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JOURNAL OF Econometrics
Journal of Econometrics 118 (2004) 111 – 127
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An omnibus test for the time series model AR(1)

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Abstract

An omnibus test is given for the hypothesis that a given time series sample comes from an autoregressive model of order 1. The test is of Cramér–von Mises type, based on the discrepancy between the standardized spectral distribution and its sample estimate. Tables are given to make the test for the case when the correlation between successive observations is known, and also for the case when this parameter is unknown and is estimated from the sample values. Two examples are given.

JEL classification: C22

Keywords: Goodness-of-fit; Spectral distribution; Cramér–von Mises test

1. Introduction

Goodness-of-fit tests are used to determine whether or not a specified model or class of models is appropriate for some observed statistical phenomena. In the time series context, several descriptions of a dependence structure are commonly used. The first is in terms of the autocorrelations, the second is the Fourier transform of the autocorrelations, giving the standardized spectral density, and a third utilizes the integral of the standardized spectral density, which is the standardized spectral distribution.

In this paper, we test the hypothesis that an observed time series is from some AR(1) process when the variance of the error is not known, but is estimated from the data. The autocorrelation parameter may be known or unknown; similarly, the mean of the error may be known or unknown. The test statistic is of Cramér–von Mises type,

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0304-4076/03/$ - see front matter
doi:10.1016/S0304-4076(03)00137-4

based on the discrepancy between the standardized spectral distribution and its sample estimate. The test should have power against every alternative.

2. The standardized spectral density and distribution

The definitions used will first be given for a general stationary stochastic process $\{y_t\}$, $t=\ldots,-1,0,1,\ldots$, with $\mathscr{E}y_t=\mu$, autocovariance function $\mathscr{E}(y_t-\mu)(y_{t+h}-\mu)=\sigma(h)$, $h=\ldots,-1,0,1,\ldots$, and autocorrelation function $\rho_h=\sigma(h)/\sigma(0)$, $h=\ldots,-1,0,1,\ldots$. The standardized spectral density is then defined as

$$f(\lambda)=\frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\rho_h\cos\lambda h,\qquad -\pi\leqslant\lambda\leqslant\pi.$$

Since $f(\lambda)=f(-\lambda)$, the standardized spectral distribution becomes

$$F(\lambda)=2\int_0^{\lambda}f(v)\,\mathrm{d}(v)=\frac{1}{\pi}\left(\lambda+2\sum_{h=1}^{\infty}\rho_h\frac{\sin\lambda h}{h}\right).$$

Note that $F(\pi)=1$; the standardized spectral distribution has the properties (non-negative increments) of a probability distribution on $[0,\pi]$.

Now suppose a sample is taken over time, say $y_1,y_2,\ldots,y_T$. When the expected value of the process is known to be μ, the sample autocovariance will be defined as

$$c_h=c_{-h}=\frac{1}{T}\sum_{t=1}^{T-h}(y_t-\mu)(y_{t+h}-\mu),\qquad h=0,1,\ldots,T-1. \tag{1}$$

When the mean of the process is not known, μ in the above expression is replaced by $\bar{y}$, the mean of the T observations. The asymptotic theory which follows in this paper will be the same whether μ or $\bar{y}$ is used in (1). The sample autocorrelation sequence is defined as $r_h=c_h/c_0$, $h=-(T-1),\ldots,T-1$; then the standardized sample spectral density is

$$I_T(\lambda)=\frac{1}{2\pi}\sum_{h=-(T-1)}^{T-1}r_h\cos\lambda h,\qquad -\pi\leqslant\lambda\leqslant\pi,$$

and the standardized sample spectral distribution function becomes

$$F_T(\lambda)=2\int_0^{\lambda}I_T(v)\,\mathrm{d}v=\frac{1}{\pi}\left(\lambda+2\sum_{h=1}^{T-1}r_h\frac{\sin\lambda h}{h}\right).$$

A test statistic can then be based on a functional of the process

$$Z_T(\lambda)=\sqrt{T}[F_T(\lambda)-F(\lambda)] \tag{2}$$

as a stochastic process over $[0,\pi]$.

The word 'standardized' is employed here because the autocorrelations are used in the definitions, following Bartlett (1954, 1978), although many authors (e.g. Grenander and Rosenblatt, 1957) use the autocovariances; this is especially so when the primary interest is in the probabilistic properties of the spectral density and distribution. A summary of these properties, with many references, is given by Anderson (1993). However, for statistical testing, when the variance of the random error is not known, the available information concerning the time series lies with the correlations, as was noted by Bartlett; hence our use of the standardized distribution.

Suppose Q is defined as $I_T(\lambda)$ but with autocovariances c_h replacing autocorrelations r_h. Grenander and Rosenblatt (1952, 1957) showed that Q converges to a Gaussian stochastic process on the interval $(-\pi, \pi)$ and gave the covariance function under the condition $E(y_t^8) < \infty$. The standardization given by replacing c_h by r_h permits convergence of $I_t(\lambda)$ under weaker conditions. Anderson and You (2000) showed that $I_T(\lambda)$ converges to a Gaussian process under the condition $E(y_t^2) < \infty$. We observe in passing that many authors have noted that the conditions for normal asymptotic distributions of autocorrelations are weaker than for autocovariances, see Anderson (1993). Küppelberg and Mikosch (1996) give a related observation concerning the periodogram.

The statistic $I_T(\lambda)$ is related to the periodogram, which, as defined by Bartlett (1978), is (with our notation)

$$I_T^*(\lambda_p) = 2 \sum_{h=-(T-1)}^{T-1} c_h \cos(h\lambda_p)$$

with $\lambda_p = 2\pi p/T;\ p = 0, 1, \ldots, T/2$ or $(T-1)/2$. Statistic $I_T^*(\lambda_p)$ is therefore defined on a discrete set of points λ_p, whereas $I_T(\lambda)$ is a continuous function of λ. Suppose $F_T^*(\lambda_p) = 2\sum_{u=0}^{p} I_T^*(\lambda_u)$, and let $U_p = F_T^*(\lambda_p)/F_T^*(\lambda_{T/2})$. Bartlett (1978) considers a test statistic based on U_p, a standardized partial sum of the periodogram corresponding to $F_T(\lambda)$ above. Bartlett suggests a test statistic of Kolmogorov–Smirnov type, based on the probability that U_p lies between two boundaries; when the test is for uncorrelated y_T, the statistic has the well-known asymptotic distribution of the Kolmogorov–Smirnov statistic used for testing a completely specified probability distribution; see Bartlett (1978). Dahlhaus (1995) has investigated U_p and has demonstrated the weak convergence properties of $Z_T(\lambda)$ above. Dahlhaus (1995) observed (with Bartlett) that the distribution of Bartlett's test statistic is not easily found, even asymptotically, when parameters are unknown. The same is true for Kolmogorov–Smirnov statistics for testing probability distributions; for several examples, see Stephens (1986).

