Poisson Processes

Particles arriving over time at a particle detector. Several ways to describe most common model.

Approach 1: numbers of particles arriving in an interval has Poisson distribution, mean proportional to length of interval, numbers in several non-overlapping intervals independent.

For $s < t$, denote number of arrivals in $(s, t]$ by $N(s, t)$. Model is

1. $N(s, t)$ has a Poisson($\lambda(t - s)$) distribution.
2. For $0 \leq s_1 < t_1 \leq s_2 < t_2 \cdots \leq s_k < t_k$ the variables $N(s_i, t_i); i = 1, \ldots, k$ are independent.

Approach 2:

Let $0 < S_1 < S_2 < \cdots$ be the times at which the particles arrive.

Let $T_i = S_i - S_{i-1}$ with $S_0 = 0$ by convention.

Then $T_1, T_2, \ldots$ are independent Exponential random variables with mean $1/\lambda$.

Note $P(T_i > x) = e^{-\lambda x}$ is called **survival function** of T_i.

Approaches are equivalent. Both are deductions of a model based on **local** behaviour of process.

Approach 3: Assume:

1. given all the points in $[0, t]$ the probability of 1 point in the interval $(t, t + h]$ is of the form
$$\lambda h + o(h)$$
2. given all the points in $[0, t]$ the probability of 2 or more points in interval $(t, t + h]$ is of the form
$$o(h)$$

All 3 approaches are equivalent. I show: 3 implies 1, 1 implies 2 and 2 implies 3. First explain o, O.

Notation: given functions f and g we write

$$f(h) = g(h) + o(h)$$

provided

$$\lim_{h \to 0} \frac{f(h) - g(h)}{h} = 0$$

[Aside: if there is a constant M such that

$$\limsup_{h\to 0}\left|\frac{f(h)-g(h)}{h}\right| \le M$$

we say

$$f(h) = g(h) + O(h)$$

Notation due to Landau. Another form is

$$f(h) = g(h) + O(h)$$

means there is $\delta > 0$ and M s.t. for all $|h| < \delta$

$$|f(h) - g(h)| \le Mh$$

Idea: $o(h)$ is tiny compared to h while $O(h)$ is (very) roughly the same size as h.]

Model 3 implies 1: Fix t, define $f_t(s)$ to be conditional probability of 0 points in $(t, t+s]$ given value of process on $[0, t]$.

Derive differential equation for f. Given process on $[0, t]$ and 0 points in $(t, t+s]$ probability of no points in $(t, t+s+h]$ is

$$f_{t+s}(h) = 1 - \lambda h + o(h)$$

Given the process on $[0, t]$ the probability of no points in $(t, t+s]$ is $f_t(s)$. Using $P(AB|C) = P(A|BC)P(B|C)$ gives

$$\begin{aligned} f_t(s+h) &= f_t(s) f_{t+s}(h) \\ &= f_t(s)(1 - \lambda h + o(h)) \end{aligned}$$

Now rearrange, divide by h to get

$$\frac{f_t(s+h) - f_t(s)}{h} = -\lambda f_t(s) + \frac{o(h)}{h}$$

Let $h \to 0$ and find

$$\frac{\partial f_t(s)}{\partial s} = -\lambda f_t(s)$$

Differential equation has solution

$$f_t(s) = f_t(0)\exp(-\lambda s) = \exp(-\lambda s)\,.$$

Notice: survival function of exponential rv..

General case. Notation: $N(t) = N(0,t)$.

$N(t)$ is a non-decreasing function of t. Let

$$P_k(t) = P(N(t) = k)$$

Evaluate $P_k(t+h)$ by conditioning on $N(s); 0 \leq s < t$ and $N(t) = j$.

Given $N(t) = j$ probability that $N(t+h) = k$ is conditional probability of $k - j$ points in $(t, t+h]$.

So, for $j \leq k-2$:

$$P(N(t+h) = k | N(t) = j, N(s), 0 \leq s < t) \\ = o(h)$$

For $j = k-1$ we have

$$P(N(t+h) = k | N(t) = k-1, N(s), 0 \leq s < t) \\ = \lambda h + o(h)$$

For $j = k$ we have

$$P(N(t+h) = k | N(t) = k, N(s), 0 \leq s < t) \\ = 1 - \lambda h + o(h)$$

N is increasing so only consider $j \leq k$.

$$\begin{aligned} P_k(t+h) &= \sum_{j=0}^{k} P(N(t+h) = k | N(t) = j) P_j(t) \\ &= P_k(t)(1 - \lambda h) + \lambda h P_{k-1}(t) + o(h) \end{aligned}$$

