Parameter Embedding Methods for Finding DC Operating Points: Formulation and Implementation

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Abstract

We address the calculation of dc operating points of nonlinear circuits by using parameter embedding methods, and we show that the usefulness of these methods depends on the type of a circuit's descriptive equations. We discuss various approaches to embedding a parameter into nonlinear equations that describe bipolar and MOS transistor circuits. Embedding algorithms were implemented in an industrial circuit simulator. We demonstrated that homotopy methods can be used as an alternative to the Newton-Raphson-type solvers and that they can be successfully applied to solving nonlinear circuit equations as well as to calculating dc operating points and transfer curves of nonlinear transistor circuits.

I. Introduction

In this paper[1] we address the calculation of dc operating points of nonlinear circuits by means of parameter embedding methods (also known as continuation and homotopy methods). One of the first implementations of continuation methods in an industrial circuit simulator was by Cermak in 1971 [1]. Other implementations followed [2]–[7]. Continuation and homotopy methods have been rediscovered several times in circuit theory (see [5] and [8] for extensive reference lists).

Nevertheless, information about implementations of homotopy methods in industrial circuit simulators is not readily available in the literature with the exception of [3]–[7]. In [5], the authors describe an implementation that is suitable for bipolar transistor circuits based on the Gummel-Poon bipolar transistor model [9] with an embedded continuation parameter. In [6], the authors discuss the application of publicly available software packages with continuation algorithms PITCON [10] and HOMPACK [11] for calculation of dc operating points. Although interesting results were obtained, the authors did not address the details of parameter embedding that are crucial for a successful application of homotopy methods for solving circuit equations by circuit simulators.

II. Embeddings and Network Equations

In contrast to classical methods for solving a system of nonlinear equations

$$\mathbf{f}(\mathbf{x}) = \mathbf{0}, \qquad (1)$$

we consider a parameterized family of nonlinear equations

$$\mathbf{F}(\mathbf{x}, \lambda) = \mathbf{0}, \quad 0 \leq \lambda \leq 1. \qquad (2)$$

If a solution $\mathbf{x}_0$ is known for a fixed value λ_0, a homotopy method tries to find a solution for another value λ. In applications, it is a nontrivial problem to find a suitable criteria to embed a parameter λ into a given set of nonlinear equations. In case of circuit equations, it is useful to decompose the circuit descriptive equations into the linear part that corresponds to the linear network, and a nonlinear part that corresponds to the nonlinear network elements. A rather simple principle can be used to generate parameterized equations from nonlinear equations (1):

$$\mathbf{F}(\mathbf{x}, \lambda) := (1-\lambda)\mathbf{L}(\mathbf{x}-\mathbf{a}) + \lambda\mathbf{f}(\mathbf{x}) = \mathbf{0}, \qquad (3)$$

where $\mathbf{a}$ is an arbitrary constant vector. If $\mathbf{L}$ is a linear operator, this type of embedding is called *standard embedding.* Often, $\mathbf{L}$ is the identity operator and $\mathbf{L}(\mathbf{x}) = \mathbf{x}$ is used as the linear part of equation (3).

[1] Part of this work was presented at the NDES'95 Workshop, Dublin, Ireland, July 1995.

It can be shown using a simple example that this approach may cause difficulties if the well-known Modified Nodal Analysis (MNA) equations [12] are used to describe networks.

We should mention that it is not necessary to restrict the decomposition of a nonlinear network that will be parameterized with an embedding (or its descriptive equations) into a linear and a nonlinear part. A more general idea is to use a decomposition of the form

$$\mathbf{F}(\mathbf{x},\lambda) = \mathbf{N}_1(\mathbf{x}) + \lambda\mathbf{N}_2(\mathbf{x}), \tag{4}$$

where $\mathbf{N}_1$ and $\mathbf{N}_2$ are nonlinear operators. If $\lambda = 0$, then $\mathbf{N}_1(\mathbf{x}) = \mathbf{0}$ describes a nonlinear network. In this case we need to demonstrate that these equations are well-conditioned and posses a unique solution that can be calculated. Such an approach is described in [3]–[6] where the gains of the Ebers-Moll transistor model are multiplied by λ. For $\lambda = 0$ the network consists of independent sources, linear resistors, and diodes, and, therefore, possesses a unique dc operating point.

