STAT 380 Lecture 22

Black Scholes

We model the price of a stock as

$$X(t) = x_0 e^{Y(t)}$$

where

$$Y(t) = \sigma B(t) + \mu t$$

is a Brownian motion with drift (B is standard Brownian motion).

If annual interest rates are $e^{\alpha} - 1$ we call α the instantaneous interest rate; if we invest \$1 at time 0 then at time t we would have $e^{\alpha t}$. In this sense an amount of money $x(t)$ to be paid at time t is worth only $e^{-\alpha t} x(t)$ at time 0 (because that much money at time 0 will grow to $x(t)$ by time t).

Present Value: If the stock price at time t is $X(t)$ per share then the present value of 1 share to be delivered at time t is

$$Z(t) = e^{-\alpha t} X(t)$$

With X as above we see

$$Z(t) = x_0 e^{\sigma B(t) + (\mu - \alpha)t}$$

Now we compute

$$\mathrm{E}\left\{ Z(t) | Z(u); 0 \le u \le s \right\} \\ = \mathrm{E}\left\{ Z(t) | B(u); 0 \le u \le s \right\}$$

for $s < t$. Write

$$Z(t) = x_0 e^{\sigma B(s) + (\mu - \alpha)t} \times e^{\sigma (B(t) - B(s))}$$

Since B has independent increments we find

$$\mathrm{E}\left\{ Z(t) | B(u); 0 \le u \le s \right\} \\ = x_0 e^{\sigma B(s) + (\mu - \alpha)t} \times \mathrm{E}\left[e^{\sigma \{B(t) - B(s)\}} \right]$$

Note: $B(t) - B(s)$ is $N(0, t-s)$; the expected value needed is the moment generating function of this variable at σ.

Suppose $U \sim N(0,1)$. The Moment Generating Function of U is

$$M_U(r) = \mathrm{E}(e^{rU}) = e^{r^2/2}$$

Rewrite

$$\sigma\{B(t) - B(s)\} = \sigma(t-s)U$$

where $U \sim N(0,1)$ to see

$$\mathrm{E}\left[e^{\sigma\{B(t)-B(s)\}}\right] = e^{\sigma^2(t-s)/2}$$

Finally we get

$$\begin{aligned}
&\mathrm{E}\{Z(t)|Z(u); 0 \le u \le s\} \\
&\quad = x_0 e^{\sigma B(s)+(\mu-\alpha)s} e^{(\mu-\alpha)(t-s)+\sigma^2(t-s)/2} \\
&\quad = Z(s)
\end{aligned}$$

provided

$$\mu + \sigma^2/2 = \alpha\,.$$

If this identity is satisfied then the present value of the stock price is a martingale.

Option Pricing

Suppose you can pay \$$c$ today for the right to pay K for a share of this stock at time t (regardless of the actual price at time t).

If, at time t, $X(t) > K$ you will**exercise** your **option** and buy the share making $X(t) - K$ dollars.

If $X(t) \le K$ you will not exercise your option; it becomes worthless.

The present value of this option is

$$e^{-\alpha t}(X(t) - K)_+ - c$$

where

$$z_+ = \begin{cases} z & z > 0 \\ 0 z \le 0 \end{cases}$$

(Called **positive part** of z.)

In a fair market:

- The discounted share price $e^{-\alpha t}X(t)$ is a martingale.
- The expected present value of the option is 0.

So:

$$c = e^{-\alpha t}\mathrm{E}\left[\{X(t) - K\}_+\right]$$

Since

$$X(t) = x_0 e^{N(\mu t, \sigma^2 t)}$$

we are to compute

$$\mathrm{E}\left\{\left(x_0 e^{\sigma t^{1/2}U + \mu t} - K\right)_+\right\}$$

This is

$$\int_a^\infty \left(x_0 e^{bu+d} - K\right) e^{-u^2/2} du/\sqrt{2\pi}$$

where

$$\begin{aligned}
a &= (\log(K/x_0) - \mu t)/(\sigma t^{1/2}) \\
b &= \sigma t^{1/2} \\
d &= \mu t
\end{aligned}$$

Evidently

$$K\int_a^\infty e^{-u^2/2} du/\sqrt{2\pi} = KP(N(0,1) > a)$$

The other integral needed is

$$\begin{aligned}
\int_a^\infty e^{-u^2/2+bu} du/\sqrt{2\pi} \\
&= \int_a^\infty \frac{e^{-(u-b)^2/2}e^{b^2/2}}{\sqrt{2\pi}} du \\
&= \int_{a-b}^\infty \frac{e^{-v^2/2}e^{b^2/2}}{\sqrt{2\pi}} dv \\
&= e^{b^2/2}P(N(0,1) > a - b)
\end{aligned}$$

Introduce the notation

$$\Phi(v) = P(N(0,1) \le v) = P(N(0,1) > -v)$$

and do all the algebra to get

$$\begin{aligned} c &= \left\{ e^{-\alpha t} e^{b^2/2+d} x_0 \Phi(b-a) - K e^{-\alpha t} \Phi(-a) \right\} \\ &= x_0 e^{(\mu+\sigma^2/2-\alpha)t} \Phi(b-a) - K e^{-\alpha t} \Phi(-a) \\ &= x_0 \Phi(b-a) - K e^{-\alpha t} \Phi(-a) \end{aligned}$$

This is the Black-Scholes option pricing formula.