Convolution: If X and Y independent rvs with densities f and g respectively and $Z = X + Y$ then

$$P(Z \leq z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f(x)g(y)dydx$$

Differentiating wrt z we get

$$f_Z(z) = \int_{-\infty}^{\infty} f(x)g(z-x)dx$$

This integral is called the **convolution** of densities f and g.

If $T_1, \ldots, T_n$ iid Exponential(λ) then $S_n = T_1 + \cdots + T_n$ has a Gamma(n, λ) distribution. Density of S_n is

$$f_{S_n}(s) = \lambda(\lambda s)^{n-1} e^{-\lambda s}/(n-1)!$$

for $s > 0$.

Proof:

$$\begin{aligned} P(S_n > s) &= P(N(0,s] < n) \\ &= \sum_{j=0}^{n-1} (\lambda s)^j e^{-\lambda s}/j! \end{aligned}$$

Then

$$\begin{aligned} f_{S_n}(s) &= \frac{d}{ds} P(S_n \leq s) \\ &= \frac{d}{ds} \left\{1 - P(S_n > s)\right\} \\ &= -\lambda \sum_{j=0} n - 1 \left\{ j(\lambda s)^{j-1} - (\lambda s)^j \right\} \frac{e^{-\lambda s}}{j!} \\ &= \lambda e^{-\lambda s} \sum_{j=0}^{n-1} \left\{ \frac{(\lambda s)^j}{j!} - \frac{(\lambda s)^{j-1}}{(j-1)!} \right\} \end{aligned}$$

This telescopes to

$$f_{S_n}(s) = \lambda(\lambda s)^{n-1} e^{-\lambda s}/(n-1)!$$

Properties of Poisson Processes

1) If N_1 and N_2 are independent Poisson processes with rates λ_1 and λ_2, respectively the $N = N_1 + N_2$ is a Poisson processes with rates $\lambda_1 + \lambda_2$.

2) Suppose N is a Poisson process with rate λ. Suppose each point is marked with a label, say one of $L_1, \ldots, L_r$ independently of all other occurences. Suppose p_i is the probability that a given point receives label L_i. Let N_i count the points with label i (so that $N = N_1 + \cdots + N_r$). Then $N_1, \ldots, N_r$ are independent Poisson processes with rates $p_i\lambda$.

3) Suppose $U_1, U_2, \ldots$ independent rvs, each uniformly distributed on $[0, T]$. Suppose M is a Poisson(λT) random variable independent of the U's. Let

$$N(t) = \sum_1^M 1(U_i \leq t)$$

Then N is a Poisson process on $[0, T]$ with rate λ.

4) Suppose N is a Poisson process with rate λ. Let $S_1 < S_2 < \cdots$ be the times at which points arrive Given $N(T) = n$ $S_1, \ldots, S_n$ have the same distribution as the order statistics of a sample of size n from the uniform distribution on $[0, T]$.

5) Given $S_{n+1} = t$, $S_1, \ldots, S_n$ have the same distribution as the order statistics of a sample of size n from the uniform distribution on $[0, T]$.

Indications of some proofs:

1) $N_1, \ldots, N_r$ independent Poisson processes rates λ_i, $N = \sum N_i$. Let A_h be the event of 2 or more points in N in the time interval $(t, t + h]$, B_h, the event of exactly one point in N in the time interval $(t, t + h]$.

Let A_{ih} and B_{ih} be the corresponding events for N_i.

Let H_t denote the history of the processes up to time t; we condition on H_t.

Note that

$$A_h \subset \bigcup_{i=1}^r A_{ih} \cup \bigcup_{i \neq j} (B_{ih} \cap B_{jh})$$

Since

$$\begin{aligned} P(B_{ih} \cap B_{jh}|H_t) &= P(B_{ih}|H_t)P(B_{jh}|H_t) \\ &= (\lambda_i h + o(h))(\lambda_j h + o(h)) \\ &= O(h^2) \\ &= o(h) \end{aligned}$$

we have checked one of the two infinitesimal conditions for a Poisson process.

Next note

$$B_h \subset \bigcup_{j=1}^{r} B_{j,h}$$

so that

$$\begin{aligned} P(B_h|H_t) &\leq \sum P(B_{j,h}|H_t) \\ &= \sum(\lambda_j h + o(h)) \\ &= (\sum \lambda_i)h + o(h) \end{aligned}$$

On the other hand let C_h be the event of no points in N in the time interval $(t, t+h]$ and C_{ih} the same for N_i. Then

$$\begin{aligned} P(C_h|H_t) &= P(\cap C_{ih}|H_t) \\ &= \prod P(C_{ih}|H_t) \\ &= \prod(1 - \lambda_i h + o(h)) \\ &= 1 - (\sum \lambda_i)h + o(h) \end{aligned}$$

shows

$$\begin{aligned} P(B_h|H_t) &\geq 1 - P(C_h|H_t) \\ &= (\sum \lambda_i)h + o(h) \end{aligned}$$

Hence N is a Poisson process with rate $\sum \lambda_i$.