Rearrange, divide by h and let $h \to 0$ t get

$$P_k'(t) = -\lambda P_k(t) + \lambda P_{k-1}(t)$$

For $k = 0$ the term P_{k-1} is dropped and

$$P_0'(t) = -\lambda P_0(t)$$

Using $P_0(0) = 1$ we get

$$P_0(t) = e^{-\lambda t}$$

Put this into the equation for $k = 1$ to get

$$P_1'(t) = -\lambda P_1(t) + \lambda e^{-\lambda t}$$

Multiply by $e^{\lambda t}$ to see

$$\left(e^{\lambda t} P_1(t)\right)' = \lambda$$

With $P_1(0) = 0$ we get

$$P_1(t) = \lambda t e^{-\lambda t}$$

For general k we have $P_k(0) = 0$ and

$$\left(e^{\lambda t} P_k(t)\right)' = \lambda e^{\lambda t} P_{k-1}(t)$$

Check by induction that

$$e^{\lambda t} P_k(t) = (\lambda t)^k / k!$$

Hence: $N(t)$ has Poisson(λt) distribution.

Similar ideas permit proof of

$$P(N(s,t) = k | N(u); 0 \le u \le s) = \frac{\{\lambda(t-s)\}^k e^{-\lambda(t-s)}}{k!}$$

From which (by induction) we can prove that N has independent Poisson increments.

Exponential Interarrival Times

If N is a Poisson Process we define $T_1, T_2, \ldots$ to be the times between 0 and the first point, the first point and the second and so on.

Fact: $T_1, T_2, \ldots$ are iid exponential rvs with mean $1/\lambda$.

We already did T_1 rigorously. The event $T > t$ is exactly the event $N(t) = 0$. So

$$P(T > t) = \exp(-\lambda t)$$

which is the survival function of an exponential rv.

I do case of T_1, T_2. Let t_1, t_2 be two positive numbers and $s_1 = t_1$, $s_2 = t_1 + t_2$. Consider event

$$\{t_1 < T_1 \le t_1 + \delta_1\} \cap \{t_2 < T_2 \le t_2 + \delta_2\}.$$

This is almost the same as the intersection of the four events:

$$\begin{aligned} N(0, t_1] &= 0 \\ N(t_1, t_1 + \delta_1] &= 1 \\ N(t_1 + \delta_1, t_1 + \delta_1 + t_2] &= 0 \\ N(s_2 + \delta_1, s_2 + \delta_1 + \delta_2] &= 1 \end{aligned}$$

which has probability

$$e^{-\lambda t_1} \times \lambda \delta_1 e^{-\lambda \delta_1} \times e^{-\lambda t_2} \times \lambda \delta_2 e^{-\lambda \delta_2}$$

Divide by $\delta_1 \delta_2$ and let δ_1 and δ_2 go to 0 to get joint density of T_1, T_2 is

$$\lambda^2 e^{-\lambda t_1} e^{-\lambda t_2}$$

which is the joint density of two independent exponential variates.

More rigor:

- Find joint density of $S_1, \ldots, S_k$.
- Use **change of variables** to find joint density of $T_1, \ldots, T_k$.

First step: Compute

$$P(0 < S_1 \leq s_1 < S_2 \leq s_2 \cdots < S_k \leq s_k)$$

This is just the event of exactly 1 point in each interval $(s_{i-1}, s_i]$ for $i = 1, \ldots, k-1$ ($s_0 = 0$) and at least one point in $(s_{k-1}, s_k]$ which has probability

$$\prod_1^{k-1} \left\{ \lambda (s_i - s_{i-1}) e^{-\lambda (s_i - s_{i-1})} \right\} \left(1 - e^{-\lambda (s_k - s_{k-1})} \right)$$

Second step: write this in terms of joint cdf of $S_1, \ldots, S_k$. I do $k = 2$:

$$\begin{aligned} &P(0 < S_1 \leq s_1 < S_2 \leq s_2) \\ &\qquad\qquad = F_{S_1,S_2}(s_1, s_2) - F_{S_1,S_2}(s_1, s_1) \end{aligned}$$

Notice tacit assumption $s_1 < s_2$.

