FINDING DC OPERATING POINTS: LIMITATIONS OF TOPOLOGICAL AND DETERMINANT CRITERIA

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Abstract— Problem of finding dc operating points of transistor circuits cannot be solved by using only approximate numerical techniques. In this paper we examine circuit properties that may help determine the number of a circuit's dc operating points, i.e., the number of solutions of the corresponding set of nonlinear equations. We revisited two well known approaches: the topological criteria based on a search for feedback structures, and the determinant criteria based on the sign of certain coefficients calculated from circuit equations. We formulate an algorithm based on these approaches, show an example of its use, and discuss its limitations.

I. INTRODUCTION

A nonlinear circuit's dc operating points need to be determined during every design or simulation cycle. Common numerical approaches for finding these operating points (e.g., Newton-Raphson method) require that the starting point of a numerical process be close enough to the unknown solution. Hence, in order to apply these methods, it is necessary to determine:

- the existence of *multiple* operating points
- a good *starting point* for each solution that need to be found.

Both problems are difficult and remain unsolved.

There are two known criteria for determining the existence of multiple operating points: topological and the determinant criteria. We investigate here the limitations of both approaches.

II. PROBLEM STATEMENT

The mathematical problem to be solved is described by a set of nonlinear algebraic equations

$$\mathbf{F}(\mathbf{x}) = \mathbf{0} \tag{1}$$

constructed by applying:

- Kirchhoff's Voltage Law: $\mathbf{B}^T\mathbf{u} = \mathbf{0}$
- Kirchhoff's Current Law: $\mathbf{A}\mathbf{i} = \mathbf{0}$
- the characteristics of the circuits elements

$$f_k(\mathbf{i}, \mathbf{u}) = 0.$$

Although we confine our analysis to bipolar-transistor circuits, our results are also applicable to other types of transistor circuits.

III. TOPOLOGICAL APPROACH

One approach to qualitatively analyze circuits is to consider only the circuit's *topology*, i.e., only the way circuit's elements are connected together.

Nielsen and Willson have found [5] a unique topological structure, called feedback structure, that may cause multiple dc operating points in transistor circuits. Such a structure, consisting of bipolar transistors, is shown in Fig. 1.

At least one feedback structure is necessary if a transistor circuit is to possess multiple dc operating points [5]. Unfortunately, this is only a necessary condition. Even if a feedback structure is present, the circuit *may* or *may not* possess multiple operating points, depending on circuit parameters and biasing conditions. Nevertheless, searching for feedback structures first, considerably decreases the effort in calculating certain coefficients that can indicate the existence of multiple operating points.

IV. DETERMINANT CRITERIA

Circuit equations can be written in the from [10]:

$$\mathbf{QTF}(\mathbf{v}) + \mathbf{P}\mathbf{v} + \mathbf{c} = \mathbf{0}. \tag{2}$$

The corresponding circuit may possess multiple operating points only if the determinant of the Jacobian matrix can change its sign. The Jacobian matrix is

$$\mathbf{J} = \mathbf{QTD} + \mathbf{P}, \tag{3}$$

where $\mathbf{D}$ is a diagonal matrix consisting of the derivatives of the nonlinear function $\mathbf{F(v)}$. Its determinant can be decomposed as [4]:

$$det(\mathbf{QTD} + \mathbf{P}) = c^{1234}d_1d_2d_3d_4 + c^{123}d_1d_2d_3 + c^{124}d_1d_2d_4 + ... + c^{12}d_1d_2 + ... + c^1d_1 + ... + c^0d_0, \tag{4}$$

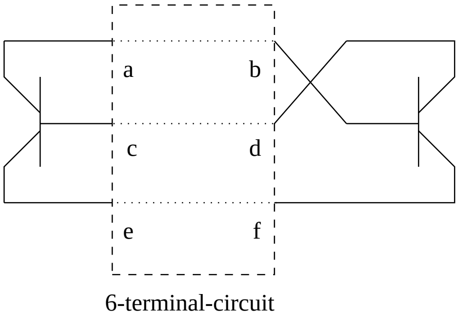

Figure 1: Feedback structure consisting of two bipolar transistors. The remaining of the circuit is represented as a black box. The type of the transistors (n type or p type) is irrelevant.

where $d_i \geq 0$, if all $f_i(v)$ from $\mathbf{F(v)}$ are monotone increasing functions.