III. Choice of Equations

We now show that the usefulness of parameter embedding method depends on the type of a circuit's descriptive equations. If a resistive circuit is described by a system of nonlinear equations (1), then the zeros of this system of equations are its dc operating points. The essential idea of the homotopy approach is to embed a parameter into equation (1) and, thus, obtain a parameterized family of equations (2) The original equation is obtained by setting $\lambda = 1$:

$$\mathbf{F}(\mathbf{x},1) := \mathbf{f}(\mathbf{x}) = \mathbf{0}. \tag{5}$$

If $\lambda = 0$, then the equation

$$\mathbf{F}(\mathbf{x},0) = \mathbf{0} \tag{6}$$

should be much simpler than the original equation (1) and finding its zeros should be an "easy" task.

We show that the *standard embedding*

$$\mathbf{F}(\mathbf{x},\lambda) := (1-\lambda)\mathbf{x} + \lambda\mathbf{f}(\mathbf{x}) = \mathbf{0} \tag{7}$$

fails if we use the MNA equations. The MNA method is used today in most SPICE-like circuit simulators [13]. The main reason for choosing the MNA approach when writing circuits' equations is to include network elements that are not current sources or do not have an admittance representation (such as certain controlled sources). For such elements one cannot introduce their currents into the vector of network variables. Therefore, in contrast to the basic nodal equations, not all MNA equations represent sums of currents. Some of these equations describe the inter-connection of network elements.

Consider the simple circuit shown in Figure 1(a). The MNA equations of this circuit (see e.g., [14]) are:

$$\begin{pmatrix} G & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ i_0 \end{pmatrix} = \begin{pmatrix} 0 \\ U_0 \end{pmatrix}. \tag{8}$$

In order to apply the homotopy approach, we define the following function:

$$\mathbf{f}(\mathbf{x}) := \begin{pmatrix} G & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x} - \begin{pmatrix} 0 \\ U_0 \end{pmatrix}, \tag{9}$$

where $\mathbf{x} = \begin{pmatrix} u_1 \\ i_0 \end{pmatrix}$. *Standard embedding* of (9) into (7) leads to the difficulties we are addressing in this paper. Even though we use a simple circuit to illustrate the cause of the problem, it is easy to extend this result to more complex circuits. If, when embedding a parameter, we add to the original MNA equations (9) the term $(1-\lambda)\mathbf{x}$ (7), we create a new circuit. The following equations are obtained by applying the *standard embedding*, after dividing both equations by λ:

$$\begin{aligned} \frac{1-\lambda}{\lambda} u_1 + (G u_1 + i_0) &= 0 \\ \frac{1-\lambda}{\lambda} i_0 + (u_1 - U_0) &= 0. \end{aligned} \tag{10}$$

A circuit interpretation of the additional terms in (10) leads to the circuit shown in Figure 1(b). It is evident from the polarity of u_2 that $\frac{1-\lambda}{\lambda}$ should be negative. That implies that in the process of finding solutions of (8) we need to find operating points of circuits with negative resistors. Therefore, for some λ, the coefficient matrix of the MNA equations may become singular. In our example, for $G = 1/2$, the coefficient matrix

$$\begin{pmatrix} \frac{1-\lambda}{\lambda} + 1/2 & 1 \\ 1 & \frac{1-\lambda}{\lambda} \end{pmatrix} \tag{11}$$

becomes singular for $\lambda \approx 0.562$. This is in contrast to the behavior of nodal equations, where similar embedding always leads to circuits with non-negative resistances [15].

