STAT 380

Final Examination

Richard Lockhart 11 April 1991

Instructions: This is an open book exam. You may use notes, books and a calculator. The exam is out of 50. Each question is worth 10 marks. Parts a) and c) of 1 are worth 4 each, part b) is worth 2. Each part of question 2 is worth 5 marks; Part c is a bonus question worth 5. Question 4a) and 4c) are worth 3 each while 4b) is worth 4. Question 5a) and 5b) are worth 5 each. Be as clear as possible about what you are doing but if you can only give an intuitive explanation for your calculations please do give it.

1. An experiment is performed to compare two brands of light bulbs. At time 0 one light bulb of brand X and one of brand Y are turned on. When the brand Y bulb burns out it is replaced immediately with a brand Z bulb. Let X, Y and Z be the corresponding lifetimes. Assume that the light bulbs have exponentially distributed lifetimes and that the mean lifetime is 1000 hours for brand X, 500 hours for brand Y and 2000 for brand Z. Let A be the event that the brand X light bulb is still burning when the brand Z bulb burns out.

   (a) Show that $\mathrm{P}(A|Y = y, Z = z) = \exp(-(y + z)/1000)$.

   The event A is just $\{X > Z + Y\}$ so we want

$$\begin{aligned} P(X > Z + Y|Y = y, Z = z) &= P(X > z + y|Y = y, Z = z) \\ &= \int_{z+y}^{\infty} f_{X|Y,Z}(x|y, z)\, dx \end{aligned}$$

   Notice that given $Z = z$ and $Y = y$ I can replace $X > Z + Y$ by $X > z + y$. Now X is independent of Y and Z; this assumption was not explicit when I asked the question – it probably should have been.
   With this assumption you get

$$f_{X|Y,Z}(x|y, z) = f_X(x) = \frac{1}{1000} \exp\{-x/1000\}.$$

   so

$$\begin{aligned} P(X > Z|Y = y, Z = z) &= \int_{z+y}^{\infty} \frac{1}{1000} \exp\{-x/1000\}\, dx \\ &= \exp\{-(z + y)/1000\}. \end{aligned}$$

   (b) Find the moment generating functions of Y and Z.

   The moment generating function of an exponential random variable U with rate λ or mean $\mu = 1/\lambda$ is

$$\begin{aligned} M(t) &= \mathrm{E}\left(e^{tU}\right) \\ &= \int_0^{\infty} \exp(tu)\lambda \exp(-\lambda u)\, du \\ &= \lambda \int_0^{\infty} \exp(-(\lambda - t)u)\, du \\ &= \frac{\lambda}{\lambda - t} \\ &= \frac{1}{1 - t/\lambda} = \frac{1}{1 - \mu t}. \end{aligned}$$

   So

$$M_Y(t) = \frac{1}{1 - 500t}$$

and

$$M_Z(t) = \frac{1}{1-2000t}.$$

(c) Use the results of a) and b) to compute $P(A)$.

We have

$$\begin{aligned}
P(A) &= \mathrm{E}(P(A|Y,Z)) \\
&= \int_0^\infty \int_0^\infty P(A|Y=y, Z=z) f_Y(y) f_Z(z)\, dy\, dz \\
&= \int_0^\infty \int_0^\infty \exp\{-(z+y)/1000\} f_Y(y) f_Z(z)\, dy\, dz \\
&= \int_0^\infty \exp(-y/1000) f_Y(y)\, dy \int_0^\infty \exp(-z/1000) f_Z(z)\, dz \\
&= M_Y(-1/1000) M_Z(-1/1000) \\
&= \frac{1}{1+500/1000} \cdot \frac{1}{1+2000/1000} \\
&= \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}.
\end{aligned}$$

2. When playing roulette two of the possible bets are to bet on red or to bet on the numbers from 1 to 12. If you bet on red the probability that you win is 18/38 while the probability of winning when betting on 1 to 12 is 12/38. Consider the following strategy: you begin by betting on red. You continue betting on red until you have lost twice in a row. Then you bet on 1 to 12 until you lose twice in a row. Then you go back to betting on red and so on.

   (a) Using 4 states like "about to bet on red with 0 consecutive losses so far" describe the state space and transition matrix of a Markov chain, $X_n, n = 0, 1, \cdots$ which records the stage you are at in following the strategy.

   (b) What is the probability that for $n = 3$ you are betting on the numbers 1 to 12?

   (c) (BONUS: Do not do this problem unless you have extra time and know what you are doing.) What is the long-run fraction of the time on which you are betting on red?

3. Consider a renewal process $N(t)$ with interarrival distribution given by $\mathrm{P}(T = k) = kp^2(1-p)^{k-1}; k = 1, 2, \cdots$. Compute $m(3)$ where $m(t) = \mathrm{E}(N(t))$ is the renewal function.

4. A Markov Chain has state space $\{1, 2, 3, 4\}$ and transition matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   (a) Identify all the communicating classes and say whether or not each is transient.

   (b) Let $q_k = \mathrm{P}(\text{there is an } n \text{ such that } X_n = 4 | X_0 = k)$. Derive the equations

$$q_2 = (q_1 + q_2 + q_3)/3$$

   and

$$q_3 = (q_2 + 2q_3 + q_4)/4.$$

(c) Solve the equations in b) (using whatever additional information is necessary).

5. Cosmic rays arrive at a particle detector according to a Poisson Process with rate 2. An arriving cosmic ray is detected with probability 0.8 independent of all other cosmic rays and the time at which the cosmic ray arrives.

   (a) Compute the probability that 12 cosmic rays arrived in a time period of length $t = 5$ given that 8 cosmic rays were detected in the time period.

   (b) Now suppose that in addition neutrinos arrive at the rate 1 and are detected with probability 0.5 (the real detection probabilities are ludicrously small numbers). Given that exactly one object was detected in a period of length $t = 2$ what is the probability that the object detected was a neutrino. (Only cosmic rays and neutrinos are detected.)