Proof of **4)**: Fix s_i, h_i for $i = 1, \ldots, n$ such that

$$0 < s_1 < s_1 + h_1 < s_2 < \cdots < s_n < s_n + h_n < T$$

Given $N(T) = n$ we compute the probability of the event

$$A = \bigcap_{i=1}^{n} \{s_i < S - i < s_I + h_i\}$$

Intersection of A and $N(T) = n$ is ($s_0 = 0$):

$$B \equiv \bigcap_{i=1}^{n} \{N(s_{i-1}, s_i] = 0, N(s_i, s_i + h] = 1\} \cap \{N(s_n + h, T] = 0\}$$

whose probability is

$$\left(\prod \lambda h_i\right) e^{-\lambda T}$$

So

$$\begin{aligned} P(A|N(t) = n) &= \frac{P(A, N(T) = n)}{(N(T) = n)} \\ &= \frac{\lambda^n e^{-\lambda T} \prod h_i}{(\lambda T)^n e^{-\lambda T}/n!} \\ &= \frac{n! \prod h_i}{T^n} \end{aligned}$$

Divide by $\prod h_i$ and let all h_i go to 0 to get joint density of $S_1, \ldots, S_n$ is

$$\frac{n!}{T^n} 1(0 < s_1 < \cdots < s_n < T)$$

which is the density of order statistics from a Uniform$[0, T]$ sample of size n.

5) Replace the event $S_{n+1} = T$ with $T < S_{n+1} < T + h$. With A as before we want

$$P(A|T < S_{n+1} < T + h) = \frac{P(B, N(T, T + h] \geq 1)}{P(T < S_{n+1} < T + h)}$$

Note that B is independent of $\{N(T, T + h] \geq 1\}$ and that we have already found the limit

$$\frac{P(B)}{\prod h_i} \to \lambda^n e^{-\lambda T}$$

We are left to compute the limit of

$$\frac{P(N(T, T + h] \geq 1)}{P(T < S_{n+1} < T + h)}$$

The denominator is

$$\int_T^{T+h} f_{S_{n+1}}(u)du$$

so

$$\lim_{h\to 0} \frac{h}{P(T < S_{n+1} < T+h)} = \lambda(\lambda T)^n e^{-\lambda T}/n!$$

Finally

$$P(N(T, T+h] \geq 1) = \lambda h + 0(h)$$

Thus

$$\frac{P(N(T,T+h] \geq 1)}{P(T < S_{n+1} < T+h)} = \frac{\lambda h + o(h)}{\frac{(\lambda T)^n}{n!} e^{-\lambda T} \lambda h + o(h)}$$
$$\to \frac{n!}{(\lambda T)^n e^{-\lambda T}}$$

This gives the conditional density of $S_1, \ldots, S_n$ given $S_{n+1} = T$ as in **4)**.

Inhomogeneous Poisson Processes

The idea of hazard rate can be used to extend the notion of Poisson Process. Suppose $\lambda(t) \geq 0$ is a function of t. Suppose N is a counting process such that

$$P(N(t+h) = k+1 | N(t) = k, H_t) = \lambda(t)h + o(h)$$

and

$$P(N(t+h) \geq k+2 | N(t) = k, H_t) = o(h)$$

Then N has independent increments and $N(t+s) - N(t)$ has a Poisson distribution with mean

$$\int_t^{t+s} \lambda(u)du$$

If we put

$$\Lambda(t) = \int_0^t \lambda(u)du$$

then mean of $N(t+s) - N(T)$ is $\Lambda(t+s) - \Lambda(t)$.

Jargon: λ is the **intensity** or **instaneous intensity** and Λ the **cumulative intensity**.

Can use the model with Λ any non-decreasing right continuous function, possibly without a derivative. This allows ties.