Finding DC Operating Points of Nonlinear Circuits Using Carleman Linearization

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***Abstract*—In this paper, we present a procedure for approximating DC operating points of nonlinear circuits. The procedure is based on the Carleman linearization that transforms polynomial algebraic equations into an equivalent infinite dimensional linear system. Hence, we first perform a polynomial approximation of the nonlinear equations describing circuits. The infinite system of linear equations is then transformed into a finite system using a self-consistent technique. The presented procedure enables derivation of an approximation for all roots within a predefined interval. The initial interval is gradually divided into sub-intervals until all roots are identified. Contrary to usually applied methods, this approach does not depend on the domain of attraction of a root and may be also applied in cases of multiple solutions.**

***Index Terms*—Nonlinear circuits, DC operating points, nonlinear analysis, Carleman linearization**

I. Introduction

Analysis of DC operating points of a nonlinear circuit requires solving a nonlinear algebraic equation. In most cases, its analytic solution cannot be derived and, hence, various approaches exist including the well known Newton-Raphson method and homotopy methods [1]. However, these methods have drawbacks. In the case of the Newton-Raphson method, a suitable starting point needs to be identified in advance. In the case of multiple DC operating points, an inappropriate guess may yield an undesired solution. Alternatively, a homotopy method may be used where the original problem is transformed by embedding a new parameter resulting in a traceable system. By varying the homotopy parameter and thus tracing the solution, the original problem may be solved. However, in case of multiple solutions, the method needs to be extended since the trace may vanish for some values of the parameter [2]. Furthermore, knowledge of the system to be solved is necessary for devising a successful embedding [3].

Another approach is the so called bisection method [4]. Hereby, the considered nonlinear algebraic equation are examined for a sign change in a given interval. By gradually dividing the interval, the roots may be enclosed and a more accurate approximation of the root in the initial interval can be obtained. However, the initial interval should be selected so that a sign change of the nonlinear function occurs at the borders of the interval. Furthermore, the method cannot be applied in case of a double root.

In this contribution, we present an alternative procedure that provides an approximation of all possible DC operating points in an initially selected interval. These solutions may then be used as starting points for the Newton-Raphson method in order to increase the accuracy of the solutions. The procedure is based on the Carleman linearization that allows transformation of polynomial algebraic equations into an equivalent infinite-dimensional linear system [5]. Since the procedure is restricted to polynomial equations, we assume that the behavior of the nonlinear circuit is approximated by a polynomial model. For example, the approximation may be performed by applying a Taylor series or a least square fit.

Due to the infinite dimension of the system of linear equations, a solution may be derived only in special cases. Therefore, we apply a self-consistent technique that approximates the infinite system of equations by a finite system over a predefined interval [6]. In contrast to local methods, this technique enables performing a global analysis of the circuit behavior. If no roots or multiply roots exist in the predefined interval, it may be observed that the solution of the finite dimensional system do not converge when increasing its dimension. This property is applied in order to isolate each root yielding separate intervals that contain a single root. Hence, a gradual division of the initial interval is performed until the solution converges within each sub-interval for an increasing order of the finite dimensional system. In contrast to the bisection method, it is not necessary that a sign change occurs in the initial interval and, furthermore, double roots may be identified. Furthermore, if the no-gain property holds, all possible solutions are limited by the supply voltage [7]. Hence, the initial interval for the self-consistent technique is predefined. The procedure is first described for the one-dimensional case. Only circuits that may be decomposed into sub-circuits to reduce the multidimensional problem [8] are considered.

The paper is organized as follows: In Section II, the Carleman linearization for algebraic equations and the self-consistent technique are presented. The procedure for the approximation of DC operating points is described in Section III. In Section IV, a tunnel-diode circuit and a CMOS astable multivibrator are analyzed using the presented procedure. Finally, a conclusion is given in Section V.