V. ALGORITHM

We present here four steps of the algorithm that combines the topological and the determinant criteria, and can be used to test whether a circuit possesses multiple dc operating points.

1. Search for feedback structures using the algorithm given in [1]. The outcome of the search can be twofold:

- No feedback structure has been detected. The circuit possesses a unique dc operating point. (*End of the test.*)
- At least one feedback structure has been detected. The circuit *may* possess multiple dc operating points. Further analysis is necessary to determine whether this can occur for the specific choice of circuit parameters and the biasing conditions.

2. Test the type of the feedback structure. We first examine which transistor terminals (collector or emitter) are connected to each other in a feedback path. These terminals are labeled *e* and *f* in Fig. 1:

- If a collector-collector feedback structure has no parallel path via another transistor (i.e., it is not affected by another transistor) the circuit possesses a unique dc operating point. The reason is that a two-transistor circuit may possess multiple dc operating points only if $1 - \alpha_e - \alpha_f < 0$ [7][1]. This is impossible for realistic reverse current gains α_e and α_f.
- If a circuit possesses a collector-emitter, emitter-collector, or emitter-emitter feedback structure, multiple dc operating points are possible.

[1]Note that we have not applied here the general condition for n-transistor case: $1 - \Sigma\Pi\alpha^s < 0$ [7]. In general, this condition cannot be interpreted in a simple topological way.

3. Apply the determinant criteria. Formulate Eq. (2) and determine coefficients $c^s = det(\mathbf{QTD_a} + \mathbf{PD_b})$, where $\mathbf{D_a}$ is a matrix with all zeros except two ones on the main diagonal, and $\mathbf{D_b} = \mathbf{I} - \mathbf{D_a}$, where $\mathbf{I}$ is an identity matrix. For example, $s = 13$ implies that $\mathbf{D_a}(1,1) = 1$ and $\mathbf{D_a}(3,3) = 1$. Then, $c^s = det(\mathbf{C})$, where $\mathbf{C}$ is a matrix comprised of the appropriate columns of $\mathbf{QT}$ and $\mathbf{P}$. For example, $s = 13$ is calculated by taking columns 1 and 3 from $\mathbf{QT}$, and all the remaining columns from $\mathbf{P}$.

Matrices $\mathbf{P}$ and $\mathbf{Q}$ in Eq. (2) can be chosen so that all coefficients c^s, except possibly one coefficient associated with a specific feedback structure, are nonnegative [4]. Condition $c^s < 0$ is necessary if the circuit is to have multiple dc operating points. Hence, the following test is possible:

- Nonnegative coefficient implies a unique operating point. (*End of the test.*)
- Negative coefficient implies possible multiple operating points.

4. The outcome of the algorithm is inconclusive. Further analysis of the circuit is necessary.

A. Illustrative example

We illustrate the algorithm by applying it to a differential amplifier circuit with active load, shown in Fig. (2). We show that this circuit fails to satisfy the

