Continuous Time Markov Chains

Consider a population of single celled organisms in a stable environment.

Fix short time interval, length h.

Each organism has some probability of dividing to produce two organisms and some other probability of dying.

We might suppose:

- Different organisms behave independently.
- Probability of division is λh plus $o(h)$.
- Probability of death is μh plus $o(h)$.
- Probability that an organism divides twice (or divides once and dies) in the interval of length h is $o(h)$.

Tacit assumptions:

Constants of proportionality do not depend on time: "stable environment".

Constants do not depend on organism: organisms are all similar and live in similar environments.

$Y(t)$: total population at time t.

$\mathcal{H}_t$: history of process up to time t.

Condition on event $Y(t) = n$.

Probability of two or more divisions (more than one division by a single organism or two or more organisms dividing) is $o(h)$.

Probability of both a division and a death or of two or more deaths is $o(h)$.

So probability of exactly 1 division by any one of the n organisms is $n\lambda h + o(h)$.

Similarly probability of 1 death is $n\mu + o(h)$.

We deduce:

$$\begin{aligned}
P(Y(t+h) &= n+1|Y(t)=n,\mathcal{H}_t) \\
&= n\lambda h + o(h) \\
P(Y(t+h) &= n-1|Y(t)=n,\mathcal{H}_t) \\
&= n\mu h + o(h) \\
P(Y(t+h) &= n|Y(t)=n,\mathcal{H}_t) \\
&= 1 - n(\lambda+\mu)h + o(h) \\
P(Y(t+h) &\notin \{n-1,n,n+1\}|Y(t)=n,\mathcal{H}_t) \\
&= o(h)
\end{aligned}$$

These equations lead to:

$$\begin{aligned}
P(Y(t+s) = j|Y(s) = i, \mathcal{H}_s)& \\
&= P(Y(t+s)=j|Y(s)=i) \\
&= P(Y(t)=j|Y(0)=i)
\end{aligned}$$

This is the **Markov Property**.

Definition:A process $\{X(t); t \geq 0\}$ taking values in S, a finite or countable state space is a Markov Chain if

$$\begin{aligned}
P(Y(t+s) = j|Y(s) = i, \mathcal{H}_s)& \\
&= P(Y(t+s)=j|Y(s)=i) \\
&\equiv \mathbf{P}_{ij}(s, s+t)
\end{aligned}$$

Definition:A Markov chain X has **stationary transitions** if

$$\mathbf{P}_{ij}(s, s+t) = \mathbf{P}_{ij}(0,t) \equiv \mathbf{P}_{ij}(t)$$

From now on: our chains have stationary transitions.