II. Carleman Linearization and Algebraic Equations

We consider one-dimensional polynomial algebraic equation $F : \mathbb{R} \to \mathbb{R}$:

$$F(x) := \sum_{i=0}^{l} \alpha_i x^i = \alpha_0 + \alpha_1 x + \cdots + \alpha_l x^l = 0, \quad (1)$$

where $l < \infty$ and $x \in \mathbb{R}$. We use transformation:

$$x_n := x^n. \quad (2)$$

Multiplying (1) by x_n yields the linear difference equation:

$$\begin{aligned} f_n(x_1, \cdots, x_{n+l}) &:= \sum_{i=0}^{l} \alpha_i x_{n+i} \\ &= \alpha_0 x_n + \alpha_1 x_{n+1} \cdots + \alpha_l x_{n+l} = 0, \end{aligned} \quad (3)$$

with $x_0 := x^0 := 1$ and $n = 0, 1, 2, \cdots$ [5]. The desired solution x in (1) corresponds to $x_1 := x^1 := x$. For each n, (3) corresponds to a linear function of several variables $f_n(x_1, x_2, \cdots, x_{n+l})$ yielding an equivalent infinite-dimensional linear system:

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \quad (4)$$

with

$$\mathbf{x} = [x_1, x_2, x_3, \cdots]^\mathsf{T} \in \mathbb{R}^\infty \quad (5)$$

$$\mathbf{b} = [-\alpha_0, 0, 0, \cdots]^\mathsf{T} \in \mathbb{R}^\infty \quad (6)$$

and the coefficient matrix:

$$\mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_l & 0 & \cdots \\ \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_l & \cdots \\ 0 & \alpha_0 & \alpha_1 & \cdots & \alpha_{l-1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbb{R}^\infty \times \mathbb{R}^\infty. \quad (7)$$

Although each row contains only a finite number of nonzero coefficients, a general solution of (4) may only be derived in special cases [9]. Therefore, an approximation of (4) is necessary. One possible approximation is to truncate the infinite-dimensional linear system (4) to a predefined dimension $N_{\max}$ thus setting:

$$x_n = x^n = 0 \,\forall\, n > N_{\max}. \quad (8)$$

This approach yields the finite dimensional linear system:

$$\hat{\mathbf{A}}\hat{\mathbf{x}} = \hat{\mathbf{b}}, \quad (9)$$

with finite vectors $\hat{\mathbf{x}} \in \mathbb{R}^{N_{\max}}$ and $\hat{\mathbf{b}} \in \mathbb{R}^{N_{\max}}$, and the finite matrix $\hat{\mathbf{A}} \in \mathbb{R}^{N_{\max}} \times \mathbb{R}^{N_{\max}}$ that may be solved using the well-known methods from linear algebra. However, this approach provides only an approximation in the vicinity of the origin since only then the assumption for higher order terms (8) is approximately satisfied. Alternatively, a self-consistent technique may be applied for finding an approximation in a predefined interval Ω for nonlinear differential equations [6]. The main idea is to adapt the coefficient matrix (7) so that the finite linear system approximates the infinite-dimensional linear system within Ω.

The procedure for the self-consistent technique follows: We first define a maximal order $N_{\max}$ and the interval Ω. In the next step, $f_n(x_1, x_2, \cdots, x_{n+l})$ is transformed using (2):

$$f_n(x, x^2, \cdots, x^{n+l}) = 0. \quad (10)$$

Each polynomial function (10) is now approximated by a polynomial; $g_n(x, \cdots, x^{N_{\max}})$ with maximal order $N_{\max}$:

$$\begin{aligned} f_n(x, x^2, \cdots, x^{n+l}) &\approx g_n(x, x^2, \cdots, x^{N_{\max}}) \\ &\approx \sum_{i=0}^{N_{\max}} \beta_{n,i} x^i, \end{aligned} \quad (11)$$

where $\beta_{n,i}$ is calculated using a least square fit [10]:

$$\min \int_{x \in \Omega} \left[f_n(x, x^2, \cdots) - \sum_{i=0}^{N_{\max}} \beta_{n,i} x^i \right]^2 \mathrm{d}x \quad (12)$$

for the predefined interval Ω. The coefficients $\beta_{n,i}$ are only calculated if the maximal degree of $f_n(x, x^2, \cdots, x^{n+l})$ is larger than $N_{\max}$. Otherwise, the original coefficients in $f_n(x, x^2, \cdots, x^{n+l})$ are kept. The new polynomial $g_n(x, x^2, \cdots, x^{N_{\max}})$ approximates $f_n(x, x^2, \cdots, x^{n+l})$ over the given interval Ω. Finally, (2) is used to transform $g_n(\cdot)$ into:

