STAT 380

Midterm Examination

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Instructions: This is an open book exam. You may use notes, books and a calculator. The exam is out of 25. Questions 1 and 3 are worth 5 marks each. Each of the 5 parts of question 2 is worth 3 marks. I will be marking for clarity of explanation as well as correctness. Without a clear explanation you should not expect to get more than half marks.

1. A Markov Chain has state space $\{1, 2, 3, 4\}$ and transition matrix

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

   Identify all the communicating classes and say whether or not each is transient. [5 marks]

   Solution: Since 2 leads only to 2 one class is $\{2\}$. Similarly 3 leads only to 3 and must be in a class of its own, $\{3\}$. Finally 1 and 4 lead to each other so $\{1, 4\}$ is the last class. The classes 2 and 3 are recurrent while $\{1, 4\}$ is transient.

2. Each day I get a random number of pieces of voice mail. I deal with, and delete, a random number of pieces of voice mail. When the mail box gets full any further messages received are lost. Here is a simplified model. Assume that my mail box can hold two messages. Each morning I get either 1 message or 0 messages. Each evening I delete either 1 message or 0 messages. The probability that I get 1 message is p regardless of what has happened in the past. The probability that I delete 1 message is π if there is a message to delete. Let X_n be the number of messages on day n in the morning *before* any message arrives. Assume that $X_0 = 0$; day 0 is the starting day.

   (a) Write out the transition matrix of the resulting Markov Chain. [3 marks]

   Solution: Clearly $P_{02} = P_{20} = 0$. If you start the morning with no voice mail then you end the day with 1 voice mail if a piece of mail comes and you don't delete it. This makes $P_{01} = p(1 - \pi)$. Similarly $P_{12} = p(1 - \pi)$. Since the rows must sum to 1 we find $P_{00} = 1 - p(1 - \pi)$. If I start the day with 1 voice mail I will end the day with 0 voice mails provided no voice mail arrives and I delete the one I already have; the probability of this is $(1 - p)\pi = P_{10}$. This makes $P_{11} = 1 - p(1 - \pi) - (1 - p)\pi = p\pi + (1 - p)(1 - \pi)$. Finally: if I start the day with 2 voice mails any arriving voice mail will be discarded so that I will end up at 1 voice mail at the end of the day if I delete a voice mail that day; thus $P_{21} = \pi$. Putting it together get

$$\mathbf{P} = \left[\begin{array}{ccc} 1-p(1-\pi) & p(1-\pi) & 0 \\ (1-p)\pi & p\pi + (1-p)(1-\pi) & p(1-\pi) \\ 0 & \pi & 1-\pi \end{array} \right]$$

A common mistake will be to think that $P_{21} = (1-p)\pi$. (This is the mistake I made in doing the problem when I made it up; with the mistake in place part c is much easier!)

(b) Suppose $p = \pi$ and compute the probability that the mailbox is empty in the morning on day 2 (before the arrival of any mail). [3 marks]

Solution: You need

$$\begin{aligned} (P^2)_{00} &= P_{00}P_{00} + P_{01}P_{10} + P_{02}P_{20} \\ &= \{1-p(1-p)\}^2 + \{p(1-p)\}^2 + 0 \\ &= 2p^4 - 4p^3 + 4p^2 - 2p + 1 \end{aligned}$$

I didn't need to see that last line.

(c) Again supposing $p = \pi$, show that in the long run the fraction of days on which I lose an email is $p/3$. [3 marks]

Solution: I made a mistake here. When I created the exam I thought I was making $\mathbf{P}$ doubly stochastic in which case the stationary initial distribution would be

$$(\alpha_0, \alpha_1, \alpha_2) = (1/3, 1/3, 1/3)$$

Instead we solve

$$\begin{aligned} \alpha_0 &= \alpha_0\{1-p(1-p)\} + \alpha_1\{p(1-p)\} \\ \alpha_2 &= \alpha_1\{p(1-p)\} + \alpha_2(1-p) \\ 1 &= \alpha_0 + \alpha_1 + \alpha_2 \end{aligned}$$