In conclusion, the embedding of a parameter should be done in a manner that avoids non-physical network instances (e.g., occurrence of negative resistors). For that reason, we opted to implement in an industrial circuit simulator [16] the embeddings suggested by Trajković et al., [3]–[6] and by Hasler et al., [17] (see also Mathis [18]).

IV. Useful Homotopies

A useful embedding approach [17] is to parameterize all nonlinear network elements with λ in a multiplicative manner.

Starting point of the approach in [17] is the solution of a linear network obtained by setting $\lambda = 0$. The continuation or homotopy approach is applied using the general formulation:

$$\mathbf{F}(\mathbf{x}, \lambda) := (\mathbf{Lx} - \mathbf{b}) + \lambda \mathbf{N}(\mathbf{x}) = \mathbf{0}, \qquad (12)$$

where $\mathbf{Lx} = \mathbf{b}$ describes the linear network (when $\lambda = 0$) and the original nonlinear network is described by $\mathbf{Lx} + \mathbf{N}(\mathbf{x}) = \mathbf{b}$ (when $\lambda = 1$).

For a successful homotopy method, it is necessary that the linear equations are well-conditioned and solvable. Therefore, a suitable choice of the values of the added resistors is essential.

We illustrate this approach by considering a nonlinear resistor $i = g(v)$ that will be replaced by

$$i = (1 - \lambda)G(v - a) + \lambda g(v), \qquad (13)$$

where a is an arbitrary voltage. (Note that the choice $\lambda = 1$ leads to the nonlinear resistor.) A simplified version can be obtained for $a = 0$:

$$i = Gv + \lambda(g(v) - Gv). \qquad (14)$$

A common nonlinear circuit element is a diode. The diode current is $i_d = g(v)$, where

$$g(v) = I_0 \left(\exp(v/V_T) - 1\right). \qquad (15)$$

This approach can be extended to other nonlinear circuit elements.

If a network contains bipolar junction transistors that are described by the Ebers-Moll model [19]:

$$\begin{aligned} I_E &= -\frac{I_S}{\alpha_F}\left[\exp\left(-v_{EB}/V_T\right) - 1\right] + \\ &\quad + I_S\left[\exp\left(-v_{CB}/V_T\right) - 1\right] \\ I_C &= I_S\left[\exp\left(-v_{EB}/V_T\right) - 1\right] - \\ &\quad - \frac{I_S}{\alpha_R}\left[\exp\left(-v_{CB}/V_T\right) - 1\right], \end{aligned} \qquad (16)$$

where I_E is the emitter and I_C is the collector current. For the embedding, we use the same approach as in the diode case.

Finally, in the case of MOS transistors we need to parameterize the nonlinear terms of the drain current. We only considered a simplified expression for the non-saturation mode:

$$I_D = C\left[(v_{gs} - V_{Th})v_{ds} - \frac{1}{2}(1 + \delta)v_{ds}^2\right], \qquad (17)$$

where v_{gs} is the gate-sources voltage, v_{ds} is the drain-source voltage, V_{Th} is the threshold voltage, and C and δ are the MOS transistor parameters [20]. Again, the nonlinear terms were replaced by suitable resistors when $\lambda = 0$.

An alternative embedding approach [3]–[6] is applicable if the nonlinear operator satisfies condition $\mathbf{N}(\mathbf{0}) = \mathbf{0}$. This embedding can be formulated by

$$\mathbf{F}(\mathbf{x}, \lambda) := (\mathbf{Lx} - \mathbf{b}) + \mathbf{N}(\lambda \mathbf{x}) = \mathbf{0}. \qquad (18)$$

Nonlinear parts of many equations describing models of semiconductor devices have this property. For example, the diode current $i_d = g(v)$ satisfies $i_d(0) = 0$. This embedding for diode equations leads to:

$$i_d(v, \lambda) = I_0 \left(\exp((\lambda v)/V_T) - 1\right). \qquad (19)$$

The linear network description obtained by setting $\lambda = 0$ corresponds to the descriptive equations of embedding given in [17].

