Example: $S = \{0, 1\}$. Then

$$\mathbf{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the chain is otherwise specivied by v_0 and v_1. The matrix $\mathbf{R}$ is

$$\mathbf{R} = \begin{bmatrix} -v_0 & v_0 \\ v_1 & -v_1 \end{bmatrix}$$

The backward equations become

$$\begin{aligned}
\mathbf{P}'_{00}(t) &= v_0\mathbf{P}_{10}(t) - v_0\mathbf{P}_{00}(t) \\
\mathbf{P}'_{01}(t) &= v_0\mathbf{P}_{11}(t) - v_0\mathbf{P}_{01}(t) \\
\mathbf{P}'_{10}(t) &= v_1\mathbf{P}_{00}(t) - v_1\mathbf{P}_{10}(t) \\
\mathbf{P}'_{11}(t) &= v_1\mathbf{P}_{01}(t) - v_1\mathbf{P}_{11}(t)
\end{aligned}$$

while the forward equations are

$$\begin{aligned}
\mathbf{P}'_{00}(t) &= v_1\mathbf{P}_{01}(t) - v_0\mathbf{P}_{00}(t) \\
\mathbf{P}'_{01}(t) &= v_0\mathbf{P}_{00}(t) - v_1\mathbf{P}_{01}(t) \\
\mathbf{P}'_{10}(t) &= v_1\mathbf{P}_{11}(t) - v_0\mathbf{P}_{10}(t) \\
\mathbf{P}'_{11}(t) &= v_0\mathbf{P}_{10}(t) - v_1\mathbf{P}_{11}(t)
\end{aligned}$$

Add $v_1\times$first plus $v_0\times$third backward equations to get

$$v_1\mathbf{P}'_{00}(t) + v_0\mathbf{P}'_{10}(t) = 0$$

so

$$v_1\mathbf{P}_{00}(t) + v_0\mathbf{P}_{10}(t) = c$$

Put $t = 0$ to get $c = v_1$. This gives

$$\mathbf{P}_{10}(t) = \frac{v_1}{v_0}\{1 - \mathbf{P}_{00}(t)\}$$

Plug this back in to the first equation and get

$$\mathbf{P}'_{00}(t) = v_1 - (v_1 + v_0)\mathbf{P}_{00}(t)$$

Multiply by $e^{(v_1+v_0)t}$ and get

$$\left\{e^{(v_1+v_0)t}\mathbf{P}_{00}(t)\right\}' = v_1 e^{(v_1+v_0)t}$$

which can be integrated to get

$$\mathbf{P}_{00}(t) = \frac{v_1}{v_0+v_1} + \frac{v_0}{v_0+v_1}e^{-(v_1+v_0)t}$$

Alternative calculation:

$$\mathbf{R} = \begin{bmatrix} -v_0 & v_0 \\ v_1 & -v_1 \end{bmatrix}$$

can be written as

$$\mathbf{M}\Lambda\mathbf{M}^{-1}$$

where

$$\mathbf{M} = \begin{bmatrix} 1 & v_0 \\ \\ 1 & -v_1 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} \frac{v_1}{v_0+v_1} & \frac{v_0}{v_0+v_1} \\ \\ \frac{1}{v_0+v_1} & \frac{-1}{v_0+v_1} \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} 0 & 0 \\ \\ 0 & -(v_0+v_1) \end{bmatrix}$$

Then

$$\begin{aligned} e^{\mathbf{R}t} &= \sum_0^\infty \mathbf{R}^n t^n/n! \\ &= \sum_0^\infty \left(\mathbf{M}\Lambda\mathbf{M}^{-1}\right)^n \frac{t^n}{n!} \\ &= \mathbf{M}\left(\sum_0^\infty \Lambda^n \frac{t^n}{n!}\right)\mathbf{M}^{-1} \end{aligned}$$

Now

$$\sum_0^\infty \Lambda^n \frac{t^n}{n!} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(v_0+v_1)t} \end{bmatrix}$$

so we get

$$\begin{aligned}
\mathbf{P}(t) &= e^{\mathbf{R}t} \\
&= \mathbf{M}\begin{bmatrix} 1 & 0 \\ 0 & e^{-(v_0+v_1)t} \end{bmatrix}\mathbf{M}^{-1} \\
&= \mathbf{P}^\infty - \frac{e^{-(v_0+v_1)t}}{v_0+v_1}\mathbf{R}
\end{aligned}$$

where

$$\mathbf{P}^\infty = \begin{bmatrix} \frac{v_1}{v_0+v_1} & \frac{v_0}{v_0+v_1} \\ \\ \frac{v_1}{v_0+v_1} & \frac{v_0}{v_0+v_1} \end{bmatrix}$$

Notice: rows of $\mathbf{P}^\infty$ are a stationary initial distribution. If rows are π then

$$\mathbf{P}^\infty = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\pi \equiv \mathbf{1}\pi$$

so

$$\pi\mathbf{P}^\infty = (\pi\mathbf{1})\pi = \pi$$

Moreover

$$\pi\mathbf{R} = \mathbf{0}$$

Fact: $\pi_0 = v_1/(v_0+v_1)$ is long run fraction of time in state 0.

