A Generalization of Brayton-Moser's Mixed Potential Function

Laurens Weiss, Wolfgang Mathis, and Ljiljana Trajković

Abstract

In this paper we give algorithms for constructing the Brayton-Moser's mixed potential function for a class of nonlinear reciprocal RLC networks, and we state necessary conditions for their existence. We have attempted to find the largest possible class of networks for which such a scalar function of state variables consisting of capacitor voltages and inductor currents can be constructed explicitly. Our results are applicable to a certain subclass of complete networks. From a mathematical point of view, we show that the corresponding network equations belong to the class of *index 1* systems.

I. Introduction

Brayton and Moser [1], [2] introduced a single scalar function of capacitor voltages and inductor currents, called the *mixed potential function*, to describe the dynamics of a large class of nonlinear RLC networks with two-terminal elements. This mixed potential function is sufficient to describe the dynamics of a given network. It may be used to investigate stability of a network's equilibrium points [3], and to describe the thermal noise [4] of complete networks. For the class of complete networks Brayton and Moser proved the existence of the mixed potential function. For the subclass of "topologically complete" networks (for a definition see next Section) they gave an algorithm for constructing the mixed potential function. Since then, many generalizations of the concept of mixed potential function to larger classes of networks have been given. Brayton [5] included capacitors and inductors with linear dependent dynamical variables. Because the corresponding networks are not necessarily complete, their network equations are not necessarily expressed in terms of capacitor voltages and inductor currents. For the same classes of networks treated in [1], [2], and [5], Flower [6], [7] proved the existence of the mixed potential function and gave algorithm for their construction, using the notion of the"total content" and "total co-content" (The content of a two-terminal element was defined as $\int v(i)di$, and its co-content as $\int i(v)dv$.) Chua [8] explicitly constructed the mixed potential function for non-complete m-ports by using the concept of a "pseudo-(co-)content" and a "pseudo-hybrid-content". He also stated necessary conditions for its existence. Horneber [9] constructed a mixed potential function for "topologically complete" networks including ideal transformers.

In this paper we generalize the original Brayton-Moser approach [1], [2]. For a fixed set of two-terminal network elements, we attempt to find the largest class of networks for which the mixed potential function can be *constructed explicitly*. The problem of finding the largest class of networks for which a mixed potential function *exists* has been discussed by Marten, Chua, and Mathis [10]. We seek the existence of state equations in terms of capacitor voltages and inductor currents. All generalizations mentioned above, except the one by Horneber, lead to violations of this condition, because they concern non-complete networks. The networks studied in this paper are necessarily complete.

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II. Brayton-Moser Equations for "Topologically Complete" Networks

In the remainder of this paper we consider nonlinear RLC networks consisting of two-terminal elements. In order to ensure the existence of state equations in terms of dynamical variables, we need to confine our study to the class of *complete networks*.

Definition 1: A network is *complete* if the capacitor voltages and the inductor currents (v, i) form a *complete* set of variables. The set of variables (v, i) is *complete* if they can be chosen independently without violating Kirchhoff's laws, and if they determine either the voltage or the current (or both) in every branch of the network ([1], p. 13).

When the equilibrium problem can be solved ([1], p. 4), i.e., each branch voltage v_μ and branch current i_μ can be (not necessarily uniquely) determined in terms of capacitor voltages v and inductor currents i from some implicit equations

$$\forall \mu : \quad v_\mu = f_\mu(v, i), \quad i_\mu = g_\mu(v, i) \tag{1}$$

the dynamics of an RLC network with two-terminal elements, including independent sources, is governed by a single scalar function $P(v, i)$ called the mixed potential function

$$\begin{aligned} C(v)\frac{dv}{dt} &= -\nabla_v P(v, i) \\ L(i)\frac{di}{dt} &= \nabla_i P(v, i). \end{aligned} \tag{2}$$

$C(v)$ denotes the (diagonal) $s \times s$-matrix of capacitances, and $L(i)$ is the (diagonal) $r \times r$-matrix of inductances. ∇_x denotes the p-dimensional gradient with respect to $x = (x_1, \ldots, x_p)$.