We therefore propose, as a test criterion, a Cramér–von Mises functional of the process $Z_T(\lambda)$. It is first convenient to transform as follows. We have

$$Z_T(\lambda) = \frac{2\sqrt{T}}{\pi} \sum_{h=1}^{T-1} \frac{\sin \lambda h}{h} (r_h - \rho_h) - \frac{2\sqrt{T}}{\pi} \sum_{h=T}^{\infty} \frac{\sin \lambda h}{h} \rho_h. \tag{3}$$

Let $u = G(\lambda)/G(\pi)$, equivalently $\lambda = G^{-1}[G(\pi)u] = \lambda(u)$, say, where

$$G(\lambda) = 2\int_0^{\lambda} f^2(v)\,\mathrm{d}v.$$

Then the left-hand side of (3) is transformed from $Z_T(\lambda)$ to

$$Y_T(u) = \sqrt{T}\{F_T[\lambda(u)] - F[\lambda(u)]\}.$$

Suppose further that $Y_T^*(u) = Y_T(u)/[2\sqrt{\pi G(\pi)}]$. These transformations convert the original $Z_T(\lambda)$ process to one on the interval (0,1), and also simplify the covariance function.

The functional of (2) that we shall use for testing the null hypothesis is the Cramér–von Mises statistic

$$W_T^2 = \int_0^1 [Y_T^*(u)]^2 \, \mathrm{d}u. \tag{4}$$

The computing formula for W_T^2 is (Anderson, 1993)

$$W_T^2 = \frac{T}{4\pi^4 G^2(\pi)} \sum_{h=1}^{H} \left\{ \sum_{g=1}^{T-1} \frac{(r_g - \rho_g)(\rho_{h+g} - \rho_{h-g})}{g} \right\}^2. \tag{5}$$

In theory, H is infinite, but in practice, H must be chosen so that the term in the sum is negligibly small for $h > H$. Large values of W_T^2 will indicate that the hypothesized model should be rejected.

3. Cramér–von Mises statistics: asymptotic theory

The asymptotic distribution of W_T^2 may be found following a well-known procedure (see, for example, Anderson and Darling, 1952), as follows. Suppose the process $Y_T^*(u)$ converges weakly to the Gaussian process $Y^*(u)$ with covariance function $\rho(u,v)$. This covariance will depend on the model and on any parameters which must be estimated. The limiting distribution is then the distribution of $W^2 = \int_0^1 \{Y^*(u)\}^2 \, \mathrm{d}u$, and takes the form

$$W^2 = \sum_{i=1}^{\infty} \omega_i X_i^2, \tag{6}$$

where the X_i are independent standard normal random variables and the weights ω_i are the solutions of the integral equation

$$\omega g(u) = \int_0^1 \rho(u,v) g(v) \, \mathrm{d}v. \tag{7}$$

The mean of W^2 is given by $\mu_W = \int_0^1 \rho(u,u) \, \mathrm{d}u$. In practice, the quantity (6) may be approximated by

$$S = \sum_{i=1}^{n} \omega_i X_i^2 + \omega_\infty,$$

where ω_∞ is a random variable independent of the X_i and with a distribution chosen so that W^2 and S have the same mean and variance. For n large enough, the variance of ω_∞ will become negligibly small, and ω_∞ can be replaced by a constant C. Then percentage points S_α of $S-C$ at level α may be found by Imhof's method, and hence percentage points of W^2 are given by $W^2_\alpha = S_\alpha + C$. Percentage points for finite sample sizes T can be found by Monte Carlo methods; see Section 4.3 below.

4. Tests for AR(1)

In Anderson and Stephens (1993), tests for both the AR(1) and MA(1) processes were developed for the case when the parameters in the model were known. For each model the limiting distribution of W_T^2 was found and tabulated in both tails. For completeness, for the test for AR(1) with known ρ, we now add Monte Carlo points to enable the tests to be made for small values of T. This will be followed by the test for the AR(1) process with an unknown parameter ρ. For the AR(1) time series, $y_t = \rho y_{t-1} + u_t$, where the u_t are uncorrelated with mean zero and variance σ^2, and where $-1 < \rho < 1$. Then, $\rho_h = \rho_{-h} = \rho^h$, and $G(\pi) = (1+\rho^2)/\{2\pi(1-\rho^2)\}$.

4.1. *Test for AR(1) when ρ is known*

The test for AR(1) with known ρ consists of the following steps:

1. Calculate the r_g for $g = 1, \ldots, T-1$.
2. Calculate W_T^2 from (5).
3. Refer W_T^2 to the points in Table 1a when μ has been used in the calculations, and to those in Table 1b when $\bar{y}$ has been used. The table should be entered at $|\rho|$ and T, and the model will be rejected at level α if W_T^2 exceeds the percentage point given for level α.

4.2. *The test for $\rho = 0$*

When the test is for the special case $\rho = 0$, that is, the observations are uncorrelated, against the AR(1) alternative, the test statistic simplifies to

$$W_T^2 = (T/\pi^2) \sum_{h=1}^{T-1} r_h^2/h^2.$$

Since $\sqrt{T} r_h$ has a limiting standard normal distribution, the limiting distribution of W_T^2 is then (6) with $\omega_i = 1/(\pi i)^2$. This distribution is the limiting distribution of the Cramér–von Mises statistic when used to test that a random sample comes from a completely specified distribution. This test is described in Stephens (1986).

