STAT 380 Section 8

Continuous Time Markov Chains

Consider a population of single celled organisms in a stable environment.

Fix short time interval, length h.

Each organism has some probability of dividing to produce two organisms and some other probability of dying.

We might suppose:

- Different organisms behave independently.
- Probability of division is λh plus $o(h)$.
- Probability of death is μh plus $o(h)$.
- Probability that an organism divides twice (or divides once and dies) in the interval of length h is $o(h)$.

Tacit assumptions:

Constants of proportionality do not depend on time: "stable environment".

Constants do not depend on organism: organisms are all similar and live in similar environments.

$Y(t)$: total population at time t.

$\mathcal{H}_t$: history of process up to time t.

Condition on event $Y(t) = n$.

Probability of two or more divisions (more than one division by a single organism or two or more organisms dividing) is $o(h)$.

Probability of both a division and a death or of two or more deaths is $o(h)$.

So probability of exactly 1 division by any one of the n organisms is $n\lambda h + o(h)$.

Similarly probability of 1 death is $n\mu h + o(h)$.

We deduce:

$$\begin{aligned}
P(Y(t+h) &= n+1|Y(t)=n,\mathcal{H}_t) \\
&= n\lambda h + o(h) \\
P(Y(t+h) &= n-1|Y(t)=n,\mathcal{H}_t) \\
&= n\mu h + o(h) \\
P(Y(t+h) &= n|Y(t)=n,\mathcal{H}_t) \\
&= 1 - n(\lambda+\mu)h + o(h) \\
P(Y(t+h) &\notin \{n-1,n,n+1\}|Y(t)=n,\mathcal{H}_t) \\
&= o(h)
\end{aligned}$$

These equations lead to:

$$\begin{aligned}
P(Y(t+s) = j|Y(s) = i,\mathcal{H}_s)& \\
&= P(Y(t+s)=j|Y(s)=i) \\
&= P(Y(t)=j|Y(0)=i)
\end{aligned}$$

This is the **Markov Property**.

Definition:A process $\{Y(t); t \geq 0\}$ taking values in S, a finite or countable state space is a Markov Chain if

$$\begin{aligned}
P(Y(t+s) = j|Y(s) = i,\mathcal{H}_s)& \\
&= P(Y(t+s)=j|Y(s)=i) \\
&\equiv \mathbf{P}_{ij}(s,s+t)
\end{aligned}$$

Definition:A Markov chain Y has **stationary transitions** if

$$\mathbf{P}_{ij}(s,s+t) = \mathbf{P}_{ij}(0,t) \equiv \mathbf{P}_{ij}(t)$$

From now on: our chains have stationary transitions.

Summary of Markov Process Results

- Chapman-Kolmogorov equations:

$$\mathbf{P}_{ik}(t+s) = \sum_j \mathbf{P}_{ij}(t)\mathbf{P}_{jk}(s)$$

- Exponential holding times: starting from state i time, T_i, until process leaves i has exponential distribution, rate denoted v_i.
- Sequence of states visited, $Y_0, Y_1, Y_2, \ldots$ is Markov chain – transition matrix has $\mathbf{P}_{ii} = 0$. Y sometimes called **skeleton**.
- **Communicating classes** defined for skeleton chain. Usually assume chain has 1 communicating class.
- Periodicity irrelevant because of continuity of exponential distribution.
- Instantaneous transition rates from i to j:

$$q_{ij} = v_i \mathbf{P}_{ij}$$

- Kolmogorov backward equations:

$$\mathbf{P}'_{ij}(t) = \sum_{k \neq i} q_{ik} \mathbf{P}_{kj}(t) - v_i \mathbf{P}_{ij}(t)$$

- Kolmogorov forward equations:

$$\mathbf{P}'_{ij}(t) = \sum_{k \neq j} q_{kj} \mathbf{P}_{ik}(t) - v_j \mathbf{P}_{ij}(t)$$

- For strongly recurrent chains with a single communicating class:

$$\mathbf{P}_{ij}(t) \to \pi_j$$

- Stationary initial probabilities π_i satisfy:

$$v_j \pi_j = \sum_{k \neq j} q_{kj} \pi_k$$

- Transition probabilities given by

$$\mathbf{P}(t) = e^{\mathbf{R}t}$$

  where $\mathbf{R}$ has entries

$$\mathbf{R}_{ij} = \begin{cases} q_{ij} & i \neq j \\ -v_i & i = j \end{cases}$$