The first equation gives $\alpha_0 = \alpha_1$. The second gives $\alpha 2 = (1-p)\alpha_1$. Thus $(3-p)\alpha_1 = 1$ or

$$(\alpha_0, \alpha_1, \alpha_2) = \left(\frac{1}{3-p}, \frac{1}{3-p}, \frac{1-p}{3-p} \right)$$

The long run fraction of days I start with the mail box full is $(1-p)/(3-p)$; I lose an email on the fraction p of those days when I get an voice mail so I lose

$$\frac{p(1-p)}{3-p}$$

(d) Again supposing $p = \pi$, in the long run what fraction of my e-mail will be lost? [3 marks]

Solution: In the first n days I will get about np pieces of voice mail and I will lose about $np(1-p)/(3-p)$ pieces of voice mail. The answer is the ratio which is $(1-p)/(3-p)$. (This answer is not valid for $p=0$ which corresponds to never getting or deleting voice mail.

(e) Let T be the number of days till I lose my first e-mail and let m_k be the expected value of T given that I start in state k (for $k = 0, 1, 2$). Use first step analysis to derive a set of equations which would be solved to compute the m_k. DO NOT SOLVE THE EQUATIONS. [3 marks]

Solution: If you start the day in state 2 there is a chance p that $T=0$ because a piece of mail arrives. If no mail arrives and you don't delete any (probability $(1-p)(1-\pi)$) then you have used one day and should still expect to require m_2 days more. If no mail arrives and you delete one piece (probability $\pi(1-p)$) then you have used a day and should expect to wait m_1 more days. Thus

$$m_2 = (1-p) + (1-p)(1-\pi)m_2 + (1-p)\pi m_1$$

If you start the day in state 0 or 1 the situation is easier. You will use your first day and then require m_j more days if you go to state j Thus

$$\begin{aligned} m_0 &= 1 + p(1-\pi)m_1 + \{1 - p(1-\pi)\}m_0 \\ m_1 &= 1 + p(1-\pi)m_2 + \{p\pi + (1-p)(1-\pi)\}m_1 + \pi(1-p)m_0 \\ m_2 &= (1-p) + (1-p)(1-\pi)m_2 + (1-p)\pi m_1 \end{aligned}$$

3. Imagine that buses arrive at a particular stop according to a Poisson process with rate 2 per hour. I start waiting at the stop at 1:00. Given that the second bus arrives between 2:00 and 3:00 what is the probability that the first bus arrived before 2:00? You may leave your answer as a formula—do not bother plugging the numbers into the calculator. [5 marks]

Solution: You are given the information $N(0,2] \geq 2$ and $N(0,1] \leq 1$ which is the union of two events:

$$A_0 = \{N(0,1] = 0\} \cap \{N(1,2] \geq 2\}$$

and

$$A_1 = \{N(0,1] = 1\} \cap \{N(1,2] \geq 1\}$$

You want

$$\begin{aligned} P(N(0,1] \geq 1 | A_0 \cup A_1) &= \frac{P(N(0,1] \geq 1; N(0,1] \leq N(0,2] \geq 2}{P(A_0) + P(A_1)} \\ &= \frac{P(N(0,1] = 1; N(1,2) \geq 1)}{P(N(0,1] = 0; N(1,2] \geq 2) + P(N(0,1] = 1; N(1,2) \geq 1)} \\ &= \frac{\lambda e^{-\lambda}(1-e^{-\lambda})}{e^{-\lambda}(1-e^{-\lambda}-\lambda e^{-\lambda}) + \lambda e^{-\lambda}(1-e^{-\lambda})} \\ &= \frac{\lambda(1-e^{-\lambda})}{1-e^{-\lambda}-2\lambda e^{-\lambda}+\lambda e^{-\lambda}} \end{aligned}$$

Plug in 2 for λ.