Definition 2:[1] A "topologically complete" network of two-terminal voltage-controlled and current-controlled elements has a graph that possesses a tree containing all of the capacitive branches and none of the inductive branches, each resistive tree branch corresponds to a current-controlled resistor, each resistive link corresponds to a voltage-controlled resistor, and finally, for which the location of the resistive branches are such that there exist no fundamental loop in which resistive branches appear *both* as tree branches and as links. [2]

Therefore, a network is "topologically complete" if the graph of the network $\mathcal{N}$ can be separated into branches $\mathcal{N}_i$ whose currents can be determined from i by KCL, and branches $\mathcal{N}_v$ whose voltages can be determined from v by KVL, and $\mathcal{N} = \mathcal{N}_i + \mathcal{N}_v$. Let $\mathcal{N}_i = (\mathcal{T} - \mathcal{T}') + (\mathcal{L} - \mathcal{L}')$ and $\mathcal{N}_v = \mathcal{T}' + \mathcal{L}'$, where

(i) $\mathcal{T}$ denotes any tree that includes all capacitive branches. $\mathcal{T}'$ denotes a subtree (not necessarily connected) that includes all capacitive branches.

(ii) $\mathcal{L}$ and $\mathcal{L}'$ denote the corresponding set of links, and $\mathcal{L} - \mathcal{L}'$ denotes a subset of all inductive branches. ($\mathcal{L}'$ denotes those links that form closed loops with elements of $\mathcal{T}'$ only.)

(iii) Independent voltage sources are elements of $\mathcal{T}$. Independent current sources are elements of $\mathcal{L}$

(iv) Current controlled resistors $v_\mu = f_\mu(i_\mu)$ are elements of $\mathcal{T} - \mathcal{T}' \subset \mathcal{N}_i$. Voltage controlled resistors $i_\mu = g_\mu(v_\mu)$ are elements of $\mathcal{L}' \subset N_v$. To ensure that content and co-content exist for all resistors, we consider only resistors that are either voltage-controlled or current-controlled.

(v) Fundamental loops must not contain resistive tree branches and resistive links at the same time.

Note that conditions (i) and (ii) imply (iii)–(v).

For the subclass of "topologically complete" networks, Brayton and Moser derived an explicit expression for the mixed potential function

$$P(v, i) = F(i) - G(v) + (i, \gamma v) \tag{3}$$

[1]This definition is identical to the definition of *complete networks* by Willson [11], with the additional term "fundamental".

[2]Complete networks in our *Definition 1* are called *topologically complete* by Chua [12], while "topologically complete" networks in our *Definition 2* are called *complete* by Chua [12].

where $F(i)$ denotes the current potential of $\mathcal{N}_i$, and $G(v)$ denotes the voltage potential of $\mathcal{N}_v$. The matrix γ can be found by considering all inductive links $\rho \in \mathcal{L} - \mathcal{L}'$ and the corresponding fundamental loop Λ_ρ. By taking the product of its current i_ρ and the sum of all capacitor voltages v_σ belonging to that fundamental loop Λ_ρ, and finally by taking the sum over all inductive links i_ρ, we can obtain

$$(i, \gamma v) \;=\; \sum_{\rho=1}^{r} i_\rho \sum_{\Lambda_\rho \cap \mathcal{T}'} \pm v_\sigma. \tag{4}$$

III. Mixed Potential Function for a Larger Class of Networks

The question we pose is: For which class of networks that are not "topologically complete" can we find a mixed potential function as a function of capacitor voltages v and inductor currents i similar to equation (4)? Assumptions (i)–(iii) ensure the independence of variables (v, i), i.e., pathological cases such as capacitive meshes and inductive cut-sets are excluded. Therefore, in the remainder of this paper, we consider networks with s capacitors and r inductors that are not "topologically complete" because there is at least one fundamental loop containing a resistive link and one or more resistive tree branches. Such violations of condition (v) may be accompanied by violations of condition (iv).

The dynamical equations for this class of networks are

$$\begin{aligned} C(v)\frac{dv}{dt} &= -\mathcal{I}(v, i, i_{add}) \\ L(i)\frac{di}{dt} &= \mathcal{V}(v, i, i_{add}) \end{aligned} \tag{5}$$

where the matrices $C(v)$ and $L(i)$ are the same as in equation (3), and $i_{add} = (i_{s+r+1}, \ldots, i_{s+r+n})^t$ denotes the vector of "additional" branch currents. $\mathcal{I}$ is the (not necessarily unique) s-dimensional function of loop currents and $\mathcal{V}$ is the (not necessarily unique) r-dimensional function of loop voltages, both determined from Kirchhoff's laws.

We use the following algorithm to derive a mixed potential function for a network that is not "topologically complete".