Fact:

$$\frac{1}{T}\int_0^T f(X(t))dt \to \sum_j \pi_j f(j)$$

Ergodic Theorem in continuous time.

Birth and Death Processes

Consider a population of $X(t)$ individuals. Suppose in next time interval $(t, t+h)$ probability of population increase of 1 (called a birth) is $\lambda_i h + o(h)$ and probability of decrease of 1 (death) is $\mu_i h + o(h)$.

Jargon: X is a birth and death process.

Special cases:

All $\mu_i = 0$; called a **pure birth** process.

All $\lambda_i = 0$ (0 is absorbing): **pure death** process.

$\lambda_n = n\lambda$ and $\mu_n = n\mu$ is a **linear** birth and death process.

$\lambda_n \equiv 1$, $\mu_n \equiv 0$: Poisson Process.

$\lambda_n = n\lambda + \theta$ and $\mu_n = n\mu$ is a **linear** birth and death process with immigration.

Queuing Theory

Ingredients of Queuing Problem:

1: Queue input process.

2: Number of servers

3: Queue discipline: first come first serve? last in first out? pre-emptive priorities?

4: Service time distribution.

Example: Imagine customers arriving at a facility at times of a Poisson Process N with rate λ. This is the input process, denoted M (for Markov) in queuing literature.

Single server case:

Service distribution: exponential service times, rate μ.

Queue discipline: first come first serve.

$X(t)$ = number of customers in line at time t.

X is a Markov process called $M/M/1$ queue:

$$v_i = \lambda + \mu$$

$$\mathbf{P}_{ij} = \begin{cases} \frac{\mu}{\mu+\lambda} & j = i-1 \geq 0 \\ \frac{\lambda}{\mu+\lambda} & j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Example: $M/M/\infty$ queue:

Customers arrive according to PP rate λ. Each customer begins service immediately. $X(t)$ is number being served at time t. X is a birth and death process with

$$v_n = \lambda + n\mu$$

and

$$\mathbf{P}_{ij} = \begin{cases} \frac{i\mu}{i\mu+\lambda} & j = i-1 \geq 0 \\ \frac{\lambda}{i\mu+\lambda} & j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Stationary Initial Distributions

We have seen that a stationary initial distribution π is a probability vector solving

$$\pi \mathbf{R} = \mathbf{0}$$

Rewrite this as

$$v_j \pi_j = \sum_{i \neq j} q_{ij} \pi_i$$

Interpretation: LHS is rate at which process leaves state j; process is in state j a fraction π_j of time and then makes transition at rate v_j. RHS is total rate of arrival in state j. For each state $i \neq j$ π_i is fraction of time spent in state i and then q_{ij} the instantaneous rate of transition from i to j.

So equation says:

Rate of departure from j balances rate of arrival to j. This is called balance.

Application to birth and death processes:

Equation is

$$(\lambda_j + \mu_j)\pi_j = \lambda_{j-1}\pi_{j-1} + \mu_{j+1}\pi_{j+1}$$

for $j \geq 1$ and

$$\lambda_0 \pi_0 = \mu_1 \pi_1$$

Notice that this permits the recursion

$$\begin{aligned}
\pi_1 &= \frac{\lambda_0}{\mu_1}\pi_0 \\
\pi_2 &= \frac{\lambda_1 + \mu_1}{\mu_2}\pi_1 - \frac{\lambda_0}{\mu_2}\pi_0 \\
&= \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2}\pi_0
\end{aligned}$$

which extends by induction to

$$\pi_k = \frac{\lambda_0 \cdots \lambda_{k-1}}{\mu_1 \cdots \mu_k}\pi_0$$

Apply $\sum \pi_k = 1$ to get

$$\pi_0 \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \right) = 1$$

This gives the formula announced:

$$\pi_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n \left(1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} \right)}$$

If

$$\sum_{n=1}^{\infty} \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} < \infty$$

then we have defined a probability vector which solves

$$\pi \mathbf{R} = \mathbf{0}$$

Since

$$\mathbf{P}' = \mathbf{R}\mathbf{P}$$

we see that

$$\{\pi \mathbf{P}(t)\}' = 0$$

so that $\pi \mathbf{P}(t)$ is constant. Put $t = 0$ to discover that the constant is π.