- Process is a **Birth and Death** process if

$$\mathbf{P}_{ij} = 0 \text{ if } |i-j| > 1$$

In this case we write λ_i for the instantaneous "birth" rate:

$$P(X(t+h) = i+1|X_t = i) = \lambda_i h + o(h)$$

and μ_i for the instantaneous "death" rate:

$$P(X(t+h) = i-1|X_t = i) = \mu_i h + o(h)$$

We have

$$q_{ij} = \begin{cases} 0 & |i-j| > 1 \\ \lambda_i & j = i+1 \\ \mu_i & j = i-1 \end{cases}$$

- If all $\mu_i = 0$ then process is a **pure birth** process. If all $\lambda_i = 0$ a **pure death** process.

- Birth and Death process have stationary distribution

$$\pi_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}\right)}$$

Necessary condition for existence of π is

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} < \infty$$

- **Linear** birth and death processes have

$$\lambda_n = (n+1)\lambda \qquad \mu_n = n\mu$$

In this case (note *immigration*)

$$\pi_n = \frac{(\lambda/\mu)^n}{1 - (\lambda/\mu)}$$

provided $\lambda < \mu$.

Detailed development

Suppose X a Markov Chain with stationary transitions. Then

$$\begin{aligned}
&P(X(t+s)=k|X(0)=i)\\
&\quad=\sum_j P(X(t+s)=k,X(t)=j|X(0)=i)\\
&\quad=\sum_j P(X(t+s)=k|X(t)=j,X(0)=i)\\
&\qquad\quad\times P(X(t)=j|X(0)=i)\\
&\quad=\sum_j P(X(t+s)=k|X(t)=j)\\
&\qquad\quad\times P(X(t)=j|X(0)=i)\\
&\quad=\sum_j P(X(s)=k|X(0)=j)\\
&\qquad\quad\times P(X(t)=j|X(0)=i)
\end{aligned}$$

This shows

$$\mathbf{P}(t+s)=\mathbf{P}(t)\mathbf{P}(s)$$

which is the Chapman-Kolmogorov equation.

Now consider the chain starting from i and let T_i be the first t for which $X(t)\neq i$. Then T_i is a stopping time.

[Technically:

$$\{T_i\le t\}\in\mathcal{H}_t$$

for each t.] Then

$$\begin{aligned}
&P(T_i>t+s|T_i>s,X(0)=i)\\
&\quad=P(T_i>t+s|X(u)=i;0\le u\le s)\\
&\quad=P(T_i>t|X(0)=i)
\end{aligned}$$

by the Markov property.

Conclusion: given $X(0)=i$, T_i has memoryless property so T_i has an exponential distribution. Let v_i be the rate parameter.

Embedded Chain: Skeleton

Let $T_1 < T_2 < \cdots$ be the stopping times at which transitions occur. Then $X_n = X(T_n)$. The sequence X_n is a Markov chain by the Markov property. That $\mathbf{P}_{ii} = 0$ reflects the fact that $P(X(T_{n+1}) = X(T_n)) = 0$ by design.

As before we say $i \rightsquigarrow j$ if $\mathbf{P}_{ij}(t) > 0$ for some t. It is fairly clear that $i \rightsquigarrow j$ for the $X(t)$ if and only if $i \rightsquigarrow j$ for the embedded chain X_n.

We say $i \leftrightarrow j$ (i and j communicate) if $i \rightsquigarrow j$ and $j \rightsquigarrow i$.