Algorithm 1:

(1) *If possible, make the network "topologically complete" by adding dynamical network elements:*
The addition of dynamical network elements needs to be done in agreement with the requirements (i)–(v) in order to ensure the "topological completeness" of the enlarged network. Hence, we add a linear inductor in series to those link resistors that are either current controlled or that are voltage controlled and possess an invertible i-v characteristics. Thus, the former link resistors can be integrated into the tree, whereas the branches corresponding to the additional inductors become the new links and the corresponding additional currents i_{add} become new state variables. In case of voltage controlled link resistors with non-invertible i-v characteristic, one may use the dual method by adding tree capacitors in parallel to a link resistor.

(2) *Construct a mixed potential for this enlarged "topologically complete" network:*
Because (v) holds, the enlarged network is now "topologically complete". As a consequence, one can construct the mixed potential function of the enlarged network[3]

$$\begin{aligned} C(v)\frac{dv}{dt} &= -\nabla_v P(v, \tilde{i}) \\ \tilde{L}(\tilde{i})\frac{d\tilde{i}}{dt} &= \nabla_{\tilde{i}} P(v, \tilde{i}) \end{aligned} \tag{6}$$

[3] For simplicity, we consider here only the case of adding inductors. The case of adding capacitors follows by duality.

with new state variables $(v, i, i_{add}) = (v, \tilde{i})$, where

$$\tilde{L}(\tilde{i}) = \begin{pmatrix} L_{s+1}(i_{s+1}) & & & & & \\ & \ddots & & & & \\ & & L_{s+r}(i_{s+r}) & & & \\ & & & L_{s+r+1}(i_{s+r+1}) & & \\ & & & & \ddots & \\ & & & & & L_{s+r+n}(i_{s+r+n}) \end{pmatrix}. \tag{7}$$

Note that $C(v)$ and v remain unchanged. Compared to equation (6), we now have n additional equations for n additional inductors $L_{s+r+1}, \ldots, L_{s+r+n}$.

(3) *Eliminate the additional state variables from the mixed potential by setting the additional dynamical elements equal to zero:*
We now need to eliminate the additional currents i_{add} from equation (6). Setting the additional inductors $L_{s+r+1} = \ldots = L_{s+r+n}$ to zero, the voltage drops across these inductors vanish and equation (6) can be written as

$$\begin{aligned} C(v)\frac{dv}{dt} &= -\nabla_v P(v, i, i_{add}) && (8) \\ L(i)\frac{di}{dt} &= \nabla_i P(v, i, i_{add}) \\ 0 &= \nabla_{i_{add}} P(v, i, i_{add}) =: \tilde{h}(v, i, i_{add}). && (9) \end{aligned}$$

Equations (8) and (9) form a system of differential-algebraic equations (DAEs). To eliminate i_{add} from (8) we need to solve the system of n nonlinear coupled algebraic equations (9) uniquely for (the n components of) i_{add} and insert the (local or global) solution $i_{add}(v, i)$ into equation (8). The existence of solutions to equation (9) will be discussed in Section IV.
Consider the case when equation (9) can be solved uniquely for i_{add}, i.e.,

$$i_{add} = h(v, i). \tag{10}$$

From equations (8) and (10), we obtain the state equations

$$\begin{aligned} C(v)\frac{dv}{dt} &= -\nabla_v \tilde{P}(v, i) && (11) \\ L(i)\frac{di}{dt} &= \nabla_i \tilde{P}(v, i) \end{aligned}$$

where we have assumed that $h(v, i)$ is *differentiable with respect to i*. $\tilde{P}(v, i) := P(v, i, h(v, i))$ denotes the mixed potential function for the original network under study. By applying the chain rule to $\tilde{P}(v, i)$ and using (8), (9), and (10), we find

$$\begin{aligned} \nabla_i \tilde{P}(v, i) &= \nabla_i P(v, i, i_{add}) + [\nabla_{i_{add}} P(v, i, i_{add})][\nabla_i h(v, i)] && (12) \\ &= \nabla_i P(v, i, i_{add}) \\ &= L(i)\frac{di}{dt} \end{aligned}$$

which proves (11).

We can now infer from equation (12) that we would have obtained the same potential function $\tilde{P}(v, i)$ using

Algorithm 2:
(1) Derive the "pseudo" mixed potential function $P(v, i, i_{add})$, by partial integration of equation (6).
(2) Find $i_{add} = h(v, i)$ by solving equation (9).
(3) Obtain the mixed potential function $\tilde{P}(v, i)$ by inserting h into P.

IV. Existence of the Mixed Potential Function

In Section III we saw that a mixed potential $\tilde{P}(v,i)$ for a network that is not "topologically complete" exists only if the algebraic equation (9) can be solved (uniquely) for $i_{add} = h(v,i)$, *and* if $h(v,i)$ is differentiable with respect to i. The first condition is equivalent to the existence of state equations. The second condition imposes further restrictions. Therefore, the existence of state equations does not guarantee the existence of a mixed potential function $P(v,i)$ in terms of state variables (v,i).