4.3. *Test for AR(1) when ρ is unknown*

In this section, the technique to find the limiting distribution of W_T^2 will be given in detail, when the test is for an AR(1) process, with the parameter ρ now unknown. This

Table 1
Monte Carlo and asymptotic points for the test for AR(1) with ρ known

ρ	$T \setminus \alpha$	0.5	0.25	0.1	0.05	0.025	0.01	0.005	0.001
(a) *The mean μ is known*									
0.00	20	0.090	0.168	0.282	0.377	0.479	0.627	0.759	0.984
	50	0.104	0.186	0.317	0.419	0.526	0.680	0.784	1.012
	100	0.111	0.197	0.328	0.436	0.555	0.706	0.839	1.108
	200	0.115	0.205	0.344	0.458	0.568	0.734	0.849	1.137
	∞	0.119	0.209	0.347	0.461	0.581	0.742	0.869	1.168
0.10	20	0.089	0.169	0.285	0.374	0.460	0.593	0.697	0.892
	50	0.104	0.191	0.326	0.427	0.532	0.689	0.824	1.094
	100	0.112	0.201	0.338	0.466	0.579	0.741	0.850	1.159
	200	0.118	0.210	0.347	0.457	0.574	0.747	0.866	1.165
	∞	0.120	0.212	0.353	0.469	0.590	0.757	0.885	1.189
0.20	20	0.090	0.170	0.287	0.380	0.472	0.587	0.661	0.953
	50	0.106	0.196	0.328	0.445	0.559	0.723	0.797	1.165
	100	0.115	0.207	0.352	0.458	0.580	0.731	0.853	1.201
	200	0.117	0.210	0.358	0.472	0.615	0.819	0.892	1.221
	∞	0.123	0.219	0.367	0.489	0.617	0.792	0.926	1.246
0.30	20	0.089	0.168	0.290	0.383	0.480	0.596	0.698	0.961
	50	0.105	0.197	0.336	0.442	0.565	0.698	0.804	1.212
	100	0.115	0.214	0.366	0.488	0.631	0.778	0.899	1.254
	200	0.121	0.220	0.369	0.494	0.613	0.793	0.938	1.288
	∞	0.127	0.229	0.386	0.517	0.653	0.839	0.983	1.324
0.40	20	0.083	0.167	0.280	0.367	0.448	0.557	0.651	0.924
	50	0.108	0.207	0.352	0.464	0.584	0.759	0.865	1.150
	100	0.117	0.222	0.375	0.496	0.627	0.763	0.927	1.250
	200	0.124	0.227	0.389	0.524	0.678	0.854	0.986	1.338
	∞	0.132	0.240	0.408	0.547	0.692	0.891	1.044	1.407
0.50	20	0.077	0.156	0.262	0.340	0.405	0.480	0.553	0.714
	50	0.103	0.203	0.349	0.462	0.559	0.720	0.835	1.173
	100	0.115	0.221	0.381	0.511	0.639	0.830	0.965	1.335
	200	0.124	0.236	0.405	0.540	0.680	0.899	1.038	1.386
	∞	0.136	0.251	0.428	0.575	0.728	0.938	1.100	1.483
0.60	20	0.070	0.145	0.241	0.305	0.355	0.410	0.445	0.557
	50	0.098	0.194	0.338	0.438	0.542	0.666	0.817	1.159
	100	0.113	0.215	0.373	0.493	0.613	0.788	0.931	1.339
	200	0.127	0.237	0.405	0.548	0.688	0.922	1.042	1.483
	∞	0.141	0.260	0.445	0.598	0.758	0.977	1.145	1.547
0.70	20	0.062	0.132	0.205	0.248	0.279	0.318	0.340	0.378
	50	0.091	0.180	0.303	0.389	0.477	0.582	0.684	0.896
	100	0.110	0.214	0.359	0.469	0.584	0.720	0.858	1.196
	200	0.124	0.241	0.406	0.539	0.680	0.850	0.993	1.370
	∞	0.144	0.266	0.457	0.615	0.780	1.005	1.179	1.590

Table 1 (*continued*)

ρ	$T \setminus \alpha$	0.5	0.25	0.1	0.05	0.025	0.01	0.005	0.001
0.80	20	0.050	0.104	0.161	0.191	0.211	0.231	0.243	0.257
	50	0.078	0.164	0.262	0.329	0.392	0.463	0.521	0.660
	100	0.097	0.194	0.329	0.431	0.525	0.663	0.755	1.056
	200	0.117	0.229	0.395	0.526	0.658	0.819	0.951	1.285
	∞	0.146	0.271	0.465	0.626	0.794	1.024	1.201	1.621
0.90	20	0.025	0.070	0.095	0.106	0.114	0.120	0.123	0.126
	50	0.055	0.115	0.175	0.210	0.238	0.263	0.277	0.307
	100	0.075	0.154	0.255	0.322	0.379	0.446	0.493	0.621
	200	0.098	0.197	0.338	0.437	0.535	0.652	0.804	1.158
	∞	0.147	0.273	0.470	0.632	0.802	1.034	1.213	1.639
(b) *The mean μ is unknown*									
0.00	20	0.096	0.175	0.294	0.387	0.486	0.629	0.764	0.992
	50	0.107	0.190	0.319	0.423	0.533	0.683	0.785	1.077
	100	0.112	0.198	0.329	0.440	0.554	0.706	0.851	1.122
	200	0.116	0.204	0.340	0.452	0.569	0.727	0.855	1.140
	∞	0.119	0.209	0.347	0.461	0.581	0.742	0.869	1.168
0.10	20	0.100	0.183	0.303	0.399	0.490	0.618	0.701	0.870
	50	0.108	0.197	0.326	0.434	0.540	0.699	0.808	1.016
	100	0.115	0.204	0.342	0.452	0.566	0.732	0.845	1.081
	200	0.117	0.208	0.348	0.460	0.578	0.749	0.865	1.143
	∞	0.120	0.212	0.353	0.469	0.590	0.757	0.885	1.189
0.20	20	0.105	0.192	0.317	0.406	0.502	0.609	0.673	0.829
	50	0.114	0.209	0.346	0.444	0.558	0.696	0.815	1.021
	100	0.118	0.212	0.356	0.462	0.582	0.734	0.869	1.155
	200	0.119	0.214	0.360	0.477	0.602	0.767	0.897	1.194
	∞	0.123	0.219	0.367	0.489	0.617	0.792	0.926	1.246
0.30	20	0.109	0.205	0.329	0.417	0.499	0.597	0.658	0.775
	50	0.117	0.217	0.356	0.463	0.582	0.715	0.802	1.072
	100	0.123	0.223	0.369	0.490	0.610	0.773	0.885	1.217
	200	0.125	0.225	0.374	0.501	0.624	0.799	0.919	1.260
	∞	0.127	0.229	0.386	0.517	0.653	0.839	0.983	1.324
0.40	20	0.117	0.218	0.338	0.419	0.496	0.569	0.623	0.734
	50	0.124	0.231	0.375	0.493	0.616	0.720	0.841	0.∞
	100	0.126	0.233	0.389	0.513	0.637	0.788	0.936	1.210
	200	0.128	0.234	0.399	0.532	0.668	0.845	0.985	1.305
	∞	0.132	0.240	0.408	0.547	0.692	0.891	1.044	1.407
0.50	20	0.125	0.227	0.341	0.406	0.458	0.527	0.563	0.644
	50	0.123	0.236	0.392	0.490	0.587	0.703	0.792	0.964
	100	0.125	0.243	0.409	0.538	0.658	0.814	0.924	1.196
	200	0.133	0.248	0.416	0.553	0.687	0.877	1.028	1.368
	∞	0.136	0.251	0.428	0.575	0.728	0.938	1.100	1.483