Now consider

$$P(X(t+h) = j | X(t) = i, \mathcal{H}_t)$$

Suppose the chain has made n transitions so far so that $T_n < t < T_{n+1}$. Then the event $X(t+h) = j$ is, except for possibilities of probability $o(h)$ the event that

$$t < T_{n+1} \le t + h \text{ and } X_{n+1} = j$$

The probability of this is

$$(v_i h + o(h))\mathbf{P}_{ij} = v_i \mathbf{P}_{ij} h + o(h)$$

Kolmogorov's Equations

The Chapman-Kolmogorov equations are

$$\mathbf{P}(t+h) = \mathbf{P}(t)\mathbf{P}(h)$$

Subtract $\mathbf{P}(t)$ from both sides, divide by h and let $h \to 0$. Remember that $\mathbf{P}(0)$ is the identity. We find

$$\frac{\mathbf{P}(t+h) - \mathbf{P}(t)}{h} = \frac{\mathbf{P}(t)(\mathbf{P}(h) - \mathbf{P}(0))}{h}$$

which gives

$$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{P}'(0)$$

The Chapman-Kolmogorov equations can also be written

$$\mathbf{P}(t+h) = \mathbf{P}(h)\mathbf{P}(t)$$

Now subtracting $\mathbf{P}(t)$ from both sides, dividing by h and letting $h \to 0$ gives

$$\mathbf{P}'(t) = \mathbf{P}'(0)\mathbf{P}(t)$$

Look at these equations in component form:

$$\mathbf{P}'(t) = \mathbf{P}'(0)\mathbf{P}(t)$$

becomes

$$\mathbf{P}'_{ij}(t) = \sum_k \mathbf{P}'_{ik}(0)\mathbf{P}_{kj}(t)$$

For $i \neq k$ our calculations of instantaneous transition rates gives

$$\mathbf{P}'_{ik}(0) = v_i \mathbf{P}_{ik}$$

For $i = k$ we have

$$P(X(h) = i | X(0) = i) = e^{-v_i h} + o(h)$$

($X(h) = i$ either means $T_i > h$ which has probability $e^{-v_i h}$ or there have been two or more transitions in $[0, h]$, a possibility of probability $o(h)$.) Thus

$$\mathbf{P}'_{ii}(0) = -v_i$$

Let $\mathbf{R}$ be the matrix with entries

$$\mathbf{R}_{ij} = \begin{cases} q_{ij} \equiv v_i \mathbf{P}_{ij} & i \neq j \\ -v_i & i = j \end{cases}$$

That is

$$\mathbf{R} = \mathbf{P}'(0).$$

$\mathbf{R}$ is the **infinitesimal generator** of the chain.

Thus

$$\mathbf{P}'(t) = \mathbf{P}'(0)\mathbf{P}(t)$$

becomes

$$\begin{aligned} \mathbf{P}'_{ij}(t) &= \sum_k \mathbf{R}_{ik}\mathbf{P}_{kj}(t) \\ &= \sum_{k \neq i} q_{ik}\mathbf{P}_{kj}(t) - v_i \mathbf{P}_{ij}(t) \end{aligned}$$

Called **Kolmogorov's backward equations**.

On the other hand

$$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{P}'(0)$$

becomes

$$\begin{aligned}\mathbf{P}'_{ij}(t) &= \sum_k \mathbf{P}_{ik}(t)\mathbf{R}_{kj} \\ &= \sum_{k\neq j} q_{kj}\mathbf{P}_{ik}(t) - v_j\mathbf{P}_{ij}(t)\end{aligned}$$

These are **Kolmogorov's forward equations**.

Remark: When the state space is infinite the forward equations may not be justified. In deriving them we interchanged a limit with an infinite sum; the interchange is always justified for the backward equations but not for forward.

Example: $S = \{0,1\}$. Then

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the chain is otherwise specified by v_0 and v_1. The matrix $\mathbf{R}$ is

$$\mathbf{R} = \begin{bmatrix} -v_0 & v_0 \\ v_1 & -v_1 \end{bmatrix}$$

The backward equations become

$$\begin{aligned}\mathbf{P}'_{00}(t) &= v_0\mathbf{P}_{10}(t) - v_0\mathbf{P}_{00}(t) \\ \mathbf{P}'_{01}(t) &= v_0\mathbf{P}_{11}(t) - v_0\mathbf{P}_{01}(t) \\ \mathbf{P}'_{10}(t) &= v_1\mathbf{P}_{00}(t) - v_1\mathbf{P}_{10}(t) \\ \mathbf{P}'_{11}(t) &= v_1\mathbf{P}_{01}(t) - v_1\mathbf{P}_{11}(t)\end{aligned}$$

while the forward equations are

$$\begin{aligned}\mathbf{P}'_{00}(t) &= v_1\mathbf{P}_{01}(t) - v_0\mathbf{P}_{00}(t) \\ \mathbf{P}'_{01}(t) &= v_0\mathbf{P}_{00}(t) - v_1\mathbf{P}_{01}(t) \\ \mathbf{P}'_{10}(t) &= v_1\mathbf{P}_{11}(t) - v_0\mathbf{P}_{10}(t) \\ \mathbf{P}'_{11}(t) &= v_0\mathbf{P}_{10}(t) - v_1\mathbf{P}_{11}(t)\end{aligned}$$