Differentiate twice, that is, take

$$\frac{\partial^2}{\partial s_1 \partial s_2}$$

to get

$$f_{S_1,S_2}(s_1,s_2) = \frac{\partial^2}{\partial s_1 \partial s_2} \lambda s_1 e^{-\lambda s_1} \left(1 - e^{-\lambda(s_2 - s_1)}\right)$$

Simplify to

$$\lambda^2 e^{-\lambda s_2}$$

Recall tacit assumption to get

$$f_{S_1,S_2}(s_1,s_2) = \lambda^2 e^{-\lambda s_2} 1(0 < s_1 < s_2)$$

That completes the first part.

Now compute the joint cdf of T_1, T_2 by

$$F_{T_1,T_2}(t_1,t_2) = P(S_1 < t_1, S_2 - S_1 < t_2)$$

This is

$$\begin{aligned} P(S_1 < t_1, & S_2 - S_1 < t_2) \\ &= \int_0^{t_1} \int_{s_1}^{s_1+t_2} \lambda^2 e^{-\lambda s_2} ds_2 ds_1 \\ &= \lambda \int_0^{t_1} \left(e^{-\lambda s_1} - e^{-\lambda(s_1+t_2)}\right) ds_1 \\ &= 1 - e^{-\lambda t_1} - e^{-\lambda t_2} + e^{-\lambda(t_1+t_2)} \end{aligned}$$

Differentiate twice to get

$$f_{T_1,T_2}(t_1,t_2) = \lambda e^{-\lambda t_1} \lambda e^{-\lambda t_2}$$

which is the joint density of two independent exponential random variables.

Summary so far:

Have shown:

Instantaneous rates model implies independent Poisson increments model implies independent exponential interarrivals.

Next: show independent exponential interarrivals implies the instantaneous rates model.

Suppose $T_1, \ldots$ iid exponential rvs with means $1/\lambda$. Define N_t by $N_t = k$ if and only if

$$T_1 + \cdots + T_k \leq t \leq T_1 + \cdots + T_{k+1}$$

Let A be the event $N(s) = n(s); 0 < s \leq t$. We are to show

$$P(N(t, t+h] = 1 | N(t) = k, A) = \lambda h + o(h)$$

and

$$P(N(t, t+h] \geq 2 | N(t) = k, A) = o(h)$$

If $n(s)$ is a possible trajectory consistent with $N(t) = k$ then n has jumps at points

$$s_1 \equiv t_1, s_2 \equiv t_1 + t_2, \ldots, s_k \equiv t_1 + \cdots + t_k < t$$

and at no other points in $(0, t]$.

So given $N(s) = n(s); 0 < s \leq t$ with $n(t) = k$ we are essentially being given

$$T_1 = t_1, \ldots, T_k = t_k, T_{k+1} > t - s_k$$

and asked the conditional probabilty in the first case of the event B given by

$$t - s_k < T_{k+1} \leq t - s_k + h < T_{k+2} + T_{k+1}\,.$$

Conditioning on $T_1, \ldots, T_k$ irrelevant (independence).

$$\begin{aligned} P(N(t, t+h] = 1 | N(t) = k, A)/h \\ &= P(B | T_{k+1} > t - s_k)/h \\ &= \frac{P(B)}{he^{-\lambda(t-s_k)}} \end{aligned}$$

The numerator may be evaluated by integration:

$$P(B) = \int_{t-s_k}^{t-s_k+h} \int_{t-s_k+h-s_1}^{\infty} \lambda^2 e^{-\lambda(s_1+s_2)} ds_2 ds_1$$

Let $h \to 0$ to get the limit

$$P(N(t, t+h] = 1 | N(t) = k, A)/h \to \lambda$$

as required.

The computation of

$$\lim_{h\to 0} P(N(t,t+h] \geq 2|N(t)=k,A)/h$$

is similar.

Properties of exponential rvs

Convolution: If X and Y independent rvs with densities f and g respectively and $Z = X + Y$ then

$$P(Z \leq z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f(x)g(y)dydx$$

Differentiating wrt z we get

$$f_Z(z) = \int_{-\infty}^{\infty} f(x)g(z-x)dx$$

This integral is called the **convolution** of densities f and g.

If $T_1, \ldots, T_n$ iid Exponential(λ) then $S_n = T_1 + \cdots + T_n$ has a Gamma(n, λ) distribution. Density of S_n is

$$f_{S_n}(s) = \lambda(\lambda s)^{n-1} e^{-\lambda s}/(n-1)!$$

for $s > 0$.