Table 1 (*continued*)

ρ	$T \setminus \alpha$	0.5	0.25	0.1	0.05	0.025	0.01	0.005	0.001
0.60	20	0.133	0.234	0.324	0.378	0.419	0.457	0.482	0.530
	50	0.128	0.243	0.393	0.486	0.571	0.661	0.723	0.882
	100	0.131	0.251	0.412	0.525	0.639	0.792	0.922	1.111
	200	0.134	0.254	0.426	0.563	0.693	0.890	1.033	1.348
	∞	0.141	0.260	0.445	0.598	0.758	0.977	1.145	1.547
0.70	20	0.149	0.230	0.296	0.327	0.354	0.377	0.389	0.419
	50	0.133	0.248	0.376	0.457	0.519	0.592	0.653	0.755
	100	0.133	0.256	0.406	0.510	0.614	0.725	0.830	0.977
	200	0.135	0.258	0.437	0.563	0.694	0.834	0.967	1.246
	∞	0.144	0.266	0.457	0.615	0.780	1.005	1.179	1.590
0.80	20	0.153	0.197	0.230	0.246	0.256	0.264	0.269	0.278
	50	0.143	0.251	0.350	0.406	0.453	0.499	0.518	0.562
	100	0.135	0.257	0.398	0.488	0.569	0.676	0.735	0.861
	200	0.137	0.261	0.426	0.548	0.677	0.824	0.935	1.206
	∞	0.146	0.271	0.465	0.626	0.794	1.024	1.201	1.621
0.90	20	0.105	0.116	0.123	0.126	0.127	0.129	0.129	0.130
	50	0.158	0.212	0.255	0.275	0.290	0.304	0.310	0.320
	100	0.145	0.251	0.346	0.406	0.445	0.498	0.522	0.557
	200	0.138	0.260	0.402	0.497	0.570	0.645	0.706	0.826
	∞	0.147	0.273	0.470	0.632	0.802	1.034	1.213	1.639

parameter will be estimated by the first-order sample autocorrelation r_1. The quantities $f(\lambda)$, $F(\lambda)$, and $G(\lambda)$ will be relabelled $f(\lambda|\rho)$, $F(\lambda|\rho)$, and $G(\lambda|\rho)$ to emphasise the dependence on the parameter ρ. For this case

$$f(\lambda|\rho) = \frac{1-\rho^2}{2\pi(1+\rho^2-2\rho\cos\lambda)}, \qquad -\pi \leqslant \lambda \leqslant \pi,$$

$$F(\lambda|\rho) = \frac{2}{\pi}\tan^{-1}\left(\frac{1+\rho}{1-\rho}\tan\frac{\lambda}{2}\right), \qquad 0 \leqslant \lambda \leqslant \pi,$$

$$G(\lambda|\rho) = \frac{2\rho\sin\lambda}{\pi(1-\rho^2)} f(\lambda|\rho) + \frac{1+\rho^2}{2\pi(1-\rho^2)} F(\lambda|\rho), \qquad 0 \leqslant \lambda \leqslant \pi.$$

Thus

$$\frac{G(\lambda|\rho)}{G(\pi|\rho)} - F(\lambda|\rho) = \frac{4\rho\sin\lambda}{1+\rho^2} f(\lambda|\rho), \qquad 0 \leqslant \lambda \leqslant \pi.$$

Let

$$Z_T(\lambda) = \sqrt{T}\{F_T(\lambda) - F(\lambda|r_1)\}$$

$$
\begin{aligned}
&= \frac{2}{\pi}\sum_{h=1}^{T-1}\frac{\sin\lambda h}{h}\sqrt{T}(r_h - r_1^h) - \frac{2}{\pi}\sum_{h=T}^{\infty}\frac{\sin\lambda h}{h}\sqrt{T}r_1^h \\
&= \frac{2}{\pi}\sum_{h=1}^{T-1}\frac{\sin\lambda h}{h}\sqrt{T}(r_h - \rho_h) \\
&\quad - \frac{2}{\pi}\sum_{h=1}^{T-1}\frac{\sin\lambda h}{h}\sqrt{T}(r_1^h - \rho_h) + \mathrm{o_p}(1) \\
&= \frac{2}{\pi}\sum_{h=1}^{T-1}\frac{\sin\lambda h}{h}v_h - \frac{2}{\pi}\sum_{h=1}^{T-1}\frac{\sin\lambda h}{h}v_1 h\rho^{h-1} + \mathrm{o_p}(1),
\end{aligned}
$$