$$g_n(x_1, \cdots, x_{N_{\max}}) = 0, \quad (13)$$

yielding a finite dimensional linear system that approximates the algebraic equation (1) over Ω. As $N_{\max}$ increases, the deviation between infinite and finite dimensional linear systems reduces over Ω because the higher order terms in (12) are available. However, it may be observed that if multiple roots exist within Ω, the solution diverges as $N_{\max}$ increases and no unique solution may be determined. Similar behavior may be observed if no root exists within the interval Ω. This criteria will be used to identify if a single root is located within the interval.

III. Approximation of DC Operating Points

The DC operating points of a nonlinear circuit are defined as real roots of nonlinear algebraic equations: $\mathbf{F} : \mathbb{R}^k \to \mathbb{R}^k$:

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}, \quad (14)$$

where $\mathbf{x} \in \mathbb{R}^k$ consists of circuit voltages and currents. By decomposing the nonlinear circuit into sub-circuits, (14) may be reduced to the one-dimensional case $x \in \mathbb{R}$, $F : \mathbb{R} \to \mathbb{R}$ [8]:

$$F(x) = 0. \quad (15)$$

If $F(x)$ includes transcendental functions, a Taylor series or a least square fit may be applied in order to obtain a polynomial approximation. Hence, in the remaining of the paper, we assume that $F(x)$ is polynomial function and apply the self-consistent Carleman linearization. If the considered circuit satisfies the no-gain property, all possible DC operating points are limited by the supply voltage V_{DD} [7]. Hence, the initial interval Ω for the self-consistent technique is defined as $\Omega = [-V_{DD} - \delta V, V_{DD} + \delta V]$ for a differential voltage

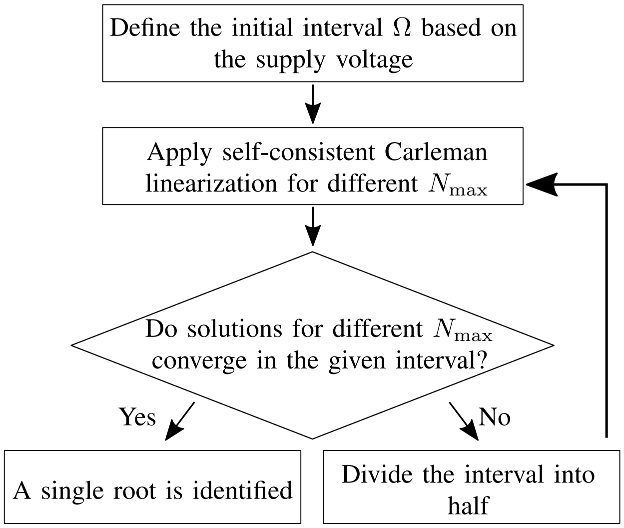

Fig. 1. Procedure for approximation of multiple DC operating points based on the self-consistent Carleman linearization.

or $\Omega = [0, V_{DD} + \delta V]$ for the single-ended supply case. An additional small offset δV is introduced to ensure that the solution $x = V_{DD}$ falls inside the interval. The self-consistent Carleman linearization is then applied with two maximal orders: $N_{\max,1}$ and $N_{\max,2}$, with $N_{\max,2} > N_{\max,1}$. If one real root exists within Ω, both solutions will converge in the vicinity of the root and the procedure is completed. However, if multiple roots exist in the interval, the solutions diverge. In this case, the initial interval Ω is divided into two equal sub-intervals: $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$. The self-consistent Carleman linearization is applied again in each sub-interval with $N_{\max,1}$ and $N_{\max,2}$ and the criteria is again verified for each case. If the criteria is still not satisfied in one of the sub-intervals, the procedure is repeated. It is possible that in a divided sub-interval no root exist. In order to check the existence of a root in a given sub-interval, the corresponding sub-interval is expanded until an already identified root falls within the new borders. If the solution converges, then the expanded sub-interval contains only the already identified root. The flowchart of the procedure is shown in Fig. 1.