It is known from the theory of DAEs that equation (9) has at least a local solution for i_{add} if the DAE system (8) and (9) is a so-called *index 1* system. A DAE system

$$A(x)\dot{x} + F(x,y,t) = 0 \qquad (13)$$
$$G(x,y,t) = 0 \qquad (14)$$

has index 1 ([13], p. 38) if it can be transformed into a system of explicit differential equations in terms of x, y

$$\begin{aligned} \dot{x} + A^{-1}(x)F(x,y,t) &= 0 \\ \dot{y} + H(x,y,t) &= 0. \end{aligned} \qquad (15)$$

As a consequence of the implicit function theorem, this implies that the algebraic equation (14) can be solved for y.

Therefore, the DAE system (8), (9) has index 1 if

(i) the matrices $C(v)$ and $L(i)$ are invertible (the differential equations (8) can be written explicitly)

(ii) the right-hand side of (8) is a continuous function

(iii) $\tilde{h}$ in (9) is differentiable with respect to i_{add}

(iv) and the Jacobian matrix $\nabla_{i_{add}}\tilde{h}$ is regular.

What do these restrictions mean for the functions describing the network elements? The invertibility of the matrices $C(v)$ and $L(i)$ is equivalent to $C_\sigma(v) \neq 0\ \forall \sigma, v$ and $L_\rho(i) \neq 0\ \forall \rho, i$. Continuity of the right-hand side of (8) is ensured if the characteristics of the nonlinear resistors are continuous functions. The differentiability of $\tilde{h}$ is ensured if the characteristics of the current-controlled nonlinear resistors in the n fundamental loops corresponding to equation (9) are differentiable functions. (This excludes piecewise linear functions.) Finally, the regularity of the Jacobian matrix $\nabla_{i_{add}}\tilde{h}$ need be checked.

In conclusion, "topologically complete" networks are a subclass of the *index 1* networks. Furthermore, any larger class of networks for which we can write the mixed potential function must also belong to the *index 1* class in order to ensure the existence of state equations, given the dynamical variables (v,i).

V. Example

Consider the circuit shown in Figure 1, with inductor L_3 short-circuited. The current i_3 cannot be expressed in terms of independent dynamical variables and the circuit is not "topologically complete". (Requirements (iv) and (v) are violated.) The state equations for this circuit are

$$\begin{aligned} C_1\frac{du_1}{dt} &= i_3 - i_2 \\ L_2\frac{di_2}{dt} &= u_1 + f_5(i_3 - i_2). \end{aligned} \qquad (16)$$

We follow the steps of *Algorithm 1*:

(1) After the inductor L_3 has been inserted, the enlarged circuit is "topologically complete". The additional dynamic equation is

$$L_3\frac{di_3}{dt} = U_0 - u_1 - f_5(i_3 - i_2) - f_4(i_3). \qquad (17)$$

(2) The mixed potential for the enlarged "topologically complete" circuit can be constructed by using equations (3) and (4)

$$P(u_1, i_2, i_3) \;=\; u_1(i_2 - i_3) + U_0 i_3 - \int\limits_0^{i_3} f_4(i)di - \int\limits_0^{i_3 - i_2} f_5(i)di. \tag{18}$$

(3) Setting $L_3 = 0$ in equation (17) leads to

$$\tilde{h}(u_1, i_2, i_3) \;:=\; U_0 - u_1 - f_5(i_3 - i_2) - f_4(i_3) = 0. \tag{19}$$

If equation (19) can be solved for i_3, then

$$i_3 \;=\; h(u_1, i_2). \tag{20}$$

Finally, with h being differentiable, the mixed potential for the original circuit becomes

$$\tilde{P}(u_1, i_2) = u_1(i_2 - h(u_1, i_2)) + U_0 h(u_1, i_2) - \int\limits_0^{h(u_1,i_2)} f_4(i)di - \int\limits_0^{h(u_1,i_2)-i_2} f_5(i)di. \tag{21}$$

VI. Concluding Remarks

We have derived algorithms for constructing Brayton-Moser's mixed potential function for a larger class of nonlinear reciprocal RLC networks. The completeness property is a necessary condition for the existence of state equations in terms of dynamical variables, and, as a consequence, the enlarged class of networks considered here, which includes the "topologically complete" networks, is a subclass of the complete networks. Mathematically, this enlarged class of networks is a subset of the *index 1* systems. Because network stability properties are based on the existence of a certain form of the mixed potential functions, rather than on the "topological completeness" of a network, Brayton-Moser's stability results are valid for the enlarged class of networks discussed in this paper. The same holds for the description of thermal noise in nonlinear networks.