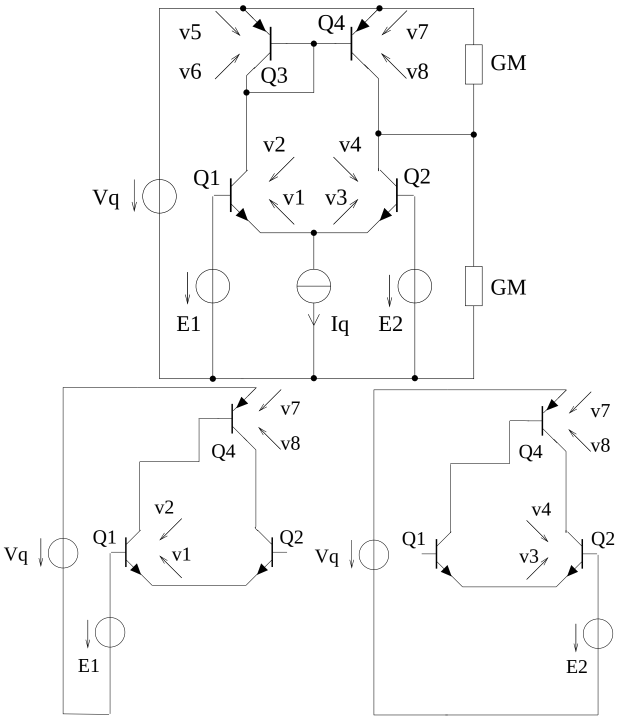

Figure 2: Differential amplifier circuit with active load (top) The circuit possesses two feedback structures (bottom).

conditions for possessing multiple operating points, and hence we will conclude that it possesses a unique dc operating point. We follow the described algorithm:

1. Two feedback structures (FS) can be detected in the circuit:

- first FS formed by transistors Q1 and Q4,
- second FS formed by transistors Q2 and Q4.

Hence, the circuit *may* possess multiple dc operating points.

2. Testing the type of the feedback structure reveals that:

- Emitter of Q1 (labeled 1) is connected to the collector of Q4 (labeled 8). Hence, transistors Q1 and Q4 form an emitter-collector feedback structure, and, therefore, multiple dc operating points are *possible*.
- Transistors Q2 and Q4 from a collector-collector feedback structure with no parallel path via another transistor. Hence, this feedback structure can only contribute to a *unique* dc operating point. (*End of the test.*)

Therefore, multiple operating points are possible only because of the existence of the first feedback structure.

3. We now apply the determinant criteria by first formulating Eq. (2), where

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G_M & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{T} = \mathbf{T_1} \oplus \mathbf{T_2} \oplus \mathbf{T_3} \oplus \mathbf{T_4}, \quad \mathbf{T_i} = \begin{pmatrix} 1 & -\alpha_{r\{2i\}} \\ -\alpha_{f\{2i-1\}} & 1 \end{pmatrix}$$

$$\mathbf{c} = \begin{pmatrix} I_q \\ 0 \\ G_M(2E_2 - V_q) \\ E_1 - E_2 \\ E_1 - V_q \\ E_2 - V_q \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{F}(\mathbf{v}) = \begin{pmatrix} f_1(v_1) \\ f_2(v_2) \\ f_3(v_3) \\ f_4(v_4) \\ f_5(v_5) \\ f_6(v_6) \\ f_7(v_7) \\ f_8(v_8) \end{pmatrix}.$$

We form matrix $\mathbf{C}$ by taking columns 1 and 8 from $\mathbf{QT}$, and the remaining six columns from matrix $\mathbf{P}$:

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_{f1} & 0 & 0 & 0 & 0 & 0 & 0 & 1-\alpha_{r8} \\ 0 & 0 & 0 & 2G_M & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We find that $det(\mathbf{C}) = 2G_M(1-\alpha_{r8}) > 0$, because the transistor's reverse current gain $\alpha_{r8} < 1$. Since all the remaining coefficients c^s are nonnegative, no possible combinations of $d_i > 0$ can cause the determinant of Eq. (4) to be negative. Hence, the circuit possesses a *unique* operating point. (*End of the test.*)

In conclusion, since the existing feedback structures cannot cause multiple operating points, the differential amplifier with active load has a unique dc operating point.