Proof:

$$\begin{aligned} P(S_n > s) &= P(N(0,s] < n) \\ &= \sum_{j=0}^{n-1} (\lambda s)^j e^{-\lambda s}/j! \end{aligned}$$

Then

$$
\begin{aligned}
f_{S_n}(s) &= \frac{d}{ds} P(S_n \leq s) \\
&= \frac{d}{ds} \left\{ 1 - P(S_n > s) \right\} \\
&= -\lambda \sum_{j=1} n - 1 \left\{ j(\lambda s)^{j-1} - (\lambda s)^j \right\} \frac{e^{-\lambda s}}{j!} \\
&\quad + \lambda e^{-\lambda s} \\
&= \lambda e^{-\lambda s} \sum_{j=1}^{n-1} \left\{ \frac{(\lambda s)^j}{j!} - \frac{(\lambda s)^{j-1}}{(j-1)!} \right\} \\
&\quad + \lambda e^{-\lambda s}
\end{aligned}
$$

This telescopes to

$$
f_{S_n}(s) = \lambda (\lambda s)^{n-1} e^{-\lambda s} / (n-1)!
$$

Extreme Values: If $X_1, \ldots, X_n$ are independent exponential rvs with means $1/\lambda_1, \ldots, 1/\lambda_n$ then $Y = \min\{X_1, \ldots, X_n\}$ has an exponential distribution with mean

$$
\frac{1}{\lambda_1 + \cdots + \lambda_n}
$$

Proof:

$$
\begin{aligned}
P(Y > y) &= P(\forall k X_k > y) \\
&= \prod e^{-\lambda_k y} \\
&= e^{-\sum \lambda_k y}
\end{aligned}
$$

Memoryless Property: conditional distribution of $X - x$ given $X \geq x$ is exponential if X has an exponential distribution.

Proof:

$$
\begin{aligned}
P(X - x > y | X \geq & x) \\
&= \frac{P(X > x + y, X \geq x)}{P(X > x)} \\
&= \frac{PX > x + y)}{P(X \geq x)} \\
&= \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}} \\
&= e^{-\lambda y}
\end{aligned}
$$

Hazard Rates

The hazard rate, or instantaneous failure rate for a positive random variable T with density f and cdf F is

$$r(t) = \lim_{\delta \to 0} \frac{P(t < T \leq t + \delta | T \geq t)}{\delta}$$

This is just

$$r(t) = \frac{f(t)}{1 - F(t)}$$

For an exponential random variable with mean $1/\lambda$ this is

$$h(t) = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

The exponential distribution has constant failure rate.

Weibull random variables have density

$$f(t|\lambda, \alpha) = \lambda(\lambda t)^{\alpha - 1} e^{-(\lambda t)^{\alpha}}$$

for $t > 0$. The corresponding survival function is

$$1 - F(t) = e^{-(\lambda t)^{\alpha}}$$

and the hazard rate is

$$r(t) = \lambda(\lambda t)^{\alpha-1}$$

which is increasing for $\alpha > 1$, decreasing for $\alpha < 1$. For $\alpha = 1$ this is the exponential distribution.

Since

$$r(t) = \frac{dF(t)/dt}{1 - F(t)} = -\frac{d\log(1 - F(t))}{dt}$$

we can integrate to find

$$1 - F(t) = \exp\{-\int_0^t r(s)ds\}$$

so that r determines F and f.

Properties of Poisson Processes

1) If N_1 and N_2 are independent Poisson processes with rates λ_1 and λ_2, respectively, then $N = N_1 + N_2$ is a Poisson process with rate $\lambda_1 + \lambda_2$.

2) Suppose N is a Poisson process with rate λ. Suppose each point is marked with a label, say one of $L_1, \ldots, L_r$, independently of all other occurences. Suppose p_i is the probability that a given point receives label L_i. Let N_i count the points with label i (so that $N = N_1 + \cdots + N_r$). Then $N_1, \ldots, N_r$ are independent Poisson processes with rates $p_i\lambda$.

3) Suppose $U_1, U_2, \ldots$ independent rvs, each uniformly distributed on $[0, T]$. Suppose M is a Poisson(λT) random variable independent of the U's. Let

$$N(t) = \sum_1^M 1(U_i \leq t)$$

Then N is a Poisson process on $[0, T]$ with rate λ.

4) Suppose N is a Poisson process with rate λ. Let $S_1 < S_2 < \cdots$ be the times at which points arrive Given $N(T) = n$ $S_1, \ldots, S_n$ have the same distribution as the order statistics of a sample of size n from the uniform distribution on $[0, T]$.

5) Given $S_{n+1} = T$, $S_1, \ldots, S_n$ have the same distribution as the order statistics of a sample of size n from the uniform distribution on $[0, T]$.

