HOMOTOPY METHODS FOR COMPUTING DC OPERATING POINTS

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Computing dc operating points is an essential task in simulating electronic circuits. It involves solving systems of nonlinear algebraic equations. Traditional methods for solving these equations often fail, exhibit convergence difficulties, and seldom find all the solutions. Recent application of parameter embedding methods, also called continuation and homotopy methods, proved successful in solving circuit nonlinear equations and, thus, in computing dc operating points of circuits that pose simulation difficulties and could not be simulated using more conventional techniques. The usefulness of the parameter embedding methods depends on the type of a circuit's descriptive equations. Such methods are an order of magnitude slower than conventional methods, but their speed can often be improved by careful implementation. They are becoming a viable alternative to the existing options in circuit simulators, where they can be used resolve convergence difficulties and to find multiple dc operating points.

1 DC Behavior of Nonlinear Electronic Circuits

An understanding of circuits' dc behavior is crucial for the analysis and design of electronic circuits. Numerous electronic circuits employ nonlinear elements. DC equations that describe such electronic circuits are systems of nonlinear algebraic equations. Such equations often have multiple solutions, which are called the circuits' dc operating points. For example, inherently nonlinear bistable circuits that possess two stable isolated equilibrium points are used in a variety of electronic designs, such as static random-access memory cells, latches, flip-flops, and shift registers. The operation of a Schmitt trigger is also intimately related to the circuit's ability to possess multiple dc operating points. Oscillator circuits employ structures that require the presence of nonlinear components. All these circuits exhibit a rich variety of nonlinear behaviors and can possess multiple operating points.

Recent advances in computer aided design (CAD) tools for circuit simulation have set designers free from the need to perform lengthy and tedious, but often only approximate, calculations to compute circuit currents and voltages. The SPICE circuit simulator [1, 2, 3] has become an industry standard, and many SPICE-like CAD tools are in use today. Nevertheless, the problem of computing the dc operating points of transistor circuits can be a difficult task. The exponential nature of the diode-type nonlinearities that model semiconductor devices can cause computational difficulties, and the fact that the equations describing transistor circuits can have multiple solutions causes concern that the "correct" solution has not been found.

2 Achieving Convergence in Circuit Simulators

Most SPICE-like circuit simulators compute dc operating points by using the Newton-Raphson (NR) method or one of its variants. These methods are robust and quadratically convergent when a good starting point, sufficiently close to a solution, is supplied. Variant schemes, such as the damped Newton's method, have been proven globally convergent only under unrealistically restrictive conditions [4, 5]. They sometimes fail because it is difficult to provide a starting point sufficiently close to an unknown solution. It has been proven that Newton's method converges in the case of arbitrary single-transistor networks with no ideal transformers, and certain "circuit-theoretic" sufficient conditions are given that guarantee convergence in the case of n-transistor networks [6]. The extension of this result beyond the one-transistor case, even for such special cases as multi-transistor circuits having no feedback structure [7] and thus possessing a unique operating point, is still an open problem [8].

To help dc convergence, experienced designers of analog circuits use several ad hoc techniques [9] that still rely on NR methods for solving nonlinear circuit equations. In the *source-stepping* algorithms, a ramping function is used for the sources and the circuit simulator provides a series of operating points until the response to the original set of driving voltages is obtained. In the *temperature-sweeping* procedure, the temperature is swept starting from a certain value (usually zero) and a dc operating point of the circuit is found. Then, using this value as a starting point, the temperature is increased and the new dc operating point calculated. The process is repeated until the dc operating point at the desired temperature is found. In another technique, known to the design community as G_{min}-*stepping*, small conductances are placed between every circuit node and ground. The initial value of the conductances is chosen large enough to ensure (if possible) the convergence of the NR method. The added conductances help convergence, since they contribute to the diagonal elements of the circuit's Jacobian matrix and can force it to become row or column sum dominant. If the operating point of the circuit is found, it is used to set initial node voltages for the next step. The auxiliary conductances are further decreased until a default minimum value is reached.