Transistors are multi-input elements. Their currents, for example, depend on two junction voltages. Therefore, the embeddings for transistor equations are performed by multiplying each input voltage by λ. Note that if the voltages v_{EB} and v_{CB} of a bipolar junction transistor are multiplied by λ, the currents I_E, I_C, and I_B become zero for $\lambda = 0$. Furthermore, additional resistors values have to be suitably chosen. Similarly, the drain current $I_D = f(v_{gs}, v_{ds})$ of a MOS transistor in the non-saturation mode satisfies the condition $\mathbf{N}(\mathbf{0}) = \mathbf{0}$ if both input voltages are multiplied by λ. There are other more complex MOS models where the drain current depends on additional internal variables (the Fermi potential in the bulk region ϕ_{FB} and the flat-band voltage V_{FB}) and external variables (the source voltage v_S) [20]:

$$\begin{aligned} \tilde{i}_D = \tilde{C}\left[v_{gs} - V_{FB} - 2\phi_{FB} - \frac{1}{2}v_{ds}\right] v_{ds} - \\ -\frac{3}{2}\gamma\left[(v_{ds} + v_S + 2\phi_{FB})^{3/2} - (v_S + 2\phi_{FB})^{3/2}\right]. \end{aligned} \qquad (20)$$

In these cases a choice an appropriate parameter embedding remains an open issue.

In many practical models for semiconductor devices the condition $\mathbf{N}(\mathbf{0}) = \mathbf{0}$ is satisfied and, therefore, the embedding approach of [3]–[6] is applicable.

Note that the implementation of these embeddings in a SPICE-like circuit simulator is rather straightforward: the embedding is performed by simply multiplying the input currents and voltages by λ before their evaluation.

V. Comparison of Two Embedding Approaches

Unfortunately, we do not have a mathematical criteria for measuring the differences and usefulness of these embeddings. Therefore, we will compare these embeddings by applying them to several nontrivial circuits.

We have shown that the *standard embedding* is not suitable in case of MNA equations. The embedding proposed by Trajković et al., [3]–[6] was much more robust then the embedding proposed by Hasler et al., [17] and lead in many cases to fewer Jacobian evaluations.

Simulation performance in terms of the number of Jacobian evaluations of several industrial circuits is shown in Table 1. In all cases, the embedding (18) suggested in [3]–[6] was more robust than the embedding approach (12) given in [17]. For example, in the case of the CMOS reference circuit for a 16 Mbit DRAM with 27 equations, the embedding (18) required only 66 Jacobian evaluations (with HOMPACK's fixpNf algorithm), while embedding (12) required 67879 Jacobian evaluations. Other examples were even more favorable.

VI. Application

A useful application of the homotopy methods is the calculation of transfer curves of circuits where the input-output characteristics exhibit folds (negative differential resistance regions) or hysteresis (e.g., Schmitt trigger circuit). These curves cannot be traced using the conventional circuit simulator (SPICE). Such curves are easy to trace using HOMPACK [11]. HOMPACK uses the arc-length parameterization of the homotopy parameter and solve the associated differential equations,

We first multiply the input (current or voltage) by λ. In contrast to the above discussed embeddings, the network equations are not linear for $\lambda = 0$. Hence, in the first step we used [3]–[6] embedding to calculate the starting point of the input-output transfer curve. HOMPACK's curve tracing algorithms is then used to generate the input-output characteristic. The details are shown in Table 2.

VII. Implementation of a Homotopy Algorithm

All analog circuits that were successfully simulated in [5] and [6] were analog designs with bipolar transistors. We have generalized the embedding method to MOS transistor circuits and have developed an embedding model for MOS transistors. The implementation was done in the industrial circuit simulator TITAN [16] using HOMPACK [11]. We have also added a modification to TITAN for calculating transfer curves. TITAN-HOMPACK Interface is shown in Figure 2.