Indications of some proofs:

1) $N_1, \ldots, N_r$ independent Poisson processes rates λ_i, $N = \sum N_i$. Let A_h be the event of 2 or more points in N in the time interval $(t, t+h]$, B_h, the event of exactly one point in N in the time interval $(t, t+h]$.

Let A_{ih} and B_{ih} be the corresponding events for N_i.

Let H_t denote the history of the processes up to time t; we condition on H_t.

We are given:

$$P(A_{ih}|H_t) = o(h)$$

and

$$P(B_{ih}|H_t) = \lambda_i h + o(h)\,.$$

Note that

$$A_h \subset \bigcup_{i=1}^{r} A_{ih} \cup \bigcup_{i \neq j} (B_{ih} \cap B_{jh})$$

Since

$$\begin{aligned} P(B_{ih} \cap B_{jh}|H_t) &= P(B_{ih}|H_t)P(B_{jh}|H_t) \\ &= (\lambda_i h + o(h))(\lambda_j h + o(h)) \\ &= O(h^2) \\ &= o(h) \end{aligned}$$

we have checked one of the two infinitesimal conditions for a Poisson process.

Next let C_h be the event of no points in N in the time interval $(t, t+h]$ and C_{ih} the same for N_i. Then

$$\begin{aligned} P(C_h|H_t) &= P(\cap C_{ih}|H_t) \\ &= \prod P(C_{ih}|H_t) \\ &= \prod (1 - \lambda_i h + o(h)) \\ &= 1 - (\sum \lambda_i)h + o(h) \end{aligned}$$

shows

$$\begin{aligned} P(B_h|H_t) &= 1 - P(C_h|H_t) - P(A_h|H_t) \\ &= (\sum \lambda_i)h + o(h) \end{aligned}$$

Hence N is a Poisson process with rate $\sum \lambda_i$.

2) The infinitesimal approach used for 1 can do part of this. See text for rest. Events defined as in **1)**: The event B_{ih} that there is one point in N_i in $(t, t+h]$ is the event, B_h that there is exactly one point in any of the r processes together with a subset of A_h where there are two or more points in N in $(t, t+h]$ but exactly one is labeled i. Since $P(A_h|H_t) = o(h)$

$$\begin{aligned} P(B_{ih}|H_t) &= p_i P(B_h|H_t) + o(h) \\ &= p_i(\lambda h + o(h)) + o(h) \\ &= p_i \lambda h + o(h) \end{aligned}$$

Similarly, A_{ih} is a subset of A_h so

$$P(A_{ih}|H_t) = o(h)$$

This shows each N_i is Poisson with rate λp_i. To get independence requires more work; see the text for the algebraic method which is easier.

3) Fix $s < t$. Let $N(s,t)$ be the number of points in $(s,t]$. Given $N = n$ the conditional distribution of $N(s,t)$ is Binomial(n,p) with $p = (t-s)/T$.

So

$$
\begin{aligned}
&P(N(s,t)=k)\\
&\qquad= \sum_{n=k}^{\infty} P(N(s,t)=k, N=n)\\
&\qquad= \sum_{n=k}^{\infty} P(N(s,t)=k|N=n)P(N=n)\\
&\qquad= \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \frac{(\lambda T)^n}{n!} e^{-\lambda T}\\
&\qquad= \frac{e^{-\lambda T}}{k!} (\lambda T p)^k \sum_{n=k}^{\infty} \frac{(1-p)^{n-k}(\lambda T)^{n-k}}{(n-k)!}\\
&\qquad= \frac{e^{-\lambda T}}{k!} (\lambda T p)^k \sum_{m=0}^{\infty} (1-p)^m (\lambda T)^m / m!\\
&\qquad= \frac{e^{-\lambda T}}{k!} (\lambda T p)^k e^{\lambda T(1-p)}\\
&\qquad= \frac{e^{-\lambda(t-s)}(\lambda(t-s))^k}{k!}
\end{aligned}
$$