where $v_h = \sqrt{T}(r_h - \rho_h)$. The v_h have a limiting normal distribution with means 0 and covariance between v_h and v_1 of $h\rho^{h-1}(1-\rho^2)$. Define $\lambda(u|\rho)$ as the value of λ satisfying $u = G\{\lambda(u|\rho)\}/G(\pi|\rho)$. Then transform $Z_T(\lambda)$ to a function of u by substituting $\lambda(u|\rho)$ for λ, and let

$$
Y_T(u) = \sqrt{T}[F_T\{\lambda(u|\rho)\} - F\{\lambda(u|\rho)|r_1\}].
$$

Then

$$
W_T^2 = \frac{1}{4\pi G^2(\pi|\rho)}\int_0^1 \{Y_T(u)\}^2\,\mathrm{d}u.
$$

Let $Y_T^*(u) = Y_T(u)/[2\sqrt{\pi G(\pi|\rho)}]$; $Y_T^*(u)$ converges weakly to the Gaussian process $Y^*(u)$ with covariance function:

$$
\rho(u,v) = EY^*(u)Y^*(v) = \min(u,v) - uv - \frac{1-\rho^2}{2\rho^2}q(u|\rho)q(v|\rho), \tag{8}
$$

where

$$
q(u|\rho) = \frac{4\rho}{1+\rho^2}\sin[\lambda(u|\rho)]f[\lambda(u|\rho)|\rho].
$$

Asymptotically W_T^2 becomes $W^2 = \int_0^1\{Y^*(u)\}^2\,\mathrm{d}u$, and the distribution of W^2 takes the form (6) as before. The integral equation (7) must now be solved for weights ω_i with the above $\rho(u,v)$ as the kernel. When $\rho(u,v)$ takes the form (8), this may be done by the following procedure:

1. Define $\omega_i^* = 1/(\pi^2 i^2)$, $i = 1, 2, \ldots$. Then ω_i^* and $f_i(t) = \sqrt{2}\sin\pi i t$ are the eigenvalues and corresponding eigenfunctions of the integral equation

   $$
   \omega f(t) = \int_0^1 [\min(t,s) - ts]\,\mathrm{d}s
   $$

   normalized so that $\int_0^1 f_i^2(t)\,\mathrm{d}t = 1$.

2. Let $a_i = c\sqrt{2}\int_0^1 q(u|\rho)\sin\pi iu\,\mathrm{d}u$, where $c = \{(1-\rho^2)/(2\rho^2)\}^{1/2}$.
3. Let

$$\Phi(\omega) = 1 + \sum_{i=1}^{\infty} \frac{a_i^2}{\omega_i^* - \omega}. \tag{9}$$

4. Then $\omega_i,\ i = 1, 2, \ldots$, in (6) are the infinitely many solutions of $\Phi(\omega) = 0$.

Clearly, $\Phi(\omega)$ has vertical asymptotes at $\omega = \omega_j^*$; also, $\Psi'(\omega)$ is always positive. It follows that a solution of $\Psi(\omega) = 0$, say ω_j, lies in the interval $(\omega_j^*, \omega_{j-1}^*)$, $j = 1, 2, \ldots$, with $\omega_0^* = \sum_{j=1}^{\infty} a_j^2$. Thus solutions ω_j can be straightforwardly found by searching this interval, and percentage points of W^2 are found as described in Section 3. These are the limiting points given in Tables 2a and b. Points for W_T^2 for finite T, derived from Monte Carlo studies, are also given in the tables.

4.3.1. Test procedure

The test follows the steps in Section 4.1, but now using Table 2a or b, and the computing formula

$$W_T^2 = \frac{T(1-r_1^2)^2}{\pi^2(1+r_1^2)^2} \sum_{h=1}^{H} \left[\sum_{g=2}^{T-1} \frac{(r_g - r_1^g)(r_1^{h+g} - r_1^{|h-g|})}{g} \right]^2. \tag{10}$$

As before, H is theoretically infinite, but in practice, it must be chosen so that the term in the sum is negligibly small for $h > H$.

An equivalent formula for W_T^2, which avoids the problem of determining H, is

$$\begin{aligned} W_T^2 = {} & \frac{T(1-r_1^2)^2}{\pi^2(1+r_1^2)^2} \sum_{g_1=2}^{T-1}\sum_{g_2=2}^{T-1} \left(\frac{r_{g_1} - r_1^{g_1}}{g_1}\right)\left(\frac{r_{g_2} - r_1^{g_2}}{g_2}\right) \\ & \times \left\{ \frac{1+r_1^2}{1-r_1^2}\left(r_1^{|g_1-g_2|} - r_1^{g_1+g_2}\right) + |g_1-g_2| r_1^{|g_1-g_2|} - (g_1+g_2) r_1^{g_1+g_2} \right\}. \end{aligned} \tag{11}$$

Table 2a is to be used when μ is known and is used in the calculations, and is entered with $|r_1|$, the absolute value of r_1. Table 2b is used when μ is estimated by $\bar{y}$ and is entered with the actual value of r_1. The null hypothesis of an AR(1) process will be rejected at significance level α if the value of W_T^2 exceeds the point in the table corresponding to r_1, T, and α.

4.3.2. Construction of the tables

The tables have been constructed so that, when the table is entered with a given T and parameter ρ or statistic value r_1, the probability that W_T^2 exceeds the point given in a specific column is approximately the value α at the head of the column.