Add $v_1\times$first plus $v_0\times$third backward equations to get

$$v_1\mathbf{P}'_{00}(t) + v_0\mathbf{P}'_{10}(t) = 0$$

so

$$v_1\mathbf{P}_{00}(t) + v_0\mathbf{P}_{10}(t) = c$$

Put $t = 0$ to get $c = v_1$. This gives

$$\mathbf{P}_{10}(t) = \frac{v_1}{v_0}\left\{1 - \mathbf{P}_{00}(t)\right\}$$

Plug this back in to the first equation and get

$$\mathbf{P}'_{00}(t) = v_1 - (v_1 + v_0)\mathbf{P}_{00}(t)$$

Multiply by $e^{(v_1+v_0)t}$ and get

$$\left\{e^{(v_1+v_0)t}\mathbf{P}_{00}(t)\right\}' = v_1 e^{(v_1+v_0)t}$$

which can be integrated to get

$$\mathbf{P}_{00}(t) = \frac{v_1}{v_0 + v_1} + \frac{v_0}{v_0 + v_1}e^{-(v_1+v_0)t}$$

Alternative calculation:

$$\mathbf{R} = \begin{bmatrix} -v_0 & v_0 \\ v_1 & -v_1 \end{bmatrix}$$

can be written as

$$\mathbf{M}\boldsymbol{\Lambda}\mathbf{M}^{-1}$$

where

$$\mathbf{M} = \begin{bmatrix} 1 & v_0 \\ & \\ 1 & -v_1 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} \frac{v_1}{v_0+v_1} & \frac{v_0}{v_0+v_1} \\ & \\ \frac{1}{v_0+v_1} & \frac{-1}{v_0+v_1} \end{bmatrix}$$

and

$$\boldsymbol{\Lambda} = \begin{bmatrix} 0 & 0 \\ & \\ 0 & -(v_0 + v_1) \end{bmatrix}$$

Then

$$
\begin{aligned}
e^{\mathbf{R}t} &= \sum_0^\infty \mathbf{R}^n t^n/n! \\
&= \sum_0^\infty \left(\mathbf{M}\mathbf{\Lambda}\mathbf{M}^{-1}\right)^n \frac{t^n}{n!} \\
&= \mathbf{M}\left(\sum_0^\infty \mathbf{\Lambda}^n \frac{t^n}{n!}\right)\mathbf{M}^{-1}
\end{aligned}
$$

Now

$$
\sum_0^\infty \mathbf{\Lambda}^n \frac{t^n}{n!} = \left[\begin{array}{cc} 1 & 0 \\ 0 & e^{-(v_0+v_1)t} \end{array}\right]
$$

so we get

$$
\begin{aligned}
\mathbf{P}(t) &= e^{\mathbf{R}t} \\
&= \mathbf{M}\left[\begin{array}{cc} 1 & 0 \\ 0 & e^{-(v_0+v_1)t} \end{array}\right]\mathbf{M}^{-1} \\
&= \mathbf{P}^\infty - \frac{e^{-(v_0+v_1)t}}{v_0+v_1}\mathbf{R}
\end{aligned}
$$

where

$$
\mathbf{P}^\infty = \left[\begin{array}{cc} \frac{v_1}{v_0+v_1} & \frac{v_0}{v_0+v_1} \\ \\ \frac{v_1}{v_0+v_1} & \frac{v_0}{v_0+v_1} \end{array}\right]
$$

Notice: rows of $\mathbf{P}^\infty$ are a stationary initial distribution. If rows are π then

$$
\mathbf{P}^\infty = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]\pi \equiv \mathbf{1}\pi
$$

so

$$
\pi\mathbf{P}^\infty = (\pi\mathbf{1})\pi = \pi
$$

Moreover

$$
\pi\mathbf{R} = \mathbf{0}
$$

Fact: $\pi_0 = v_1/(v_0+v_1)$ is long run fraction of time in state 0.

