STAT 380 Lecture 19

End of Summary for Continuous Time Markov Chains:

- Process is a **Birth and Death** process if

$$\mathbf{P}_{ij} = 0 \text{ if } |i-j| > 1$$

  In this case we write λ_i for the instantaneous "birth" rate:

$$P(X(t+h) = i+1 | X_t = i) = \lambda_i h + o(h)$$

  and μ_i for the instantaneous "death" rate:

$$P(X(t+h) = i-1 | X_t = i) = \mu_i h + o(h)$$

  We have

$$q_{ij} = \begin{cases} 0 & |i-j| > 1 \\ \lambda_i & j = i+1 \\ \mu_i & j = i-1 \end{cases}$$

- If all $\mu_i = 0$ then process is a **pure birth** process. If all $\lambda_i = 0$ a **pure death** process.

- Birth and Death process have stationary distribution

$$\pi_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}\right)}$$

  Necessary condition for existence of π is

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} < \infty$$

- **Linear** birth and death processes have

$$\lambda_n = n\lambda \qquad \mu_n = n\mu$$

  In this case

$$\pi_n = \frac{(\lambda/\mu)^n}{1-(\lambda/\mu)}$$

  provided $\lambda < \mu$.

Detailed development

Suppose X a Markov Chain with stationary transitions. Then

$$\begin{aligned}
&P(X(t+s)=k|X(0)=i) \\
&\quad = \sum_j P(X(t+s)=k, X(t)=j|X(0)=i) \\
&\quad = \sum_j P(X(t+s)=k|X(t)=j, X(0)=i) \\
&\qquad \times P(X(t)=j|X(0)=i) \\
&\quad = \sum_j P(X(t+s)=k|X(t)=j) \\
&\qquad \times P(X(t)=j|X(0)=i) \\
&\quad = \sum_j P(X(s)=k|X(0)=j) \\
&\qquad \times P(X(t)=j|X(0)=i)
\end{aligned}$$

This shows

$$\mathbf{P}(t+s) = \mathbf{P}(t)\mathbf{P}(t)$$

which is the Chapman-Kolmogorov equation.

Now consider the chain starting from i and let T_i be the first t for which $X(t) \neq i$. Then T_i is a stopping time.

[Technically:

$$\{T_i \le t\} \in \mathcal{H}_t$$

for each t.] Then

$$\begin{aligned}
&P(T_i > t+s|T_i > s, X(0)=i) \\
&\quad = P(T_i > t+s|X(u)=i; 0 \le u \le s) \\
&\quad = P(T_i > t|X(0)=i)
\end{aligned}$$

by the Markov property.

Conclusion: given $X(0)=i$, T_i has memoryless porperty so T_i has an exponential distribution. Let v_i be the rate parameter.

Embedded Chain: Skeleton

Let $T_1 < T_2 < \cdots$ be the stopping times at which tranisitons occur. Then $X_n = X(T_n)$. The sequence X_n is a Markov chain by the Markov property. That $\mathbf{P}_{ii} = 0$ reflects the fact that $P(X(T_{n+1}) = X(T_n)) = 0$ by design.

As before we say $i \rightsquigarrow j$ if $\mathbf{P}_{ij}(t) > 0$ for some t. It is fairly clear that $i \rightsquigarrow j$ for the $X(t)$ if and only if $i \rightsquigarrow j$ for the embedded chain X_n.

We say $i \leftrightarrow j$ (i and j communicate) if $i \rightsquigarrow j$ and $j \rightsquigarrow i$.

Now consider

$$P(X(t+h) = j | X(t) = i, \mathcal{H}_t)$$

Suppose the chain has made n transitions so far so that $T_n < t < T_{n+1}$. Then the event $X(t+h) = j$ is, except for possibilities of probability $o(h)$ the event that

$$t < T_{n+1} \le t + h \text{ and } X_{n+1} = j$$

The probability of this is

$$(v_i h + o(h))\mathbf{P}_{ij} = v_i \mathbf{P}_{ij} + o(h)$$

Kolmogorov's Equations

The Chapman-Kolmogorov equations are

$$\mathbf{P}(t+h) = \mathbf{P}(t)\mathbf{P}(h)$$

Subtract $\mathbf{P}(t)$ from both sides, divide by h and let $h \to 0$. Remember that $\mathbf{P}(0)$ is the identity. We find

$$\frac{\mathbf{P}(t+h) - \mathbf{P}(t)}{h} = \frac{\mathbf{P}(t)(\mathbf{P}(h) - \mathbf{P}(0))}{h}$$

which gives

$$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{P}'(0)$$

The Chapman-Kolmogorov equations can also be written

$$\mathbf{P}(t+h) = \mathbf{P}(h)\mathbf{P}(t)$$

Now subtracting $\mathbf{P}(t)$ from both sides, dividing by h and letting $h \to 0$ gives