Another known method for improving dc convergence that is used by circuit designers is *pseudo-transient* analysis [10]. The original resistive network is transformed into a dynamical network by adding pseudo capacitors and inductors to the original circuit. The excitation is set to a ramped function that saturates at the desired dc value. Therefore, a set of nonlinear algebraic equations

$$\mathbf{0} = \mathcal{F}(\mathbf{x}) \tag{1}$$

describing the resistive circuits is replaced by a set of singular perturbed differential equations

$$\epsilon \dot{\mathbf{x}} = \mathcal{F}(\mathbf{x}) \tag{2}$$

The time-domain response of the circuit is found and the analysis performed until the circuit's transient response vanishes. Unfortunately, this type of embeddings can produce "stiff" problems and cause various stability problems that can fail to produce desired solution.

The described methods implicitly exploit the idea of embedding or continuation: a variation of a parameter over a range of values until it reaches the value for which the operating point is desired. The methods often work because each subsequent dc operating point is found by using the previous result as the starting point for the NR method. Nevertheless, in many circuits that possess multiple dc operating points, these ad hoc methods fail because of the presence of *bifurcations* in the continuation path leading to the solution.

3 Parameter Embedding, Continuation, and Homotopy Methods

Parameter embedding methods [11], also known as continuation methods and homotopy methods, have shown promise in resolving the computational difficulties often encountered in transistor network simulations [12] - [20]. To employ a continuation method we embed a "continuation parameter" in the circuit's nonlinear equations. By setting the parameter to zero, the system is reduced to one whose equations can be solved easily or whose solution is known. The solution to this simple problem becomes the starting point of a continuation path. The augmented equations are then continuously deformed, as the parameter is varied, until they finally describe the originally posed difficult problem. For example, let

$$\mathcal{F}(\mathbf{x}) = \mathbf{0} \tag{3}$$

be the nonlinear equation to be solved, where

$$\mathcal{F} : R^n \to R^n. \tag{4}$$

Let us create a homotopy mapping

$$\mathcal{H} : R^{n+1} \to R^n. \tag{5}$$

We then solve the equation

$$\mathcal{H}(\mathbf{x}, \lambda) = \mathbf{0} \tag{6}$$

while varying λ.

An example of a simple homotopy is

$$\mathcal{H}(\mathbf{x}, \lambda) = (1 - \lambda)(\mathbf{x} - \mathbf{a}) + \lambda\mathcal{F}(\mathbf{x}), \tag{7}$$

where $\lambda \in R^1$ is the continuation parameter, $\mathbf{a} \in R^n$ is the starting vector for the homotopy paths, and $\mathcal{F}(\mathbf{x}) = \mathbf{0}$ is the nonlinear equation to be solved. This homotopy mapping $\mathcal{H}(\mathbf{x}, \lambda)$ has the following properties

$$\mathcal{H}(\mathbf{x}, 0) = (\mathbf{x} - \mathbf{a}), \qquad \mathcal{H}(\mathbf{x}, 1) = \mathcal{F}(\mathbf{x}). \tag{8}$$

If the nonlinear equations satisfy certain coercivity conditions [21] homotopy methods can be made globally convergent and bifurcation free; that is, they will converge to a solution from an arbitrary starting point [22]. By exploiting the passivity [23] and no-gain [24] properties of the circuit elements, transistor circuit equations can be shown to satisfy such conditions [25]. It has also been proven that other forms of equations describing transistor circuits also satisfy such conditions [20].

Continuation methods described here rely on the continuous characteristics of the nonlinear circuit elements. An alternate approach to solving circuit equations is provided by simplicial methods, which deal with piecewise-liners characteristics of circuits elements [26]. Simplicial methods provide alternative techniques to follow homotopy path and the underlying homotopies are very similar. One of their disadvantages is that in practice models of nonlinear circuit elements come in continuous form.

4 Homotopies for Solving a Circuit's DC Equations

Various homotopies can be constructed from the circuit's nodal or modified nodal formulation. They can often find all the solutions to a circuit's dc equations.

The *fixed-point* homotopy is based on the equation

$$\mathcal{H}(\mathbf{x}, \lambda) = (1-\lambda)\mathbf{G}(\mathbf{x}-\mathbf{a}) + \lambda\mathcal{F}(\mathbf{x}), \tag{9}$$

where, in addition to the parameter λ, a random vector $\mathbf{a}$ and a new parameter (a diagonal matrix) $\mathbf{G} \in R^n \times R^n$ is embedded. With probability one a random choice of $\mathbf{a}$ gives a bifurcation-free homotopy path [22]. This homotopy has an interesting circuit interpretation. If $\mathcal{F}(\mathbf{x})$ describes a transistor circuit's nodal equations, this homotopy represents equations written for an augmented circuit. A branch consisting of a conductance $G_k(1-\lambda)/\lambda$ connected in series with a grounded voltage source a_k is connected to every node for which a nodal equation is written. At $\lambda = 0$, the starting point of the homotopy path, the added branches contain only a voltage source and hence force the nodal voltages to be equal to the elements a_k of the random vector $\mathbf{a}$. As λ increases, the added conductances come into play. When $\lambda = 1$, the added branches get disconnected from the circuit and the augmented circuit reverts to the original circuit.