The usual method of creating such a table is to generate Monte Carlo samples for the model, with fixed T and ρ, and from these to find the estimated distribution

Table 2
Points for the test for AR(1) with ρ unknown

$r1$	$T \setminus \alpha$	0.5	0.25	0.1	0.05	0.025	0.01
(a) *The mean μ is known*[a]							
0.00	50	0.043	0.068	0.102	0.128	0.158	0.190
	100	0.048	0.072	0.106	0.135	0.160	0.200
	200	0.051	0.077	0.113	0.142	0.172	0.211
	500	0.054	0.080	0.116	0.151	0.180	0.218
	∞	0.056	0.085	0.124	0.155	0.188	0.233
0.10	50	0.043	0.071	0.105	0.134	0.164	0.200
	100	0.049	0.076	0.114	0.144	0.176	0.220
	200	0.053	0.080	0.120	0.149	0.187	0.226
	500	0.054	0.090	0.130	0.153	0.193	0.231
	∞	0.056	0.094	0.134	0.165	0.197	0.238
0.20	50	0.047	0.078	0.122	0.158	0.196	0.243
	100	0.054	0.086	0.130	0.166	0.206	0.256
	200	0.057	0.089	0.137	0.174	0.215	0.266
	500	0.059	0.093	0.141	0.178	0.221	0.276
	∞	0.061	0.095	0.144	0.183	0.224	0.281
0.30	50	0.052	0.089	0.142	0.186	0.223	0.270
	100	0.058	0.097	0.155	0.202	0.243	0.312
	200	0.062	0.104	0.161	0.210	0.258	0.323
	500	0.067	0.108	0.168	0.216	0.267	0.333
	∞	0.069	0.111	0.172	0.222	0.275	0.346
0.40	50	0.054	0.096	0.157	0.208	0.259	0.324
	100	0.064	0.112	0.178	0.238	0.295	0.385
	200	0.069	0.118	0.190	0.248	0.314	0.401
	500	0.074	0.125	0.199	0.253	0.324	0.408
	∞	0.078	0.129	0.204	0.266	0.331	0.420
0.50	50	0.059	0.107	0.174	0.231	0.286	0.363
	100	0.069	0.125	0.198	0.269	0.336	0.402
	200	0.076	0.133	0.214	0.284	0.356	0.444
	500	0.083	0.140	0.229	0.298	0.369	0.470
	∞	0.088	0.148	0.237	0.311	0.388	0.494
0.60	50	0.060	0.111	0.184	0.245	0.303	0.382
	100	0.073	0.130	0.213	0.279	0.350	0.433
	200	0.081	0.147	0.244	0.322	0.396	0.491
	500	0.091	0.157	0.252	0.337	0.421	0.523
	∞	0.096	0.165	0.268	0.352	0.441	0.562
0.70	50	0.056	0.108	0.181	0.236	0.289	0.363
	100	0.071	0.132	0.223	0.290	0.354	0.459
	200	0.084	0.153	0.256	0.339	0.411	0.548
	500	0.094	0.166	0.273	0.362	0.446	0.586
	∞	0.104	0.180	0.294	0.389	0.487	0.622

Table 2 (*continued*)

$r1$	$T \setminus \alpha$	0.5	0.25	0.1	0.05	0.025	0.01
0.80	50	0.046	0.097	0.162	0.211	0.258	0.309
	100	0.066	0.123	0.211	0.271	0.336	0.426
	200	0.082	0.154	0.258	0.334	0.420	0.524
	500	0.094	0.170	0.286	0.378	0.485	0.620
	∞	0.110	0.192	0.316	0.419	0.526	0.672
0.90	50	0.026	0.060	0.102	0.134	0.158	0.190
	100	0.047	0.094	0.162	0.206	0.256	0.317
	200	0.068	0.132	0.220	0.287	0.358	0.458
	500	0.089	0.162	0.278	0.368	0.461	0.587
	∞	0.115	0.202	0.334	0.443	0.557	0.712
(b) *The mean μ is unknown*[b]							
−0.90	50	0.026	0.059	0.101	0.131	0.158	0.191
	100	0.045	0.093	0.161	0.202	0.257	0.321
	200	0.067	0.129	0.216	0.286	0.362	0.472
	500	0.087	0.163	0.273	0.360	0.468	0.626
	∞	0.115	0.202	0.334	0.443	0.557	0.712
−0.80	50	0.046	0.094	0.161	0.206	0.250	0.301
	100	0.065	0.128	0.217	0.278	0.340	0.419
	200	0.081	0.150	0.255	0.337	0.414	0.526
	500	0.095	0.172	0.285	0.377	0.460	0.592
	∞	0.110	0.192	0.316	0.419	0.526	0.672
−0.70	50	0.056	0.111	0.185	0.243	0.295	0.380
	100	0.069	0.133	0.216	0.287	0.356	0.459
	200	0.084	0.156	0.253	0.338	0.414	0.521
	500	0.095	0.164	0.273	0.370	0.457	0.571
	∞	0.104	0.180	0.294	0.389	0.487	0.622
−0.60	50	0.060	0.114	0.187	0.243	0.303	0.368
	100	0.072	0.134	0.219	0.293	0.351	0.429
	200	0.081	0.146	0.242	0.312	0.394	0.489
	500	0.089	0.157	0.257	0.332	0.418	0.527
	∞	0.096	0.165	0.268	0.352	0.441	0.562
−0.50	50	0.058	0.107	0.179	0.232	0.283	0.370
	100	0.068	0.121	0.197	0.261	0.335	0.424
	200	0.077	0.136	0.215	0.285	0.354	0.454
	500	0.083	0.140	0.225	0.299	0.370	0.474
	∞	0.088	0.148	0.237	0.311	0.388	0.494
−0.40	50	0.056	0.101	0.162	0.209	0.260	0.324
	100	0.064	0.110	0.178	0.234	0.294	0.351
	200	0.070	0.118	0.188	0.250	0.312	0.384
	500	0.074	0.123	0.193	0.260	0.323	0.401
	∞	0.078	0.129	0.204	0.266	0.331	0.420

Table 2 (*continued*)