$$\mathbf{P}'(t) = \mathbf{P}'(0)\mathbf{P}(t)$$

Look at these equations in component form:

$$\mathbf{P}'(t) = \mathbf{P}'(0)\mathbf{P}(t)$$

becomes

$$\mathbf{P}'_{ij}(t) = \sum_k \mathbf{P}'_{ik}(0)\mathbf{P}_{kj}(t)$$

For $i \neq k$ our calculations of instantaneous transition rates gives

$$\mathbf{P}'_{ik}(0) = v_i \mathbf{P}_{ik}$$

For $i = k$ we have

$$P(X(h) = i | X(0) = i) = e^{-v_i h} + o(h)$$

($X(h) = i$ either means $T_i > h$ which has probability $e^{-v_i h}$ or there have been two or more transitions in $[0, h]$, a possibility of probability $o(h)$.) Thus

$$\mathbf{P}'_{ii}(0) = -v_i$$

Let $\mathbf{R}$ be the matrix with entries

$$\mathbf{R}_{ij} = \begin{cases} q_{ij} \equiv v_i \mathbf{P}_{ij} & i \neq j \\ -v_i & i = j \end{cases}$$

$\mathbf{R}$ is the **infinitesimal generator** of the chain.

Thus

$$\mathbf{P}'(t) = \mathbf{P}'(0)\mathbf{P}(t)$$

becomes

$$\begin{aligned} \mathbf{P}'_{ij}(t) &= \sum_k \mathbf{R}_{ik}\mathbf{P}_{kj}(t) \\ &= \sum_{k \neq i} q_{ik}\mathbf{P}_{kj}(t) - v_i \mathbf{P}_{ik}(t) \end{aligned}$$

Called **Kolmogorov's backward equations**.

On the other hand

$$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{P}'(0)$$

becomes

$$\begin{aligned} \mathbf{P}'_{ij}(t) &= \sum_k \mathbf{P}_{ik}(t)\mathbf{R}_{kj} \\ &= \sum_{k \neq j} q_{kj}\mathbf{P}_{ik}(t) - v_j \mathbf{P}_{ij}(t) \end{aligned}$$

These are **Kolmogorov's forward equations**.

Remark: When the state space is infinite the forward equations may not be justified. In deriving them we interchanged a limit with an infinite sum; the interchange is always justified for the backward equations but not for forward.

Example: $S = \{0, 1\}$. Then

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the chain is otherwise specivied by v_0 and v_1. The matrix $\mathbf{R}$ is

$$\mathbf{R} = \begin{bmatrix} -v_0 & v_0 \\ v_1 & -v_1 \end{bmatrix}$$

The backward equations become

$$\begin{aligned} \mathbf{P}'_{00}(t) &= v_0\mathbf{P}_{10}(t) - v_0\mathbf{P}_{00}(t) \\ \mathbf{P}'_{01}(t) &= v_0\mathbf{P}_{11}(t) - v_0\mathbf{P}_{01}(t) \\ \mathbf{P}'_{10}(t) &= v_1\mathbf{P}_{00}(t) - v_1\mathbf{P}_{10}(t) \\ \mathbf{P}'_{11}(t) &= v_1\mathbf{P}_{01}(t) - v_1\mathbf{P}_{11}(t) \end{aligned}$$

while the forward equations are

$$\begin{aligned} \mathbf{P}'_{00}(t) &= v_1\mathbf{P}_{01}(t) - v_0\mathbf{P}_{00}(t) \\ \mathbf{P}'_{01}(t) &= v_0\mathbf{P}_{00}(t) - v_1\mathbf{P}_{01}(t) \\ \mathbf{P}'_{10}(t) &= v_1\mathbf{P}_{11}(t) - v_0\mathbf{P}_{10}(t) \\ \mathbf{P}'_{11}(t) &= v_0\mathbf{P}_{10}(t) - v_1\mathbf{P}_{11}(t) \end{aligned}$$

Add first and third backward equations to get

$$v_1\mathbf{P}'_{00}(t) + v_0\mathbf{P}'_{10}(t)$$

so

$$v_1\mathbf{P}_{00}(t) + v_0\mathbf{P}_{10}(t) = c$$

Put $t = 0$ to get $c = v_1 + v_0$. This gives

$$\mathbf{P}_{10}(t) = \frac{v_1}{v_0}\{1 - \mathbf{P}_{00}(t)\}$$

Plug this back in to the first equation and get

$$\mathbf{P}'_{00}(t) = v_1 - (v_1 + v_0)\mathbf{P}_{00}(t)$$

Multiply by $e^{(v_1+v_0)t}$ and get

$$\left\{e^{(v_1+v_0)t}\mathbf{P}_{00}(t)\right\}' = v_1 e^{(v_1+v_0)t}$$

which can be integrated to get

$$\mathbf{P}_{00}(t) = \frac{v_1}{v_0 + v_1} + \frac{v_0}{v_0 + v_1} e^{-(v_1+v_0)t}$$

