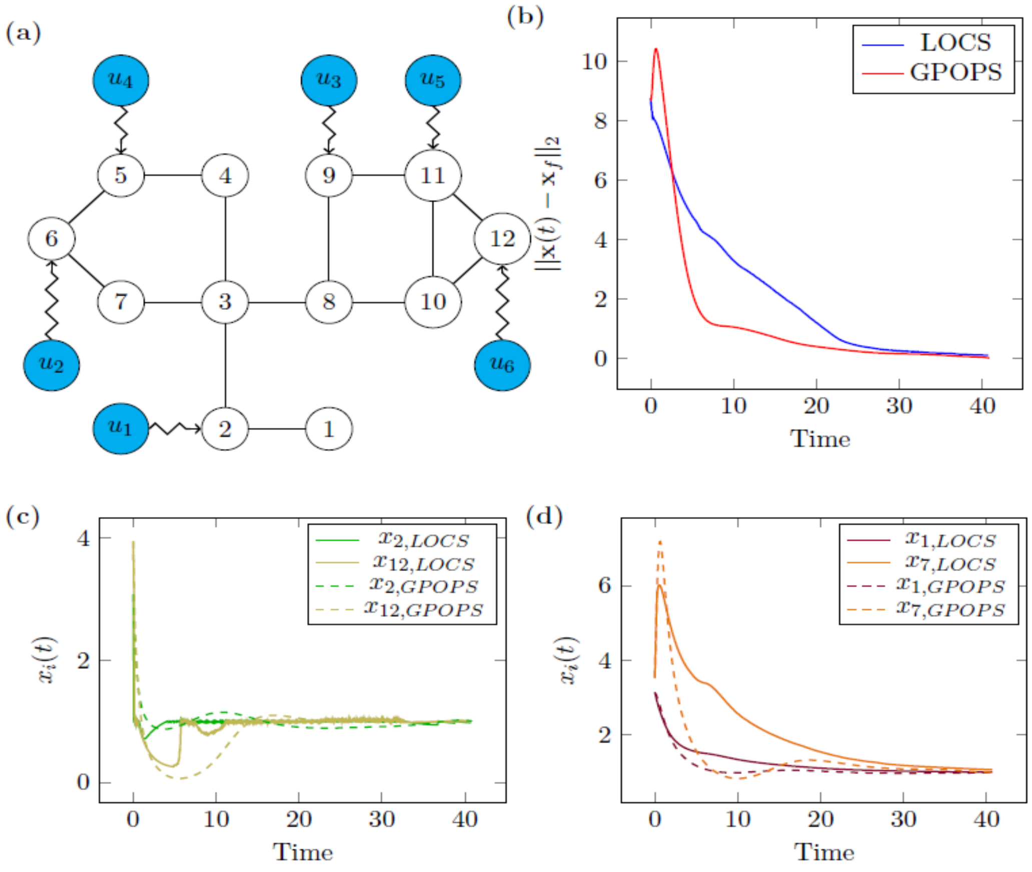

$$\dot{\mathbf{x}}(t) = \text{diag}\{\mathbf{r}\}\mathbf{x}(t) - \text{diag}\{\mathbf{x}(t)\}A\mathbf{x}(t) + B\mathbf{u}(t)$$

Mammalian Circadian Rhythm

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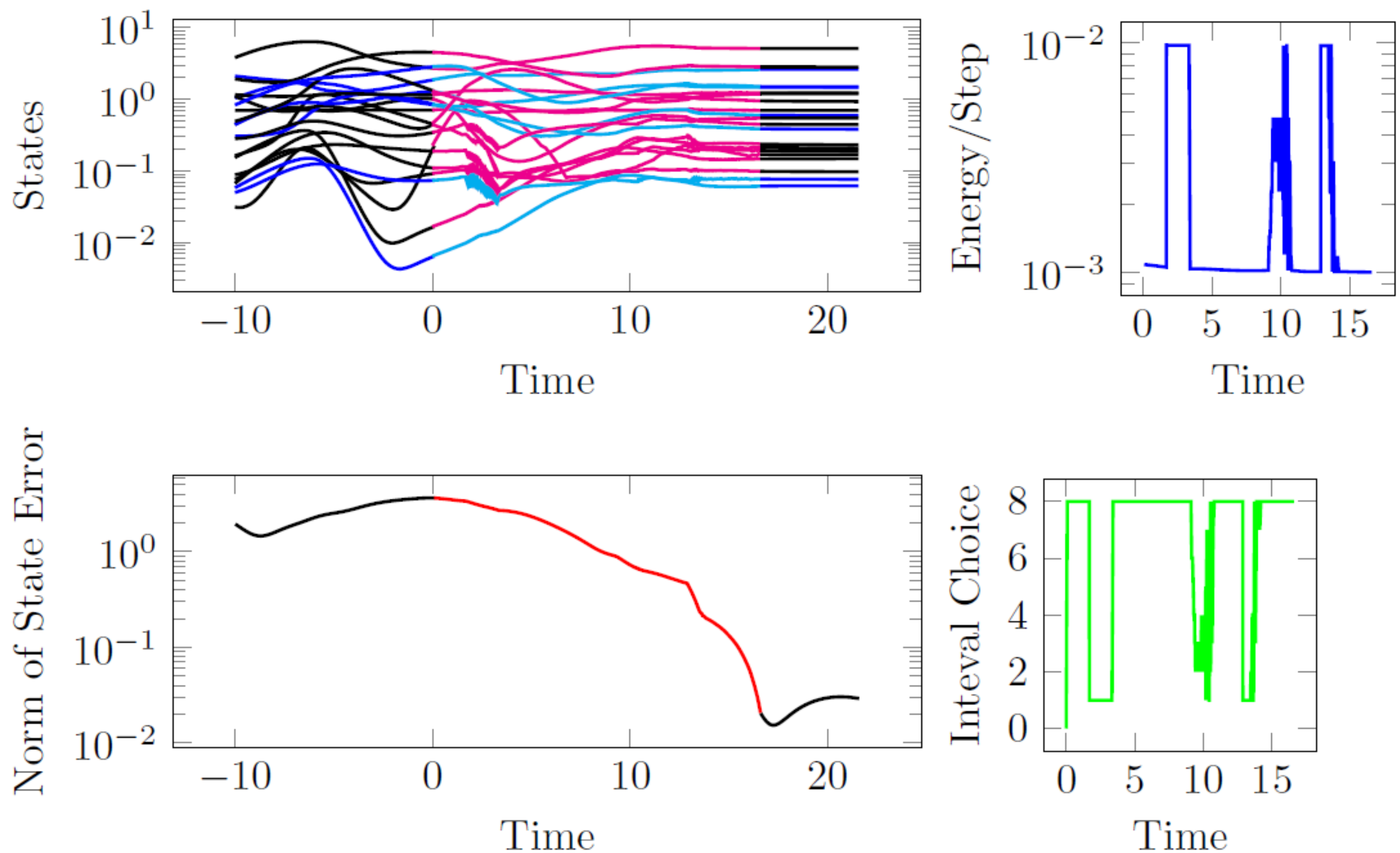

