STAT 380 Lecture 22

Properties of Brownian motion

- Suppose $t > s$. Then

$$\begin{aligned}
\mathrm{E}(X(t)|X(s)) &= \mathrm{E}\left\{X(t) - X(s) + X(s)|X(s)\right\} \\
&= \mathrm{E}\left\{X(t) - X(s)|X(s)\right\} \\
&\quad + \mathrm{E}\left\{X(s)|X(s)\right\} \\
&= 0 + X(s) = X(s)
\end{aligned}$$

Notice the use of independent increments and of $\mathrm{E}(Y|Y) = Y$.

- Again if $t > s$:

$$\begin{aligned}
&\mathrm{Var}\left\{X(t)|X(s)\right\} \\
&\quad = \mathrm{Var}\left\{X(t) - X(s) + X(s)|X(s)\right\} \\
&\quad = \mathrm{Var}\left\{X(t) - X(s)|X(s)\right\} \\
&\quad = \mathrm{Var}\left\{X(t) - X(s)\right\} \\
&\quad = t - s
\end{aligned}$$

Suppose $t < s$. Then $X(s) = X(t) + \{X(t) - X(s)\}$ is a sum of two independent normal variables. Do following calculation:

$X \sim N(0, \sigma^2)$, and $Y \sim N(0, \tau^2)$ independent. $Z = X + Y$.

Compute conditional distribution of X given Z:

$$\begin{aligned}
f_{X|Z}(x|z) &= \frac{f_{X,Z}(x,z)}{f_Z(z)} \\
&= \frac{f_{X,Y}(x, z-x)}{f_Z(z)} \\
&= \frac{f_X(x) f_Y(z-x)}{f_Z(z)}
\end{aligned}$$

Now Z is $N(0, \gamma^2)$ where $\gamma^2 = \sigma^2 + \tau^2$ so

$$\begin{aligned}
f_{X|Z}(x|z) &= \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)} \frac{1}{\tau\sqrt{2\pi}} e^{-(z-x)^2/(2\tau^2)}}{\frac{1}{\gamma\sqrt{2\pi}} e^{-z^2/(2\gamma^2)}} \\
&= \frac{\gamma}{\tau\sigma\sqrt{2\pi}} \exp\{-(x-a)^2/(2b^2)\}
\end{aligned}$$

for suitable choices of a and b. To find them compare coefficients of x^2, x and 1.

Coefficient of x^2:

$$\frac{1}{b^2} = \frac{1}{\sigma^2} + \frac{1}{\tau^2}$$

so $b = \tau\sigma/\gamma$.

Coefficient of x:

$$\frac{a}{b^2} = \frac{z}{\tau^2}$$

so that

$$a = b^2 z/\tau^2 = \frac{\sigma^2}{\sigma^2 + \tau^2} z$$

Finally you should check that

$$\frac{a^2}{b^2} = \frac{z^2}{\tau^2} - \frac{z^2}{\gamma^2}$$

to make sure the coefficients of 1 work out as well.

Conclusion: given $Z = z$ the conditional distribution of X is $N(a, b^2)$ with a and b as above.

Application to Brownian motion:

- For $t < s$ let X be $X(t)$ and Y be $X(s) - X(t)$ so $Z = X + Y = X(s)$. Then $\sigma^2 = t$, $\tau^2 = s - t$ and $\gamma^2 = s$. Thus

  $$b^2 = \frac{(s-t)t}{s}$$

  and

  $$a = \frac{t}{s} X(s)$$

  SO:

  $$\mathrm{E}(X(t)|X(s)) = \frac{t}{s} X(s)$$

  and

  $$\mathrm{Var}(X(t)|X(s)) = \frac{(s-t)t}{s}$$

The Reflection Principle

Tossing a fair coin:

HTHHHTHTHHTHHHTTHTH	5 more heads than tails
THTTTHTHTTHTTTHHTHT	5 more tails than heads

Both sequences have the same probability.

So: for random walk starting at stopping time:

Any sequence with k more heads than tails in next m tosses is matched to sequence with k more tails than heads. Both sequences have same prob.

Suppose Y_n is a fair ($p = 1/2$) random walk. Define

$$M_n = \max\{Y_k, 0 \le k \le n\}$$

Compute $P(M_n \ge x)$? Trick: Compute

$$P(M_n \ge x, Y_n = y)$$

First: if $y \ge x$ then

$$\{M_n \ge x, Y_n = y\} = \{Y_n = y\}$$

Second: if $M_n \ge x$ then

$$T \equiv \min\{k : Y_k = x\} \le n$$

Fix $y < x$. Consider a sequence of H's and T's which leads to say $T = k$ and $Y_n = y$.

Switch the results of tosses $k+1$ to n to get a sequence of H's and T's which has $T = k$ and $Y_n = x + (x - y) = 2x - y > x$. This proves

$$P(T = k, Y_n = y) = P(T = k, Y_n = 2x - y)$$

This is true for each k so

$$\begin{aligned} P(M_n \ge x, Y_n = y) &= P(M_n \ge x, Y_n = 2x - y) \\ &= P(Y_n = 2x - y) \end{aligned}$$

Finally, sum over all y to get

$$\begin{aligned} P(M_n \ge x) = &\sum_{y \ge x} P(Y_n = y) \\ &+ \sum_{y < x} P(Y_n = 2x - y) \end{aligned}$$

Make the substitution $k = 2x - y$ in the second sum to get

$$\begin{aligned} P(M_n \geq x) &= \sum_{y \geq x} P(Y_n = y) \\ &\quad + \sum_{k > x} P(Y_n = k) \\ &= 2 \sum_{k > x} P(Y_n = k) + P(Y_n = x) \end{aligned}$$

Brownian motion version:

$$M_t = \max\{X(s); 0 \leq s \leq t\}$$

$$T_x = \min\{s : X(s) = x\}$$

(called hitting time for level x). Then

$$\{T_x \leq t\} = \{M_t \geq x\}$$

Any path with $T_x = s < t$ and $X(t) = y < x$ is matched to an equally likely path with $T_x = s < t$ and $X(t) = 2x - y > x$.