Alternative calculation:

$$\mathbf{R} = \begin{bmatrix} -v_0 & v_0 \\ v_1 & -v_1 \end{bmatrix}$$

can be written as

$$\mathbf{M}\Lambda\mathbf{M}^{-1}$$

where

$$\mathbf{M} = \begin{bmatrix} 1 & v_0 \\ \\ 1 & -v_1 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} \frac{v_1}{v_0+v_1} & \frac{v_0}{v_0+v_1} \\ \\ \frac{1}{v_0+v_1} & \frac{-1}{v_0+v_1} \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} 0 & 0 \\ \\ 0 & -(v_0 + v_1) \end{bmatrix}$$

Then

$$\begin{aligned} e^{\mathbf{R}t} &= \sum_0^\infty \mathbf{R}^n t^n / n! \\ &= \sum_0^\infty \left(\mathbf{M}\Lambda\mathbf{M}^{-1}\right)^n \frac{t^n}{n!} \\ &= \mathbf{M}\left(\sum_0^\infty \Lambda^n \frac{t^n}{n!}\right)\mathbf{M}^{-1} \end{aligned}$$

Now

$$\sum_0^\infty \Lambda^n \frac{t^n}{n!} = \left[\begin{array}{cc} 1 & 0 \\ 0 & e^{-(v_0+v_1)t} \end{array}\right]$$

so we get

$$\begin{aligned} \mathbf{P}(t) &= e^{\mathbf{R}t} \\ &= \mathbf{M}\left[\begin{array}{cc} 1 & 0 \\ 0 & e^{-(v_0+v_1)t} \end{array}\right]\mathbf{M}^{-1} \\ &= \mathbf{P}^\infty - \frac{e^{-(v_0+v_1)t}}{v_0+v_1}\mathbf{R} \end{aligned}$$

where

$$\mathbf{P}^\infty = \left[\begin{array}{cc} \frac{v_1}{v_0+v_1} & \frac{v_0}{v_0+v_1} \\ \\ \frac{v_1}{v_0+v_1} & \frac{v_0}{v_0+v_1} \end{array}\right]$$

Notice: rows of $\mathbf{P}^\infty$ are a stationary initial distribution. If rows are π then

$$\mathbf{P}^\infty = \left[\begin{array}{c} 1 \\ 1 \end{array}\right]\pi \equiv \mathbf{1}\pi$$

so

$$\pi\mathbf{P}^\infty = (\pi\mathbf{1})\pi = \pi$$

Moreover

$$\pi\mathbf{R} = \mathbf{0}$$

Fact: $\pi_0 = v_1/(v_0+v_1)$ is long run fraction of time in state 0.

Fact:

$$\frac{1}{T}\int_0^T f(X(t))dt \to \sum_j \pi_j f(j)$$

Ergodic Theorem in continuous time.

Birth and Death Processes

Consider a population of $X(t)$ individuals. Suppose in next time interval $(t, t+h)$ probability of population increase of 1 (called a birth) is $\lambda_i h + o(h)$ and probability of decrease of 1 (death) is $\mu_i h + o(h)$.

Jargon: X is a birth and death process.

Special cases:

All $\mu_i = 0$; called a **pure birth** process.

All $\lambda_i = 0$ (0 is absorbing): **pure death** process.

$\lambda_n = n\lambda$ and $\mu_n = n\mu$ is a **linear** birth and death process.

$\lambda_n \equiv 1$, $\mu_n \equiv 0$: Poisson Process.

$\lambda_n = n\lambda + \theta$ and $\mu_n = n\mu$ is a **linear** birth and death process with immigration.

Queuing Theory

Ingredients of Queuing Problem:

1: Queue input process.

2: Number of servers

3: Queue discipline: first come first serve? last in first out? pre-emptive priorities?

4: Service time distribution.

Example: Imagine customers arriving at a facility at times of a Poisson Process N with rate λ. This is the input process, denoted M (for Markov) in queuing literature.

Single server case:

Service distribution: exponential service times, rate μ.

Queue discipline: first come first serve.

$X(t)$ = number of customers in line at time t.

X is a Markov process called $M/M/1$ queue:

$$v_i = \lambda + \mu$$

$$\mathbf{P}_{ij} = \begin{cases} \frac{\mu}{\mu+\lambda} & j = i-1 \geq 0 \\ \frac{\lambda}{\mu+\lambda} & j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Example: $M/M/\infty$ queue:

Customers arrive according to PP rate λ. Each customer begins service immediately. $X(t)$ is number being served at time t. X is a birth and death process with

$$v_n = \lambda + n\mu$$

and

$$\mathbf{P}_{ij} = \begin{cases} \frac{i\mu}{i\mu+\lambda} & j = i-1 \geq 0 \\ \frac{\lambda}{i\mu+\lambda} & j = i+1 \\ 0 & \text{otherwise} \end{cases}$$