The *variable-stimulus* homotopy is based on the equation

$$\mathcal{H}(\mathbf{x}, \lambda) = (1-\lambda)\mathbf{G}(\mathbf{x}-\mathbf{a}) + \mathcal{F}(\mathbf{x}, \lambda), \tag{10}$$

where the node voltages of the nonlinear elements are multiplied by λ. The starting point of the homotopy corresponds to the setting of all voltages across the nonlinear elements to zero, and hence it is the solution to a linear circuit.

The fastest converging homotopy for bipolar circuits is the *variable-gain* homotopy:

$$\mathcal{H}(\mathbf{x}, \lambda) = (1 - \lambda)\mathbf{G}(\mathbf{x} - \mathbf{a}) + \mathcal{F}(\mathbf{x}, \lambda\alpha), \tag{11}$$

where α is a vector consisting of transistor forward and reverse current gains. These current gains are multiplied by λ. Setting $\lambda = 0$ forces all transistor current gains to zero at the beginning of the continuation process. This solution point corresponds to the dc operating point of a circuit consisting of resistors and diodes only. Such a circuit always possesses a unique dc operating point, and its dc equations can be solved easily. For example, an efficient way to find the starting point of the homotopy path is to employ the variable-stimulus homotopy to solve the nonlinear circuit consisting of resistors and diodes only. Then the variable-gain homotopy is used to find the dc operating points of the original circuit. Through small and carefully selected changes of λ, the circuit is slowly deformed and the coupling of the transistors' p-n junctions is introduced. For each instance of λ, the circuit equations are solved (i.e., a dc operating point) is found. The original circuit and its dc operating point is obtained when $\lambda = 1$. This combination of variable-stimulus and variable-gain homotopies has been named the *hybrid* homotopy.

The *modified variable-stimulus* homotopy, chosen for its simplicity and ease of implementation, has been used in a production version of a circuit simulator [15, 16]:

$$\mathcal{H}(\mathbf{x}, \lambda) = (1 - \lambda)\mathbf{G}(\mathbf{x} - \mathbf{a}) + \mathcal{F}(\lambda\mathbf{x}). \tag{12}$$

Choosing a good starting point for the homotopy method is essential to assure fast convergence of the algorithm. In circuit terms, a good starting point may be a solution to a linear circuit or to a nonlinear circuit that has a unique solution. The choice of the starting point is important because it will influence the length of the path and the number of iterations needed to reach the destination. It is advantageous to choose a starting point that is easy to compute and that is physically related to the final state of the circuit, such as that which results by setting all transistor current gains to zero. This new circuit consists of resistors and diodes only. It has been proven that the equations describing such a circuit are diffeomorphisms, and hence a modified Newton method [4] can be used to solve these equations and to produce a good starting point for the homotopy algorithm.

The usefulness of the parameter embedding methods depends also on the type of a circuit's descriptive equations. SPICE-like simulators use modified nodal formulations where the unknowns are nodal voltages and currents. Embeddings for modified nodal equations do not always have convenient circuit interpretations such as the *fixed-point* homotopy embedding (9) has when applied to nodal equations. Modified nodal equations with embeddings that cause presence of nonphysical circuit elements (e.g., occurrence of negative resistors) can sometimes pose numerical difficulties and will have worse performance than the nodal equations [16]. For such equations the embedding of a parameter should be done in a manner that avoids nonphysical network instances.

An important issue in using homotopies to find dc operating points is the type of nonlinear functions that characterize nonlinear circuit elements. Most homotopy algorithms require that these nonlinear functions be at least $\mathcal{C}^2$ continuous. Furthermore, the passivity and the no-gain properties of the models used for semiconductor devices should be preserved, since they prove essential in assuring that the conditions required by the path-following algorithms [21] will be satisfied [25]. The transistor models implemented in circuit simulators often satisfy these properties.