IV. Examples

The described procedure is applied to two nonlinear circuits: a tunnel-diode circuit and a CMOS astable multivibrator.

A. Tunnel-Diode Circuit

We analyze DC operating points of the tunnel-diode circuit shown in Fig. 2. The behavior of the tunnel-diode may be described by the polynomial equation [11]:

$$I = g(V) := (17.76V - 103.79V^2 + 229.62V^3 - 226.31V^4 + 83.72V^5)\,\text{mA}, \tag{16}$$

yielding the algebraic equation:

$$f(V) := \frac{V_{DD} - V}{R} - g(V) = 0. \tag{17}$$

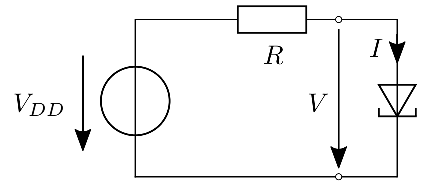

Fig. 2. Tunnel-diode circuit with $V_{DD} = 1.2\,\text{V}$ and $R = 1.5\,\text{k}\Omega$.

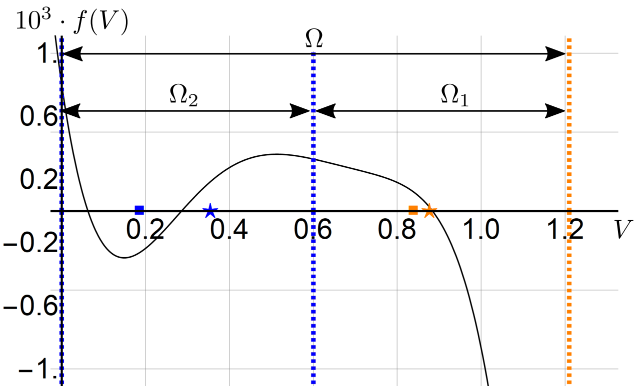

Fig. 3. Algebraic function $f(V)$ (line) of the tunneling-diode circuit shown in Fig. 2. Real roots are calculated by self-consistent Carleman linearization for two values of maximal order $N_{\max}$ (■ $= 2, \star = 4$). The initial interval Ω is divided into Ω_1 and Ω_2.

The DC operating points are obtained by solving (17). We apply the procedure described in Section III. We used the initial interval $\Omega = [0, V_{DD} + \delta V]$ with $\delta V = 10\,\text{mV}$ and apply the procedure for $N_{\max} = 2$ and $N_{\max} = 4$.

As shown in Fig. 3, multiple roots exist within the interval Ω. Hence, the initial interval is divided into two sub-intervals: Ω_1 and Ω_2. Only one root exists within Ω_1 and, hence, the solution converges as $N_{\max}$ increases. In contrast, two roots lie within the interval Ω_2 and the solutions do not converge. Hence, the criteria is not satisfied and the interval Ω_2 needs to be further divided.

Note that the initial interval Ω_2 has to be divided twice until all roots are approximated.

The resulting sub-intervals are shown in Fig. 4. In each sub-interval, the self-consistent Carleman linearization converges in the vicinity of roots. Furthermore, Ω_2' is extended to $\hat{\Omega}_2$ in order to examine if an additional root lies within Ω_2'. As shown, the solution within $\hat{\Omega}_2$ converges as $N_{\max}$ increases so that no additional root exists within the interval Ω_2'. Since the criteria is satisfied in each sub-interval, all roots in the initial interval Ω have been identified and their approximate solutions calculated.

B. CMOS Astable Multivibrator

The CMOS astable mutlivibrator realized with four MOS transistors M_1 to M_4 is shown in Fig. 5.