So for $y > x$

$$P(M_t \geq x, X(t) > y) = P(X(t) > y)$$

while for $y < x$

$$P(M_t \geq x, X(t) < y) = P(X(t) > 2x - y)$$

Let $y \to x$ to get

$$\begin{aligned} P(M_t \geq x, X(t) > x) &= P(M_t \geq x, X(t) < x) \\ &= P(X(t) > x) \end{aligned}$$

Adding these together gives

$$\begin{aligned} P(M_t > x) &= 2P(X(t) > x) \\ &= 2P(N(0,1) > x/\sqrt{t}) \end{aligned}$$

Hence M_t has the distribution of $|N(0,t)|$.

On the other hand in view of

$$\{T_x \le t\} = \{M_t \ge x\}$$

the density of T_x is

$$\frac{d}{dt} 2P(N(0,1) > x/\sqrt{t})$$

Use the chain rule to compute this. First

$$\frac{d}{dy} P(N(0,1) > y) = -\phi(y)$$

where ϕ is the standard normal density

$$\phi(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

because $P(N(0,1) > y)$ is 1 minus the standard normal cdf.

So

$$\begin{aligned}
&\frac{d}{dt} 2P(N(0,1) > x/\sqrt{t}) \\
&\qquad = -2\phi(x/\sqrt{t}) \frac{d}{dt}(x/\sqrt{t}) \\
&\qquad = \frac{x}{\sqrt{2\pi} t^{3/2}} \exp\{-x^2/(2t)\}
\end{aligned}$$

This density is called the **Inverse Gaussian** density. T_x is called a **first passage time**

NOTE: the preceding is a density when viewed as a function of the variable t.

Martingales

A stochastic process $M(t)$ indexed by either a discrete or continuous time parameter t is a **martingale** if:

$$\mathrm{E}\{M(t) | M(u); 0 \le u \le s\} = M(s)$$

whenever $s < t$.

Examples

- A fair random walk is a martingale.
- If $N(t)$ is a Poisson Process with rate λ then $N(t) - \lambda t$ is a martingale.
- Standard Brownian motion (defined above) is a martingale.

Note: Brownian motion with drift is a process of the form

$$X(t) = \sigma B(t) + \mu t$$

where B is **standard** Brownian motion, introduced earlier. X is a martingale if $\mu = 0$. We call μ the **drift**

- If $X(t)$ is a Brownian motion with drift then

  $$Y(t) = e^{X(t)}$$

  is a geometric Brownian motion. For suitable μ and σ we can make $Y(t)$ a martingale.
- If a gambler makes a sequence of fair bets and M_n is the amount of money s/he has after n bets then M_n is a martingale – even if the bets made depend on the outcomes of previous bets, that is, even if the gambler plays a strategy.

Some evidence for some of the above:
Random walk: $U_1, U_2, \ldots$ iid with

$$P(U_i = 1) = P(U_i = -1) = 1/2$$

and $Y_k = U_1 + \cdots + U_k$ with $Y_0 = 0$. Then

$$\begin{aligned}
&\mathrm{E}(Y_n|Y_0, \ldots, Y_k) \\
&\qquad = \mathrm{E}(Y_n - Y_k + Y_k|Y_0, \ldots, Y_k) \\
&\qquad = \mathrm{E}(Y_n - Y_k|Y_0, \ldots, Y_k) + Y_k \\
&\qquad = \sum_{k+1}^{n} \mathrm{E}(U_j|U_1, \ldots, U_k) + Y_k \\
&\qquad = \sum_{k+1}^{n} \mathrm{E}(U_j) + Y_k \\
&\qquad = Y_k
\end{aligned}$$

Things to notice:

Y_k treated as constant given $Y_1, \ldots, Y_k$.

Knowing $Y_1, \ldots, Y_k$ is equivalent to knowing $U_1, \ldots, U_k$.

For $j > k$ we have U_j independent of $U_1, \ldots, U_k$ so conditional expectation is unconditional expectation.

Since Standard Brownian Motion is limit of such random walks we get martingale property for standard Brownian motion.

Poisson Process: $X(t) = N(t) - \lambda t$. Fix $t > s$.

$$
\begin{aligned}
&\mathrm{E}(X(t)|X(u); 0 \le u \le s) \\
&\qquad = \mathrm{E}(X(t) - X(s) + X(s)|\mathcal{H}_s) \\
&\qquad = \mathrm{E}(X(t) - X(s)|\mathcal{H}_s) + X(s) \\
&\qquad = \mathrm{E}(N(t) - N(s) - \lambda(t-s)|\mathcal{H}_s) + X(s) \\
&\qquad = \mathrm{E}(N(t) - N(s)) - \lambda(t-s) + X(s) \\
&\qquad = \lambda(t-s) - \lambda(t-s) + X(s) \\
&\qquad = X(s)
\end{aligned}
$$

Things to notice:

I used independent increments.

$\mathcal{H}_s$ is shorthand for the conditioning event.

Similar to random walk calculation.