$r1$	$T \setminus \alpha$	0.5	0.25	0.1	0.05	0.025	0.01
−0.30	50	0.052	0.089	0.144	0.186	0.228	0.283
	100	0.057	0.095	0.153	0.197	0.242	0.304
	200	0.061	0.101	0.162	0.206	0.255	0.324
	500	0.066	0.106	0.165	0.211	0.262	0.338
	∞	0.069	0.111	0.172	0.222	0.275	0.346
−0.20	50	0.047	0.078	0.123	0.157	0.189	0.240
	100	0.052	0.085	0.131	0.164	0.201	0.255
	200	0.057	0.091	0.137	0.175	0.213	0.270
	500	0.059	0.093	0.141	0.180	0.220	0.277
	∞	0.061	0.095	0.144	0.183	0.224	0.281
−0.10	50	0.045	0.072	0.111	0.139	0.166	0.198
	100	0.050	0.078	0.116	0.142	0.175	0.213
	200	0.052	0.087	0.126	0.147	0.180	0.219
	500	0.055	0.091	0.131	0.160	0.192	0.228
	∞	0.056	0.094	0.134	0.165	0.197	0.238
0.00	50	0.044	0.070	0.105	0.134	0.164	0.176
	100	0.049	0.075	0.110	0.138	0.169	0.208
	200	0.051	0.079	0.115	0.145	0.175	0.218
	500	0.054	0.083	0.117	0.149	0.180	0.226
	∞	0.056	0.085	0.124	0.155	0.188	0.233
0.10	50	0.046	0.072	0.110	0.138	0.168	0.203
	100	0.049	0.078	0.120	0.141	0.181	0.221
	200	0.053	0.082	0.124	0.155	0.188	0.227
	500	0.054	0.090	0.131	0.157	0.194	0.233
	∞	0.056	0.094	0.134	0.165	0.197	0.238
0.20	50	0.051	0.085	0.130	0.166	0.203	0.254
	100	0.055	0.088	0.133	0.171	0.209	0.267
	200	0.057	0.090	0.136	0.175	0.214	0.274
	500	0.060	0.093	0.142	0.180	0.221	0.278
	∞	0.061	0.095	0.144	0.183	0.224	0.281
0.30	50	0.055	0.093	0.147	0.189	0.234	0.286
	100	0.062	0.100	0.156	0.206	0.255	0.314
	200	0.064	0.106	0.164	0.214	0.264	0.332
	500	0.067	0.108	0.168	0.219	0.271	0.340
	∞	0.069	0.111	0.172	0.222	0.275	0.346
0.40	50	0.062	0.108	0.174	0.224	0.264	0.320
	100	0.069	0.119	0.186	0.241	0.300	0.359
	200	0.073	0.125	0.193	0.248	0.320	0.384
	500	0.076	0.128	0.200	0.260	0.326	0.407
	∞	0.078	0.129	0.204	0.266	0.331	0.420

Table 2 (*continued*)

$r1$	$T \setminus \alpha$	0.5	0.25	0.1	0.05	0.025	0.01
0.50	50	0.069	0.123	0.194	0.246	0.304	0.365
	100	0.076	0.133	0.212	0.278	0.340	0.399
	200	0.079	0.137	0.224	0.291	0.352	0.430
	500	0.084	0.143	0.231	0.304	0.377	0.473
	∞	0.088	0.148	0.237	0.311	0.388	0.494
0.60	50	0.069	0.128	0.204	0.260	0.319	0.386
	100	0.083	0.145	0.233	0.301	0.362	0.460
	200	0.087	0.154	0.244	0.318	0.391	0.484
	500	0.090	0.158	0.260	0.338	0.424	0.537
	∞	0.096	0.165	0.268	0.352	0.441	0.562
0.70	50	0.074	0.135	0.213	0.265	0.315	0.370
	100	0.086	0.156	0.251	0.318	0.382	0.458
	200	0.091	0.161	0.264	0.339	0.407	0.508
	500	0.095	0.174	0.284	0.370	0.461	0.583
	∞	0.104	0.180	0.294	0.389	0.487	0.622
0.80	50	0.072	0.128	0.198	0.239	0.273	0.315
	100	0.086	0.155	0.240	0.294	0.356	0.417
	200	0.091	0.164	0.265	0.343	0.410	0.513
	500	0.104	0.183	0.298	0.380	0.479	0.597
	∞	0.110	0.192	0.316	0.419	0.526	0.672
0.90	50	0.061	0.120	0.159	0.184	0.216	0.243
	100	0.071	0.134	0.204	0.245	0.282	0.329
	200	0.084	0.157	0.244	0.305	0.359	0.421
	500	0.105	0.173	0.280	0.366	0.460	0.561
	∞	0.115	0.202	0.334	0.443	0.557	0.712

[a]The points for each finite T were obtained from 100,000 Monte Carlo samples as described in Section 4.3.

[b]The points for each finite T were obtained from 200,000 Monte Carlo samples as described in Section 4.3.

of W_T^2. From the Monte Carlo samples, the sample percentage points can be found and put into the table. Tables 1a and b, where ρ is known, were constructed in this way. Also, similarly constructed tables, for ρ unknown, have been given by Anderson et al. (1995, 1997).

In practice, when ρ is not known, the table would be entered with an estimate, and problems arise in using such tables. We shall focus on Table 2b. In the AR(1) model, with large samples, the estimate r_1 is approximately normally distributed with mean ρ and variance $(1-\rho^2)/T$; thus for T sufficiently large the probability is low that r_1 differs much from ρ. Then, and especially if the significance points for the whole range of ρ were roughly linear in ρ, the achieved significance level obtained by entering the table with r_1 can be expected, on average, to be approximately equal to the nominal level α.

However, in the present case, a plot of, say, the 5% point of W_T^2 against ρ is convex near $\rho = 0$. Thus, for example, when true $\rho = 0$ the curve will be entered at r_1 near 0, but not equal to 0; the apparent significance point will then be larger than it should be, and the achieved α will be less than the nominal value. Therefore a different computational procedure was used. For each fixed T, 200,000 values of ρ were drawn from the uniform distribution between -1 and 1. For each such ρ and T an AR(1) time series was generated and the values of r_1 and W_T^2 were stored. The values of r_1 were then binned; for example, for a recorded value $r_1 = 0.3$, the values of r_1 between 0.25 and 0.35 were placed in a vector. The corresponding values of W_T^2 were placed in another vector and the percentage points found. These are the points recorded against $r_1 = 0.3$. The points given for other values of r_1 were found similarly. The number of samples used for a particular r_1 is therefore approximately 200,000/(number of bins). Only values of r_1 between -0.95 and 0.95 were considered for binning; if a value of r_1 were beyond these limits the model might be judged as being nonstationary.