4): Fix s_i, h_i for $i = 1, \ldots, n$ such that

$$0 < s_1 < s_1 + h_1 < s_2 < \cdots < s_n < s_n + h_n < T$$

Given $N(T) = n$ we compute the probability of the event

$$A = \bigcap_{i=1}^{n} \{s_i < S_i < s_i + h_i\}$$

Intersection of A, $N(T) = n$ is ($s_0 = h_0 = 0$):

$$
\begin{aligned}
B \equiv\ & \\
&\bigcap_{i=1}^{n} \{N(s_{i-1} + h_{i-1}, s_i] = 0, N(s_i, s_i + h_i] = 1\}\\
&\qquad\qquad\qquad\qquad \cap \{N(s_n + h_n, T] = 0\}
\end{aligned}
$$

whose probability is

$$\left(\prod \lambda h_i\right) e^{-\lambda T}$$

So

$$\begin{aligned} P(A|N(t)=n) &= \frac{P(A, N(T)=n)}{P(N(T)=n)} \\ &= \frac{\lambda^n e^{-\lambda T} \prod h_i}{(\lambda T)^n e^{-\lambda T}/n!} \\ &= \frac{n! \prod h_i}{T^n} \end{aligned}$$

Divide by $\prod h_i$ and let all h_i go to 0 to get joint density of $S_1, \ldots, S_n$ is

$$\frac{n!}{T^n} 1(0 < s_1 < \cdots < s_n < T)$$

which is the density of order statistics from a Uniform$[0, T]$ sample of size n.

5) Replace the event $S_{n+1} = T$ with $T < S_{n+1} < T + h$. With A as before we want

$$P(A|T < S_{n+1} < T+h) = \frac{P(B, N(T, T+h] \geq 1)}{P(T < S_{n+1} < T+h)}$$

Note that B is independent of $\{N(T, T+h] \geq 1\}$ and that we have already found the limit

$$\frac{P(B)}{\prod h_i} \to \lambda^n e^{-\lambda T}$$

We are left to compute the limit of

$$\frac{P(N(T, T+h] \geq 1)}{P(T < S_{n+1} < T+h)}$$

The denominator is

$$\sum_{k=0}^{n} P(N(0,T] = k, N(T, T+h] = n+1-k) + o(h) = P(N(0,T] = n)\lambda h + o(h)$$

Thus

$$\frac{P(N(T,T+h] \geq 1)}{P(T < S_{n+1} < T+h)} = \frac{\lambda h + o(h)}{\frac{(\lambda T)^n}{n!} e^{-\lambda T} \lambda h + o(h)}$$
$$\to \frac{n!}{(\lambda T)^n e^{-\lambda T}}$$

This gives the conditional density of $S_1, \ldots, S_n$ given $S_{n+1} = T$ as in **4)**.

Inhomogeneous Poisson Processes

The idea of hazard rate can be used to extend the notion of Poisson Process. Suppose $\lambda(t) \geq 0$ is a function of t. Suppose N is a counting process such that

$$P(N(t+h) = k+1 | N(t) = k, H_t) = \lambda(t)h + o(h)$$

and

$$P(N(t+h) \geq k+2 | N(t) = k, H_t) = o(h)$$

Then N has independent increments and $N(t+s) - N(t)$ has a Poisson distribution with mean

$$\int_t^{t+s} \lambda(u)du$$

If we put

$$\Lambda(t) = \int_0^t \lambda(u)du$$

then mean of $N(t+s) - N(t)$ is $\Lambda(t+s) - \Lambda(t)$.

Jargon: λ is the **intensity** or **instaneous intensity** and Λ the **cumulative intensity**.

Can use the model with Λ any non-decreasing right continuous function, possibly without a derivative. This allows ties.

Compound Poisson Processes

Imagine insurance claims arise at times of a Poisson process, $N(t)$, (more likely for an inhomogeneous process).

Let Y_i be the value of the ith claim associated with the point whose time is S_i.

Assume that the Y's are independent of each other and of N.

Let

$$\mu = \mathrm{E}(Y_i) \text{ and } \sigma^2 = \mathrm{var}(Y_i)$$

Let

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

be the total claim up to time t. We call X a compound Poisson Process.

Useful properties:

$$\begin{aligned}
\mathrm{E}\left\{X(t)|N(t)\right\} &= N(t)\mu \\
\mathrm{Var}\left\{X(t)|N(t)\right\} &= N(t)\sigma^2 \\
\mathrm{E}\left\{X(t)\right\} &= \mu \mathrm{E}\left\{N(t)\right\} \\
&= \mu\lambda t \\
\mathrm{Var}\left\{X(t)\right\} &= \mathrm{Var}\left[\mathrm{E}\left\{X(t)|N(t)\right\}\right] \\
&\qquad + \mathrm{E}\left[\mathrm{Var}\left\{X(t)|N(t)\right\}\right] \\
&= \lambda t\mu^2 + \lambda t\sigma^2
\end{aligned}$$

(Look at all familiar? See homework.)