4.1 Example: Schmitt trigger circuit

We illustrate the application of homotopy methods by solving nonlinear equation that describe the Schmitt trigger circuit shown in Fig. 1. A set of nonlinear equations describing the circuit, based on the modified nodal formulation [27], is:

$$
\begin{aligned}
\frac{x_1}{R_e} + i_{e_1} + i_{e_2} &= 0 \\
\frac{x_2 - x_4}{R_1} + \frac{x_2 - x_6}{R_{c_1}} + i_{c_1} &= 0 \\
\frac{x_3 - x_6}{R_{c_2}} + i_{c_2} &= 0 \\
\frac{x_4 - x_2}{R_1} + \frac{x_4}{R_4} - i_{e_2} - i_{c_2} &= 0 \\
\frac{x_5 - x_6}{R_2} + \frac{x_5}{R_3} - i_{c_1} - i_{e_1} &= 0 \\
\frac{x_6 - x_2}{R_{c_1}} + \frac{x_6 - x_3}{R_{c_2}} + \frac{x_6 - x_5}{R_2} + x_7 &= 0 \\
x_6 - V_{cc} &= 0.
\end{aligned}
\tag{13}
$$

Bipolar-junction transistors are modeled using the Ebers-Moll transistor model [28]:

$$
\begin{pmatrix} i_e \\ i_c \end{pmatrix} = \begin{pmatrix} 1 & -\alpha_r \\ -\alpha_f & 1 \end{pmatrix} \begin{pmatrix} f_e(v_e) \\ f_c(v_c) \end{pmatrix} \tag{14}
$$

where

$$
f_e(x) = m_e(e^{nv_e} - 1) \quad \text{and} \quad f_c(x) = m_c(e^{nv_c} - 1) \tag{15}
$$

and

$$m_e\alpha_f = m_c\alpha_r. \tag{16}$$

For transistor T_1

$$\begin{aligned} v_1 &= x_1 - x_5 \\ v_2 &= x_2 - x_5 \end{aligned} \tag{17}$$

and for transistor T_2

$$\begin{aligned} v_3 &= x_1 - x_4 \\ v_4 &= x_3 - x_4. \end{aligned} \tag{18}$$

For the two *npn* transistors that were used in the example $m_e < 0$, $m_c < 0$, and $n < 0$.

By using the fixed-point homotopy (9) we have successfully found all three solutions to Eq. (13). The elements of the diagonal matrix $\mathbf{G}$ were set to 10^{-3}, and the starting vector $\mathbf{a}$ was chosen by a random number generator. The solutions for the circuit's node voltages and the current flowing through the independent source are listed in Table 1. The solution paths for voltages x_1 through x_4, and the current x_7 versus the homotopy parameter λ are shown in Fig. 2(a) and (b), respectively. The three solutions are found when the paths intersect the vertical line corresponding to the value $\lambda = 1$.

5 Practical Implementations

Several techniques for tracking homotopy paths are implemented in publicly available software packages [29, 30, 31]. The dependence of the homotopy parameter λ on the path length

Three DC Operating Points of the Schmitt Trigger Circuit			
Unknown	Solution 1	Solution 2	Solution 3
$x_1(V)$	0.6682	1.1388	1.1763
$x_2(V)$	0.7398	2.6204	5.4897
$x_3(V)$	10.0000	3.5785	1.2689
$x_4(V)$	0.7325	1.9587	2.0055
$x_5(V)$	1.4905	1.9515	1.9734
$x_6(V)$	10.0000	10.0000	10.0000
$x_7(A)$	-0.0079	-0.0130	-0.0133

Table 1: Three solutions were found by solving the circuit's modified nodal equations using the fixed-point homotopy. Unknown variables x_1 through x_6 are node voltages, and variable x_7 is the current flowing through the source V_{cc}.

s was implemented in the HOMPACK software package [31].

HOMPACK solves a set of nonlinear algebraic equations of the form $\mathcal{F}(\mathbf{x}) = \mathbf{0}$ that describes the dc behavior of a nonlinear circuit. HOMPACK provides a *curve-tracing* option, which allows formulating a customized homotopy. Various embedding functions can be constructed from the circuit modified nodal formulation and used with the homotopy curve tracing option. We constructed homotopies that require minimal modifications to the matrices obtained from the circuit simulator and, hence, ensure the simplicity of the interface.