If r_1 were sufficient for ρ, the percentage points for W^2 would be independent of the true ρ. (Note that the normal density of the observations is a function of the quadratic expression $(1+\rho^2)Tc_0 - 2\rho(T-1)c_1 - \rho^2(y_1^2 + y_T^2)$; thus c_0 and c_1 are nearly sufficient for σ^2 and ρ, and so r_1 is nearly sufficient for ρ). The independence was tested by Monte Carlo sampling from fixed values of ρ, and, as expected, the percentage points for the different sets of r_1 were quite consistent.

Anderson and Stephens (2000) have shown that the distribution of W_T^2 is the same for ρ and $-\rho$, but because r_1 is a biased estimate of true ρ, the points in Table 2b are not symmetric about $r_1 = 0$. The same type of construction was used for Table 2a, when the mean μ of the process is known, but now the table is symmetric in r_1. Table 2a should be entered with the absolute value of r_1. Because of the symmetry, only 100,000 Monte Carlo samples were used to construct this table.

It should be mentioned that other methods can be used to find the significance level of the test statistic, based on Monte Carlo or resampling techniques such as the bootstrap; see for example Chen and Romano (1999) and Paparoditis (2000). An advantage of the tables is that the applied worker may not have the facilities available for such computer-intensive methods, which require at least time if not also memory; also with complicated models difficulties can arise concerning the way to apply such techniques efficiently.

5. Examples

Example 1: The data for this example consists of catches of fish in the Atlantic and Gulf of the St. Lawrence River in Canada. The data, recorded in Table 3, consists of 84 values, monthly data for the years 1990–1996. When a plot is made, there is an obvious seasonal effect; the values after this effect has been removed are given in Table 3 of the appendix. The resulting series will be tested to be AR(1). For these data the estimate of ρ is $r_1 = 0.39$, and the test statistic $W_T^2 = 0.05$; with reference to Table 2b the statistic is not significant at the 0.25 level, and the AR(1) model therefore is concluded to give a good fit to the data.

Table 3
Data derived from fish landings

Year						
1990	1991	1992	1993	1994	1995	1996
3502.8	6855.9	5276.6	−1382.7	−9797.9	−4177.5	−276.9
−5664.7	3885.5	3253.2	−2028.9	−700.4	−2454.5	3710.0
4797.0	−3705.2	2896.0	−8.6	−2718.3	2965.9	−4226.9
−7011.4	3821.7	6317.1	−8537.4	10110.1	−2034.0	−2666.1
−8518.5	−16192.7	−16010.9	11166.7	45597.9	−23399.2	7356.5
5485.7	1994.6	−7528.6	−15274.9	19362.0	−2374.5	−1664.4
20869.6	42171.1	−29647.4	19074.1	18389.8	−5517.0	−65340.1
4229.2	3262.7	−41357.1	4252.8	52270.7	−9369.9	−13288.5
13450.6	−23444.2	−40340.3	−7721.4	44012.4	−369.2	14412.1
−3102.9	−11548.8	−9760.9	33484.6	4080.4	−11284.0	−1868.3
−10770.4	−178.7	−14787.6	15090.2	−4821.1	3712.6	11755.0
19800.6	14334.3	−2610.0	−6506.1	−17037.7	3208.6	−11189.7

Table 4
Wolfer's sunspot numbers, 1749–1924

1749	80.9	83.4	47.7	47.8	30.7	12.2	9.6	10.2	32.4	47.6
1759	54.0	62.9	85.9	61.2	45.1	36.4	20.9	11.4	37.8	69.8
1769	106.1	100.8	81.6	66.5	34.8	30.6	7.0	19.8	92.5	154.4
1779	125.9	84.8	68.1	38.5	22.8	10.2	24.1	82.9	132.0	130.9
1789	118.1	89.9	66.6	60.0	46.9	41.0	21.3	16.0	6.4	4.1
1799	6.8	14.5	34.0	45.0	43.1	47.5	42.2	28.1	10.1	8.1
1809	2.5	0.0	1.4	5.0	12.2	13.9	35.4	45.8	41.1	30.4
1819	23.9	15.7	6.6	4.0	1.8	8.5	16.6	36.3	49.7	62.5
1829	67.0	71.0	47.8	27.5	8.5	13.2	56.9	121.5	138.3	103.2
1839	85.8	63.2	36.8	24.2	10.7	15.0	40.1	61.5	98.5	124.3
1849	95.9	66.5	64.5	54.2	39.0	20.6	6.7	4.3	22.8	54.8
1859	93.8	95.7	77.2	59.1	44.0	47.0	30.5	16.3	7.3	37.3
1869	73.9	139.1	111.2	101.7	66.3	44.7	17.1	11.3	12.3	3.4
1879	6.0	32.3	54.3	59.7	63.7	63.5	52.2	25.4	13.1	6.8
1889	6.3	7.1	35.6	73.0	84.9	78.0	64.0	41.8	26.2	26.7
1899	12.1	9.5	2.7	5.0	24.4	42.0	63.5	53.8	62.0	48.5
1909	43.9	18.6	5.7	3.6	1.4	9.6	47.4	57.1	103.9	80.6
1919	63.6	37.6	26.1	14.2	5.8	16.7				

The first entry on each line is the beginning year for that line.

Example 2. Here the data are 176 sunspot numbers, taken from Table 8 of Anderson (1971), and reproduced in Table 4. The estimate of ρ is $r_1 = 0.81$, and the test statistic $W_T^2 = 0.84$; reference to Table 2b shows that the statistic is significant at the 0.01 level, and the null hypothesis of an AR(1) model should be rejected. It is of interest to note that the lag-2 autocorrelation is 0.43.

6. Final remarks

It is possible to extend the above tests to more complicated time series models such as more general autoregressive moving average processes (ARMA), where the coefficients of the ARMA equation define the dependence structure of the process. The general theory for completely specified hypotheses was developed in Anderson (1993); see also Anderson (1997). The test statistic W_T^2 remains as defined in (5), but the functions $F(\lambda)$ and $G(\lambda)$ become more complicated. Another function of the discrepancy between the sample and theoretical spectral distributions is the Anderson–Darling statistic. This would give more weight to the difference $Z_T(\lambda)$ near $\lambda = 0$ and $\lambda = \pi$, but the asymptotic theory is more complicated and the advantages of using this statistic are less obvious than when the tests are for probability distributions.

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