The multivibrator is described by a polynomial equation [12]:

$$I(V) = -c_1 V + c_3 V^3, \tag{18}$$

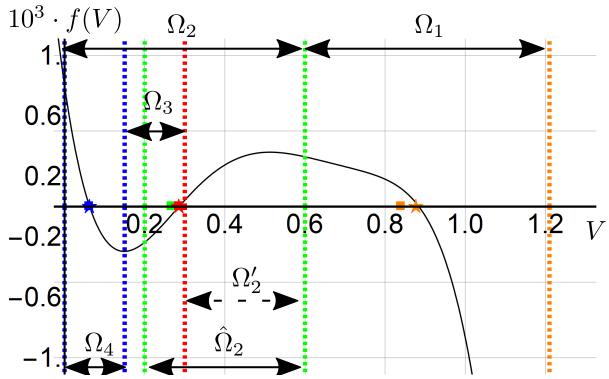

Fig. 4. Algebraic function $f(V)$ (line) of the tunneling-diode circuit in Fig. 2 and calculated real roots by self-consistent Carleman linearization for two values of maximal order $N_{\max}$ ($\blacksquare = 2, \star = 4$) for each interval Ω_i (i = 1, 2, 3, and 4).

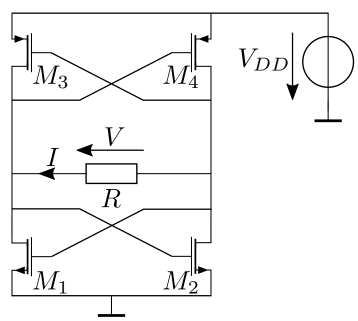

Fig. 5. CMOS astable multivibrator with $R = 10\,\mathrm{k}\Omega$ and $V_{DD} = 2.5\,\mathrm{V}$.

where

$$c_1 = \frac{8}{3}\frac{I_M}{V_{DD}}$$
$$c_3 = \frac{c_1}{V_{DD}^2}, \tag{19}$$

yielding the algebraic equation:

$$\begin{aligned} f(V) &:= \frac{V}{R} - I(V) \\ &= V\left(\frac{1}{R} + c_1\right) - c_3 V^3 = 0. \end{aligned} \tag{20}$$

In this example, $I_M = 4.72\,\mathrm{mA}$, $V_{DD} = 2.5\,\mathrm{V}$, and $R = 10\,\mathrm{k}\Omega$. Due to its differential structure, a solution of (20) lies in the range $-V_{DD} \leq V \leq V_{DD}$. Hence, we assume the initial interval $\Omega = [-V_{DD} - \delta V, V_{DD} + \delta V]$ with $\delta V = 100\,\mathrm{mV}$. The trivial root $V = 0\,\mathrm{V}$ (20) may be found directly. Three sub-intervals are used as shown in Fig. 6. The self-consistent Carleman linearization is applied in each sub-interval with $N_{\max} = 7$ and $N_{\max} = 9$. As shown in Fig. 6, the solution converges within each interval as $N_{\max}$ increases, yielding an approximation of the three roots. Since a single root is identified in each sub-interval, all possible roots in the initial interval Ω have been located and their approximate solutions calculated.

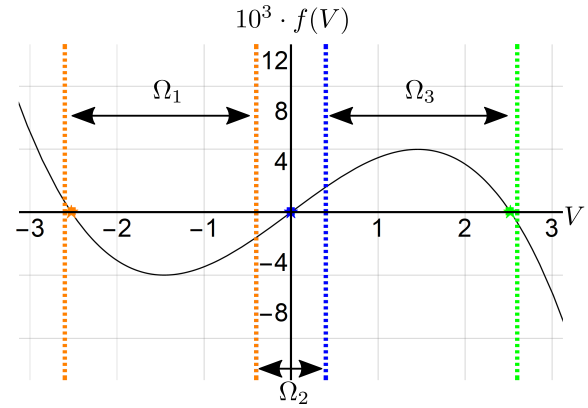

Fig. 6. Algebraic function $f(V)$ (line) of the CMOS astable multivibrator shown in Fig. 5. Real roots are calculated by self-consistent Carleman linearization for two values of maximal order $N_{\max}$ ($\blacksquare = 7, \star = 9$) for each interval Ω_i (i=1, 2, and 3).

V. Conclusion

In this paper, we presented a procedure for approximating DC operating points of nonlinear circuits by using a self-consistent Carleman linearization over a predefined interval. If the interval contains a single DC operating point, solution converges as the maximal order of the approximating polynomial increases. In the case of multiple solutions, the initial interval is gradually divided into sub-intervals in order to isolate all DC operating points. The method may be applied to finding multiple solutions independently of the domain of attraction and may be extended to the higher dimensional cases.

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