Three methods for solving nonlinear systems of equations are implemented in HOMPACK. Corresponding differential equations are created based on the ordinary differential equations, the normal flow algorithms, and the augmented Jacobian matrix algorithms. They differ by their robustness, and their convergence depends on the smoothness of the nonlinear functions used to model transistors and diodes.

Homotopy methods have been used [15, 16] to simulate various circuits that could not be simulated using conventional methods available in simulators. The software package HOMPACK [31] was interfaced to SPICE-like simulators such as ADVICE (AT&T) [15], the TITAN (Siemens) [16], and Spice 3F5 (UC Berkeley) [17] simulator's engines. When existing methods for finding dc operating points fail, the dc operating points of a transistor circuit are obtained using HOMPACK. DC operating points of various circuits that could not be simulated using conventional methods available in simulators were successfully found using homotopies. These circuits are often highly sensitive to the choice of parameters and the biasing voltages.

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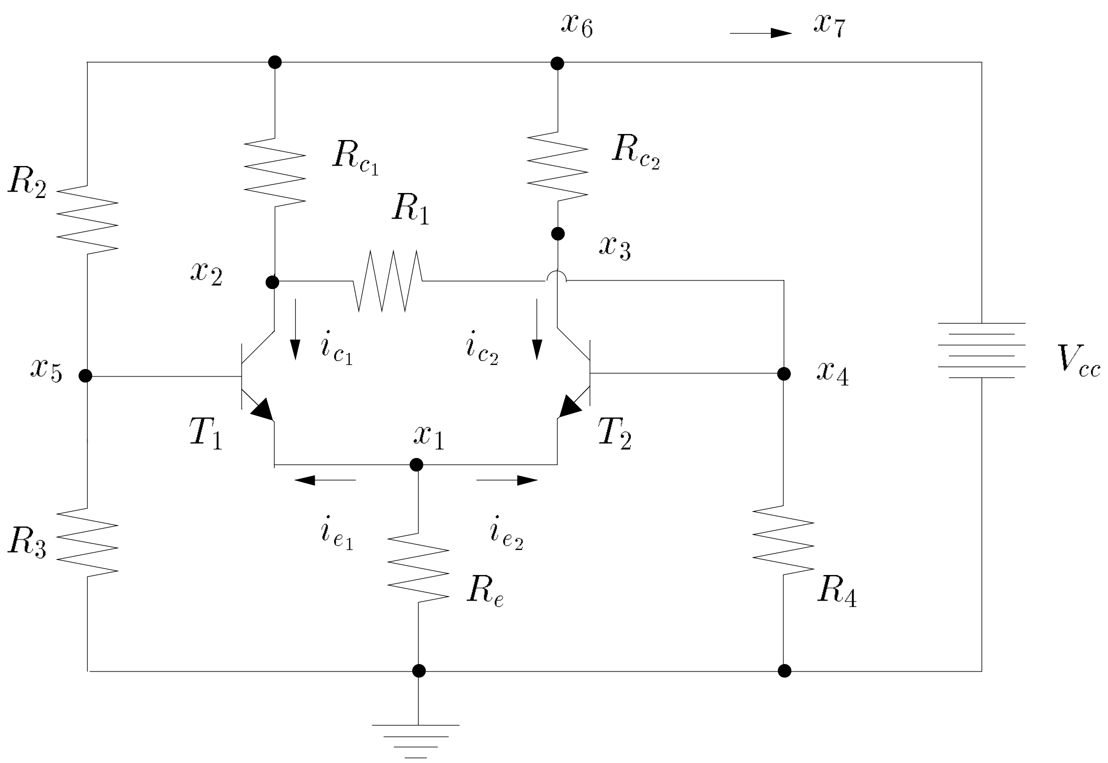

Figure 1: Schmitt trigger circuit whose equations were solved by using homotopy method. The circuits possess three dc operating points. All three solutions to the circuit's modified nodal equations were successfully found by using the fixed-point homotopy (9). Circuits parameters are: $V_{cc} = 10$ V, $R_1 = 10$ kΩ, $R_2 = 5$ kΩ, $R_3 = 1.25$ kΩ, $R_4 = 1$ MΩ, $R_{c_1} = 1.5$ kΩ, $R_{c_2} = 1$ kΩ, $R_e = 100$ Ω. The two bipolar transistors are identical with parameters: $m_e\alpha_f = m_c\alpha_r = -10^{-16}$ A, $\alpha_f = 0.99$, $\alpha_r = 0.5$, and $n = -38.78$ 1/V.