Fact:

$$
\frac{1}{T}\int_0^T f(X(t))dt \to \sum_j \pi_j f(j)
$$

Ergodic Theorem in continuous time.

Birth and Death Processes

Consider a population of $X(t)$ individuals. Suppose in next time interval $(t, t+h)$ probability of population increase of 1 (called a birth) is $\lambda_i h + o(h)$ and probability of decrease of 1 (death) is $\mu_i h + o(h)$.

Jargon: X is a birth and death process.

Special cases:

All $\mu_i = 0$; called a **pure birth** process.

All $\lambda_i = 0$ (0 is absorbing): **pure death** process.

$\lambda_n = n\lambda$ and $\mu_n = n\mu$ is a **linear** birth and death process.

$\lambda_n \equiv 1$, $\mu_n \equiv 0$: Poisson Process.

$\lambda_n = n\lambda + \theta$ and $\mu_n = n\mu$ is a **linear** birth and death process with immigration.

Queuing Theory

Ingredients of Queuing Problem:

1: Queue input process.

2: Number of servers

3: Queue discipline: first come first serve? last in first out? pre-emptive priorities?

4: Service time distribution.

Example:Imagine customers arriving at a facility at times of a Poisson Process N with rate λ. This is the input process, denoted M (for Markov) in queuing literature.

Single server case:

Service distribution: exponential service times, rate μ.

Queue discipline: first come first serve.

$X(t)$ = number of customers in line at time t.

X is a Markov process called $M/M/1$ queue:

$$v_i = \lambda + \mu$$

$$\mathbf{P}_{ij} = \begin{cases} \frac{\mu}{\mu+\lambda} & j = i-1 \geq 0 \\ \frac{\lambda}{\mu+\lambda} & j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Example:$M/M/\infty$ queue:

Customers arrive according to PP rate λ. Each customer begins service immediately. $X(t)$ is number being served at time t. X is a birth and death process with

$$v_n = \lambda + n\mu$$

and

$$\mathbf{P}_{ij} = \begin{cases} \frac{i\mu}{i\mu+\lambda} & j = i-1 \geq 0 \\ \frac{\lambda}{i\mu+\lambda} & j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Stationary Initial Distributions

We have seen that a stationary initial distribution π is a probability vector solving

$$\pi\mathbf{R} = \mathbf{0}$$

Rewrite this as

$$v_j \pi_j = \sum_{i \neq j} q_{ij} \pi_i$$

Interpretation: LHS is rate at which process leaves state j; process is in state j a fraction π_j of time and then makes transition at rate v_j. RHS is total rate of arrival in state j. For each state $i \neq j$ π_i is fraction of time spent in state i and then q_{ij} the instantaneous rate of transition from i to j.

So equation says:

Rate of departure from j balances rate of arrival to j. This is called balance.

Application to birth and death processes:

Equation is

$$(\lambda_j + \mu_j)\pi_j = \lambda_{j-1}\pi_{j-1} + \mu_{j+1}\pi_{j+1}$$

for $j \geq 1$ and

$$\lambda_0 \pi_0 = \mu_1 \pi_1$$

Notice that this permits the recursion

$$\begin{aligned} \pi_1 &= \frac{\lambda_0}{\mu_1}\pi_0 \\ \pi_2 &= \frac{\lambda_1 + \mu_1}{\mu_2}\pi_1 - \frac{\lambda_0}{\mu_2}\pi_0 \\ &= \frac{\lambda_0\lambda_1}{\mu_1\mu_2}\pi_0 \end{aligned}$$

which extends by induction to

$$\pi_k = \frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k} \pi_0$$

Apply $\sum \pi_k = 1$ to get

$$\pi_0 \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \right) = 1$$

This gives the formula announced:

$$\pi_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \right)}$$

If

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} < \infty$$

then we have defined a probability vector which solves

$$\pi \mathbf{R} = \mathbf{0}$$

Since

$$\mathbf{P}' = \mathbf{R}\mathbf{P}$$

we see that

$$\{\pi \mathbf{P}(t)\}' = 0$$

so that $\pi \mathbf{P}(t)$ is constant. Put $t = 0$ to discover that the constant is π.