Brownian Motion

For fair random walk Y_n = number of heads minus number of tails,

$$Y_n = U_1 + \cdots + U_n$$

where the U_i are independent and

$$P(U_i = 1) = P(U_i = -1) = \frac{1}{2}$$

Notice:

$$\mathrm{E}(U_i) = 0$$
$$\mathrm{Var}(U_i) = 1$$

Recall central limit theorem:

$$\frac{U_1 + \cdots + U_n}{\sqrt{n}} \Rightarrow N(0,1)$$

Now: rescale time axis so that n steps take 1 time unit and vertical axis so step size is $1/\sqrt{n}$.

We now turn these pictures into a stochastic process:

For $\frac{k}{n} \le t < \frac{k+1}{n}$ we define

$$X_n(t) = \frac{U_1 + \cdots + U_k}{\sqrt{n}}$$

Notice:

$$\mathrm{E}(X_n(t)) = 0$$

and

$$\mathrm{Var}(X_n(t)) = \frac{k}{n}$$

As $n \to \infty$ with t fixed we see $k/n \to t$. Moreover:

$$\frac{U_1 + \cdots + U_k}{\sqrt{k}} = \sqrt{\frac{n}{k}} X_n(t)$$

converges to $N(0,1)$ by the central limit theorem. Thus

$$X_n(t) \Rightarrow N(0,t)$$

Another observation: $X_n(t+s) - X_n(t)$ is independent of $X_n(t)$ because the two rvs involve sums of different U_i.

Conclusions.

As $n \to \infty$ the processes X_n converge to a process X with the properties:

1. $X(t)$ has a $N(0,t)$ distribution.
2. X has independent increments: if

$$0 = t_0 < t_1 < t_2 < \cdots < t_k$$

then

$$X(t_1) - X(t_0), \ldots , X(t_k) - X(t_{k-1})$$

are independent .
3. The increments are **stationary**:

$$X(t+s) - X(s) \sim N(0,t)$$

regardless of s.
4. $X(0) = 0$.

Def'n: Any process satisfying 1-4 above is a Brownian motion.

Properties of Brownian motion

- Suppose $t > s$. Then

$$\begin{aligned} \mathrm{E}(X(t)|X(s)) &= \mathrm{E}\left\{X(t) - X(s) + X(s)|X(s)\right\} \\ &= \mathrm{E}\left\{X(t) - X(s)|X(s)\right\} \\ &\quad + \mathrm{E}\left\{X(s)|X(s)\right\} \\ &= 0 + X(s) = X(s) \end{aligned}$$

Notice the use of independent increments and of $\mathrm{E}(Y|Y) = Y$.

- Again if $t > s$:

$$\begin{aligned} &\mathrm{Var}\left\{X(t)|X(s)\right\} \\ &\quad = \mathrm{Var}\left\{X(t) - X(s) + X(s)|X(s)\right\} \\ &\quad = \mathrm{Var}\left\{X(t) - X(s)|X(s)\right\} \\ &\quad = \mathrm{Var}\left\{X(t) - X(s)\right\} \\ &\quad = t - s \end{aligned}$$

Suppose $t < s$. Then $X(s) = X(t) + \{X(t) - X(s)\}$ is a sum of two independent normal variables. Do following calculation:

$X \sim N(0, \sigma^2)$, and $Y \sim N(0, \tau^2)$ independent. $Z = X + Y$.

Compute conditional distribution of X given Z:

$$\begin{aligned} f_{X|Z}(x|z) &= \frac{f_{X,Z}(x,z)}{f_Z(z)} \\ &= \frac{f_{X,Y}(x, z-x)}{f_Z(z)} \\ &= \frac{f_X(x) f_Y(z-x)}{f_Z(z)} \end{aligned}$$

Now Z is $N(0, \gamma^2)$ where $\gamma^2 = \sigma^2 + \tau^2$ so

$$\begin{aligned} f_{X|Z}(x|z) &= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)} \frac{1}{\tau\sqrt{2\pi}} e^{-(z-x)^2/(2\tau^2)}}{\frac{1}{\gamma\sqrt{2\pi}} e^{-z^2/(2\gamma^2)}} \\ &= \frac{\gamma}{\tau\sigma\sqrt{2\pi}} \exp\{-(x-a)^2/(2b^2)\} \end{aligned}$$

for suitable choices of a and b. To find them compare coefficients of x^2, x and 1.

Coefficient of x^2:

$$\frac{1}{b^2} = \frac{1}{\sigma^2} + \frac{1}{\tau^2}$$

so $b = \tau\sigma/\gamma$.

Coefficient of x:

$$\frac{a}{b^2} = \frac{z}{\tau^2}$$

so that

$$a = b^2 z/\tau^2 = \frac{\sigma^2}{\sigma^2 + \tau^2} z$$

Finally you should check that

$$\frac{a^2}{b^2} = \frac{z^2}{\tau^2} - \frac{z^2}{\gamma^2}$$

to make sure the coefficients of 1 work out as well.

Conclusion: given $Z = z$ the conditional distribution of X is $N(a, b^2)$ with a and b as above.

Application to Brownian motion:

- For $t < s$ let X be $X(t)$ and Y be $X(s) - X(t)$ so $Z = X + Y = X(s)$. Then $\sigma^2 = t$, $\tau^2 = s - t$ and $\gamma^2 = s$. Thus

  $$b^2 = \frac{(s-t)t}{s}$$

  and

  $$a = \frac{t}{s} X(s)$$

  SO:

  $$\mathrm{E}(X(t)|X(s)) = \frac{t}{s} X(s)$$

  and

  $$\mathrm{Var}(X(t)|X(s)) = \frac{(s-t)t}{s}$$