New England Power grid after failure and recovery of a line

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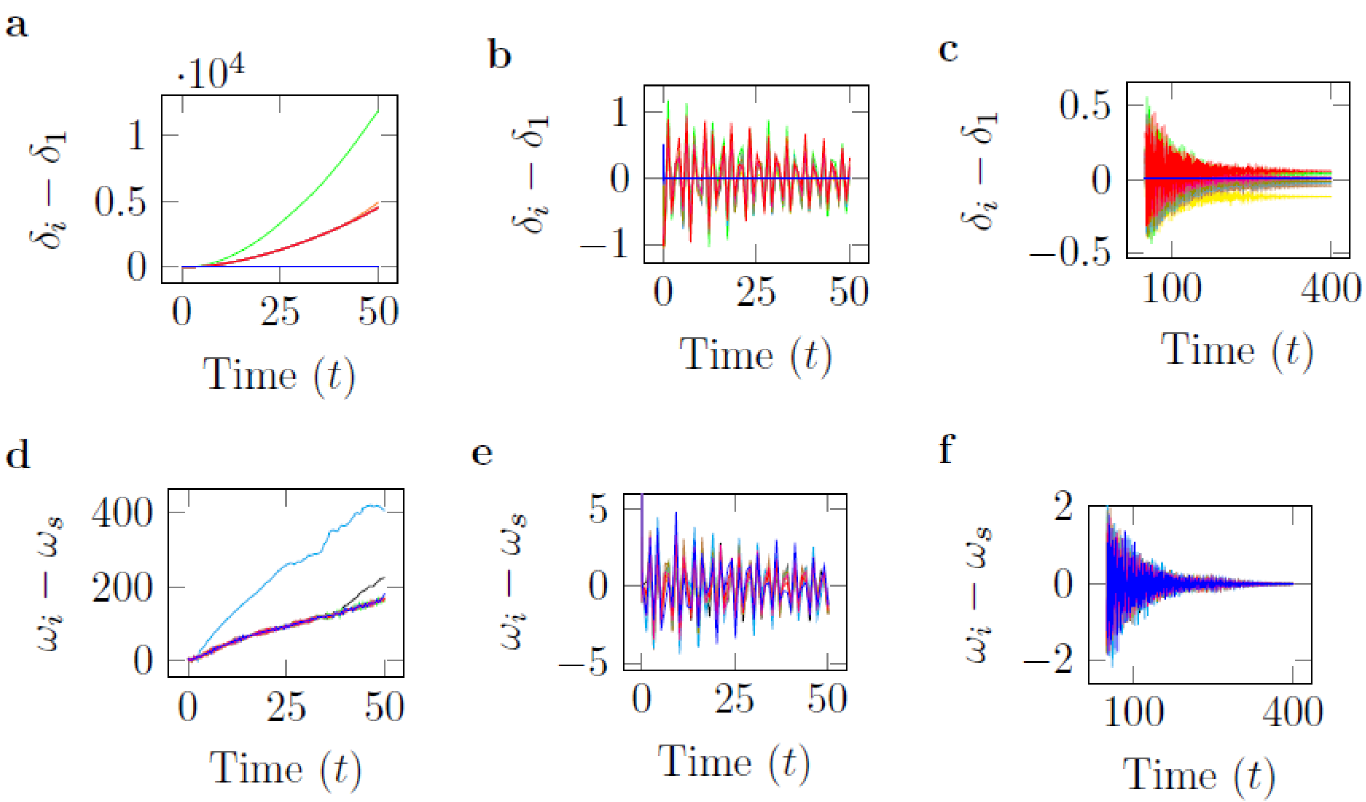

NO LOCS **LOCS**

Conclusions

We consider control of large dimensional dynamical networks with applications to biological, technological, ecological systems and so on

By choosing targets, the control energy can be reduced exponentially with respect to the size of the target set.

Upper bounds to the minimum control energy can be obtained by considering the infinite path graph for which an analytical solution is available

Optimal control of a nonlinear network (to some nonlocal point) can be achieved by performing a sequence of local optimal controls

Main References

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