During our simulations, we observed difficulties with choosing the starting points for the homotopy paths. Further research is needed to design a user-friendly tool that can help circuit designers select a "good" starting point.

VIII. Conclusions

We show that the usefulness of the parameter embedding methods depends on the type of a circuit's descriptive equations. By carefully choosing the embedding equation, we demonstrated that homotopy methods can be successfully applied to solving nonlinear equations and to calculating dc operating points of nonlinear transistor circuits. These methods can be used as an alternative to the Newton-Raphson-type solvers.

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Simulation Performance			
Circuit	No. of Equations	No. of Jacobian Evaluations	Algorithm
Bipolar Op-Amp	30	508	fixpNf
		187	fixpQf
Bipolar Op-Amp 741 (1)	39	325	fixpQf
Bipolar Adder	449	473	fixpNf
		173	fixpQf
CMOS Reference Circuit	27	66	fixpNf
CMOS Master-Slave Flip-Flop FD2.15.F (1)	47	128	fixpQf
CMOS Op-Amp OP3V3 (1)	55	59	fixpQf
Small MOS Circuit (2)	6	50004	fixpNf
		49993	fixpQf

Table 1: Circuits successfully simulated by the new TITAN dc operating point solver. (1) HOMPACK's fixpNf algorithm failed. (2) TITAN's Newton method failed.

Calculation of Transfer Curves				
	Schmitt Trigger Circuit		Siemens Circuit WSCH1	
Parameters	Starting Point	Transfer Curve	Starting Point	Transfer Curve
No. of Equations	8	8	252	252
No. of Jacobian Evaluations	121	184	237	829
CPU time (seconds)	1.2	3.2	570	2970

Table 2: Performance of the transfer curve tracer for two test circuits. In all cases HOMPACK fixpNf algorithm with starting vector $\mathbf{a} = \mathbf{0}$ and HOMPACK error parameters $arcre = arcae = 10^{-6}$ and $ansre = ansae = 10^{-9}$ were used. Implementation was done on the Sun SPARC2 platform.

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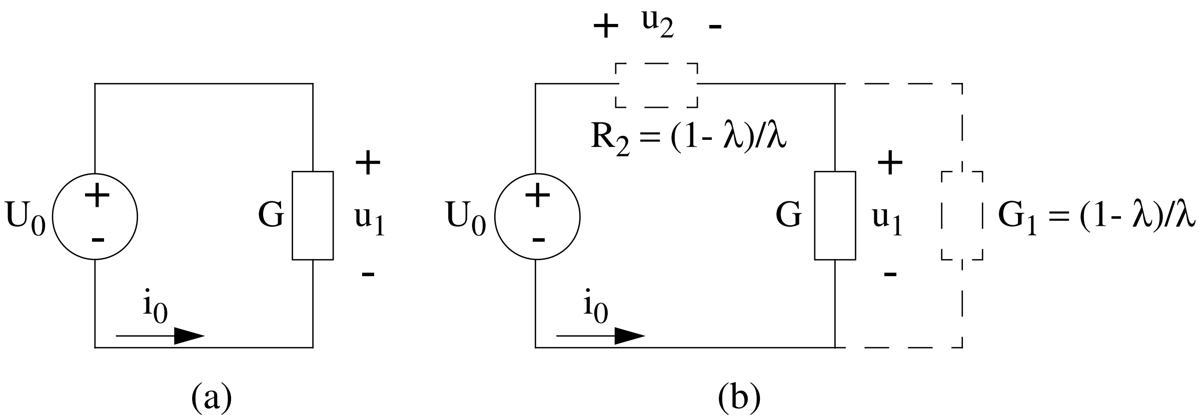

Figure 1: A simple example network for applying MNA equations. (a) Original circuit. (b) Circuit to be solved after parameter embedding.