2) The infinitesimal approach used for 1 can do part of this. See text for rest. Events defined as in **1)**: The event B_{ih} that there is one point in N_i in $(t, t+h]$ is the event, B_h that there is exactly one point in any of the r

processes together with a subset of A_h where there are two or more points in N in $(t, t+h]$ but exactly one is labeled i. Since $P(A_h|H_t) = o(h)$

$$\begin{aligned} P(B_{ih}|H_t) &= p_i P(B_h|H_t) + o(h) \\ &= p_i(\lambda h + o(h)) + o(h) \\ &= p_i \lambda h + o(h) \end{aligned}$$

Similarly, A_{ih} is a subset of A_h so

$$P(A_{ih}|H_t) = o(h)$$

This shows each N_i is Poisson with rate λp_i. To get independence requires more work; see the text for the algebraic method which is easier.

3) Fix $s < t$. Let $N(s,t)$ be the number of points in $(s,t]$. Given $N = n$ the conditional distribution of $N(s,t)$ is Binomial(n,p) with $p = (s-t)/T$. So

$$\begin{aligned} &P(N(s,t) = k) \\ &\quad = \sum_{n=k}^{\infty} P(N(s,t) = k, N = n) \\ &\quad = \sum_{n=k}^{\infty} P(N(s,t) = k | N = n) P(N = n) \\ &\quad = \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \frac{(\lambda T)^n}{n!} e^{-\lambda T} \\ &\quad = \frac{e^{-\lambda T}}{k!} p^k \sum_{n=k}^{\infty} (1-p)^{n-k} (\lambda T)^{n-k} \\ &\quad = \frac{e^{-\lambda T}}{k!} (\lambda T p)^k \sum_{m=0}^{\infty} (1-p)^m (\lambda T)^m \\ &\quad = \frac{e^{-\lambda T}}{k!} \times (\lambda T p)^k e^{-\lambda T(1-p)} \\ &\quad = \frac{e^{-\lambda(s-t)} (\lambda(s-t))^k}{k!} \end{aligned}$$

4): Fix s_i, h_i for $i = 1, \ldots, n$ such that

$$0 < s_1 < s_1 + h_1 < s_2 < \cdots < s_n < s_n + h_n < T$$

Given $N(T) = n$ we compute the probability of the event

$$A = \bigcap_{i=1}^{n} \{s_i < S - i < s_I + h_i\}$$

Intersection of A and $N(T) = n$ is ($s_0 = 0$):

$$B \equiv \bigcap_{i=1}^{n} \{N(s_{i-1}, s_i] = 0, N(s_i, s_i + h] = 1\}$$
$$\cap \{N(s_n + h, T] = 0\}$$

whose probability is

$$\left(\prod \lambda h_i\right) e^{-\lambda T}$$

So

$$\begin{aligned} P(A|N(t) = n) &= \frac{P(A, N(T) = n)}{(N(T) = n)} \\ &= \frac{\lambda^n e^{-\lambda T} \prod h_i}{(\lambda T)^n e^{-\lambda T}/n!} \\ &= \frac{n! \prod h_i}{T^n} \end{aligned}$$

Divide by $\prod h_i$ and let all h_i go to 0 to get joint density of $S_1, \ldots, S_n$ is

$$\frac{n!}{T^n} 1(0 < s_1 < \cdots < s_n < T)$$

which is the density of order statistics from a Uniform$[0, T]$ sample of size n.

5) Replace the event $S_{n+1} = T$ with $T < S_{n+1} < T + h$. With A as before we want

$$P(A|T < S_{n+1} < T + h)$$
$$= \frac{P(B, N(T, T + h] \geq 1)}{P(T < S_{n+1} < T + h)}$$

Note that B is independent of $\{N(T, T + h] \geq 1\}$ and that we have already found the limit

$$\frac{P(B)}{\prod h_i} \to \lambda^n e^{-\lambda T}$$

We are left to compute the limit of

$$\frac{P(N(T,T+h] \geq 1)}{P(T < S_{n+1} < T+h)}$$

The denominator is

$$\sum_{k=0}^{n} P(N(0,T] = k, N(T,T+h] = n+1-k)$$

$$= P(N(0,T] = n)\lambda h + o(h)$$

Thus

$$\frac{P(N(T,T+h] \geq 1)}{P(T < S_{n+1} < T+h)} = \frac{\lambda h + o(h)}{\frac{(\lambda T)^n}{n!} e^{-\lambda T} \lambda h + o(h)}$$

$$\to \frac{n!}{(\lambda T)^n e^{-\lambda T}}$$

This gives the conditional density of $S_1, \ldots, S_n$ given $S_{n+1} = T$ as in **4)**.

Inhomogeneous Poisson Processes

The idea of hazard rate can be used to extend the notion of Poisson Process. Suppose $\lambda(t) \geq 0$ is a function of t. Suppose N is a counting process such that

$$P(N(t+h) = k+1 | N(t) = k, H_t) = \lambda(t)h + o(h)$$

and

$$P(N(t+h) \geq k+2 | N(t) = k, H_t) = o(h)$$

Then N has independent increments and $N(t+s) - N(t)$ has a Poisson distribution with mean

$$\int_t^{t+s} \lambda(u)du$$

If we put

$$\Lambda(t) = \int_0^t \lambda(u)du$$

then mean of $N(t+s) - N(T)$ is $\Lambda(t+s) - \Lambda(t)$.

Jargon: λ is the **intensity** or **instaneous intensity** and Λ the **cumulative intensity**.

Can use the model with Λ any non-decreasing right continuous function, possibly without a derivative. This allows ties.

Continuous Time Markov Chains

Model for population growth: at time t population size is $N(t)$. In next h time units: population goes up 1 or down 1 (or stays same). Larger movements unlikely.

Natural model has probability of birth large when population size large, small when population size small.