References

[1] R. K. Brayton and J. K. Moser, "A theory of nonlinear networks - I," *Quarterly of Applied Mathematics*, vol. 22, pp. 1–33, April 1964.

[2] ——, "A theory of nonlinear networks - II," *Quarterly of Applied Mathematics*, vol. 22, pp. 81–104, July 1964.

[3] J. K. Moser, "Bistable systems of differential equations with applications to tunnel diode circuits," *IBM J. Res. Dev.*, vol. 5, pp. 226–240, July 1961.

[4] L. Weiss and W. Mathis, "A thermodynamical approach to noise in nonlinear networks," to be published in *Int. J. Circuit Theory and Appl.*

[5] R. K. Brayton, "A canonical form for nonlinear RLC networks," *Proc. Symp. on System Theory*, Polytechnic Inst. of Brooklyn, Polytechnic Press, New York, pp. 57–65, April 1965.

[6] J. O. Flower, "The topology of the mixed potential function," *Proc. IEEE*, vol. 56, pp. 1721–1722, October 1968.

[7] ——, "The existence of the mixed potential function," *Proc. IEEE*, vol. 56, pp. 1735–1736, October 1968.

[8] L. O. Chua, "Stationarity principles and potential functions," *Journal of The Franklin Institute*, vol. 296, no. 2, pp. 91–114, August 1973.

[9] E. H. Horneber, "Application of the mixed potential for formulating normal form equations for nonlinear RLC networks with ideal transformers," *Int. J. Circuit Theory and Appl.*, vol. 6, pp. 345–362, 1978.

[10] W. Marten, L. O. Chua, W. Mathis, "On the geometrical meaning of pseudo hybrid content and mixed potential," *Int. J. of Electronics and Communications (AEÜ)*, vol. 46, no. 4, pp. 305–309, 1992.

[11] A. N. Willson, Jr., "Some aspects of the theory of nonlinear networks," *Proc. IEEE*, vol. 61, pp. 1092–1113, August 1973.

[12] L.O.Chua, "Nonlinear circuit theory," in *Modern Network Theory - An Introduction*, Guest lectures of the 1978 ECCTD, G. S. Moschytz and J. Neirynck (Eds.), St. Saphorin, Switzerland: Georgi, 1978.

[13] W. Kampowsky, P. Rentrop, and W. Schmidt, "Classification and numerical simulation of electric circuits," *Survey on Mathematics for Industry*, vol. 2, no. 1, pp. 23–65, 1992.

LIST OF FOOTNOTES

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3. For simplicity, we consider here only the case of adding inductors. The case of adding capacitors follows by duality.

BIOGRAPHIES

Laurens Weiss (M'96) received his diploma in Mathematical Physics in 1994 from the Technical University of Braunschweig, Germany. From 1995 to 1996, he worked as a research assistant at the Dept. of EE, University of Wuppertal, Germany. Since 1996 he has been working as a research assistant at the Dept. of EE, University of Magdeburg, Germany. Since 1995, he is working towards the Ph.D. His research interests include the theory of nonlinear networks, noise in electronic circuits and the foundations of quantum theory. He is a member of IEEE and VDE/ITG.

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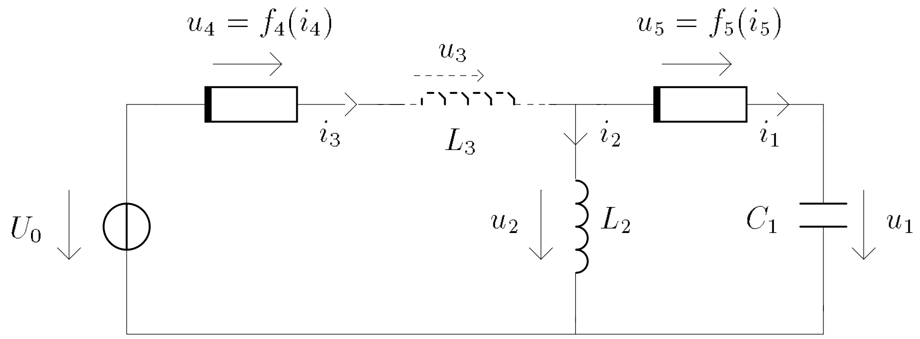
FIGURE CAPTIONS

Fig. 1. Circuit used in the example. A circuit that was not "topologically complete" has been enlarged with an inductor L_3.



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