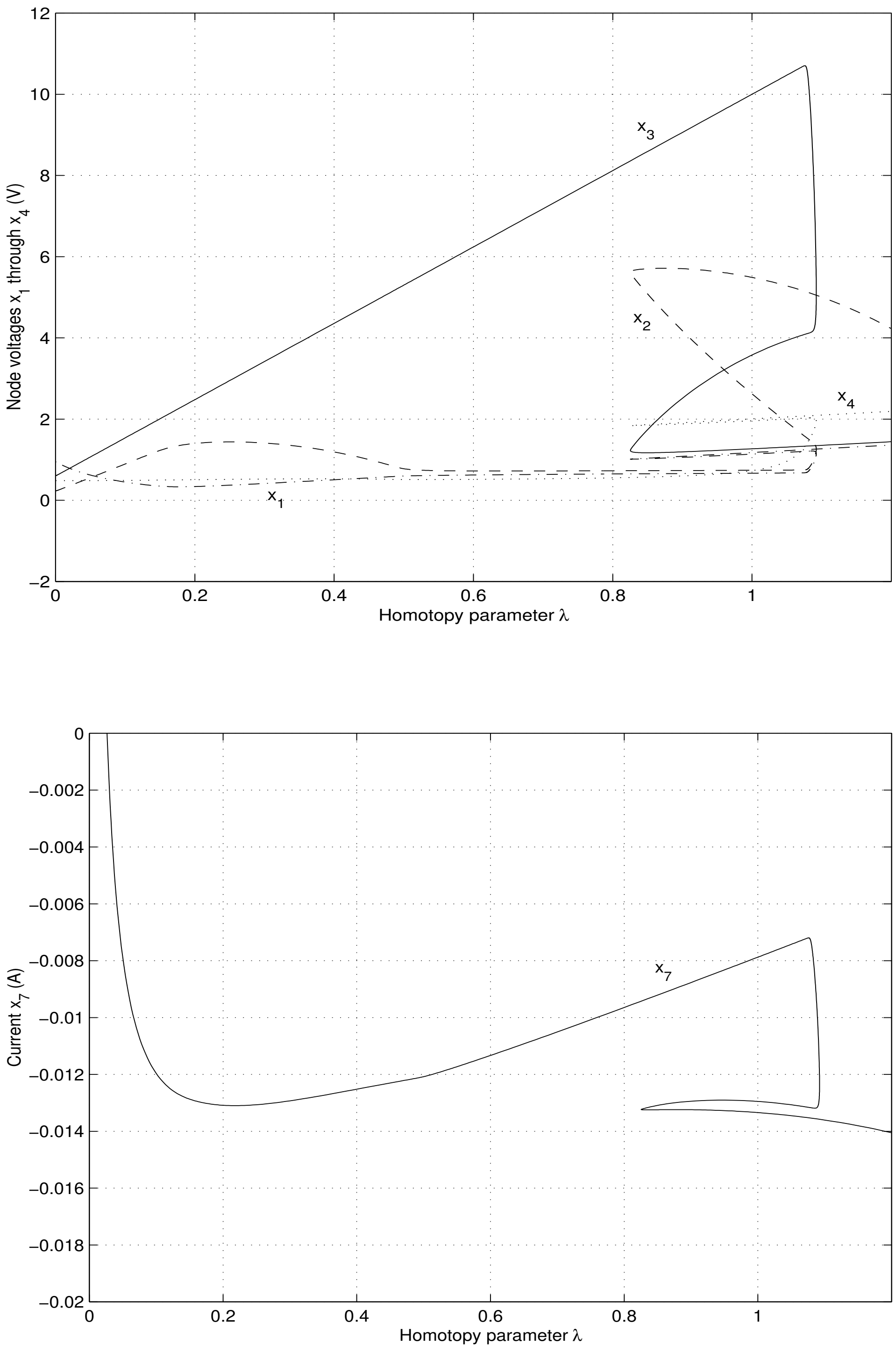

Figure 2: Homotopy paths for (a) the four node voltages x_1 through x_4, and (b) the current x_7 of the Schmitt trigger circuit. The paths were obtained by solving circuit's modified nodal equations with a simple homotopy embedding (9). The plots show solutions of the homotopy equations vs. the value of the the homotopy parameter λ. The three solutions are found when the homotopy paths intersect the vertical line corresponding to the value $\lambda = 1$.