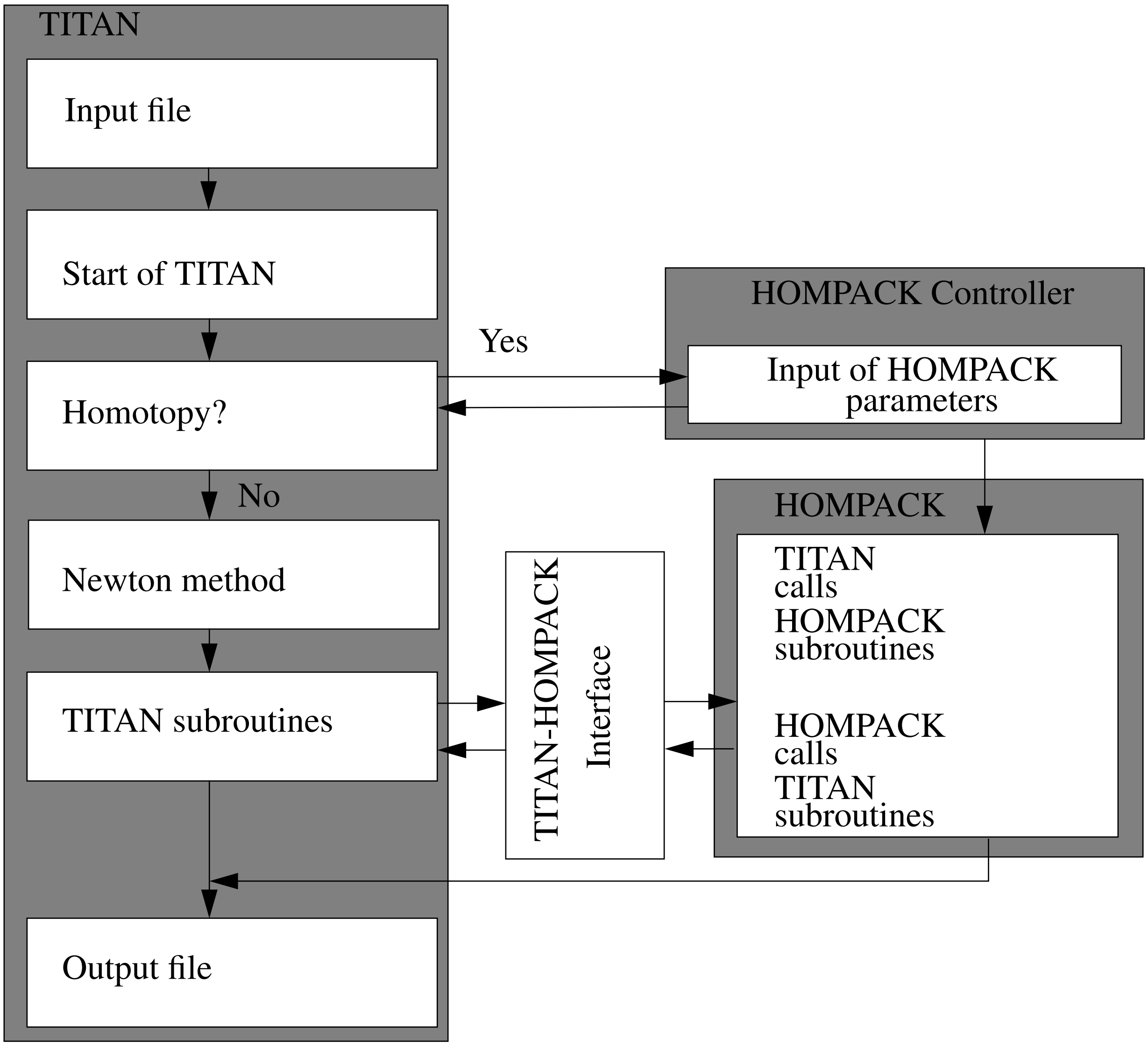

Figure 2: TITAN-HOMPACK Interface diagram.