B. Limitations of the algorithm

Matrices $\mathbf{P}$ and $\mathbf{Q}$ can be generated automatically [9], and the described algorithm can be programmed. It can be used to eliminate, step by step, all feedback structures that cannot cause multiple dc operating points. Nevertheless, the algorithm still "lacks exactness" because it relies only on several necessary criteria.

Whether a circuit possess multiple operating points or not, crucially depends on the transistors biasing. This biasing, in turns, depends on the circuit's topology *and* circuit element values (resistances, dc sources, and transistor parameters) [8]. A two-transistor circuit with no dc sources or resistors can possess only one dc operating point [6]. If dc sources are permitted, counterexamples exist.

It is still an open question which topology that involves transistors, resistors, and dc sources is sufficient for multiple operating points. We can only provide isolated examples, because no criteria exists separating circuits into those with multiple operating points and those with a unique operating point.

One limitation of the described algorithm is that the determinant criteria fails to indicate accurately the number of operating points. The reason is that the corresponding coefficients, being evaluated from matrices $\mathbf{P}$ and $\mathbf{QT}$, do not depend on the values of dc sources, which contribute only to the column-vector $\mathbf{c}$ in Eq. (2). Hence, dc sources are not taken into consideration when applying this criteria. Consequently, if the transistors biasing depends on source values, the determinant criteria fails.

VI. DISCUSSION

While numerical methods to be successful require good initial starting points, topological methods ignore useful circuit information dealing with circuit parameters and biasing. The reason why the determinant criteria seem favorable is because they can be applied before the circuit's operating point is calculated. Hence, they do not require solving Eq. (1).

We have found severe disadvantages concerning the reliability of the determinant criteria:

- The criteria does not take into account values of dc sources. If the number of dc operating points varies because of sources' values, it can not be indicated by the determinant criteria.
- Certain effects of topology on the values of coefficients c^s are unknown. Matrices $\mathbf{P}$ and $\mathbf{Q}$ are not unique, and calculating c^s requires that the columns of $\mathbf{P}$ and $\mathbf{QT}$ be combined depending on the existing feedback structure. In the process of applying the coefficients criteria, this exact, yet limited, information contributed by the feedback structure criteria may partly "vanish".
- Even if we find that the coefficients can change their sign, the only statement possible is that the circuit *may* possibly have multiple dc operating points for some value of $\mathbf{D}$.

The ultimate condition for having multiple operating points is that the determinant $det(\mathbf{QTD}+\mathbf{P})$ needs to change its sign. Hence, the fundamental reason behind the uncertainty of the determinant criteria is this determinant's dependence on $\mathbf{D}$, i.e., on the slopes d_i of the nonlinear transistor characteristics. Condition $c^s < 0$ (with all other coefficients being nonnegative) only implies that *one* term in $det(\mathbf{C})$ is negative. Hence, the biasing that is necessary for achieving multiple operating points is still unknown. Furthermore, we can not assume the existence of such a dc operating point, because the values d_i, necessary for $det(\mathbf{QTD}+\mathbf{P}) < 0$ may not be possible with realistic transistor parameters.

In summary, the determinant criteria (apart from being quite numerically expensive) is not sufficient for drawing conclusions concerning the number of dc operating points of transistor circuits. As expected, the number of operating points of a given circuit depends on the biasing of every transistor at the respective dc operating point [8]. Hence, test algorithms that do not take into account *actual* value of dc operating points may not accurately capture the behavior of the circuit under test. We have recently proposed [2, 3] a method that accurately predicts the existence of multiple dc operating points in transistor circuits.

Due to space limitations, we have omitted here circuit examples that illustrate that the determinant criteria cannot be used to predict a circuit's behavior caused by varying *resistor* values.

VII. CONCLUSIONS

In this paper we revisited two methods for testing whether a circuit possesses multiple dc operating points: topological and determinant criteria. We devised an algorithm that combines these two approaches, and we illustrated its application by testing whether a differential amplifier circuit possesses multiple operating point.

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