**STAT 450: Statistical Theory**

**Independence, conditional distributions**

So far density of $X$ specified explicitly. Often modelling leads to a specification in terms of marginal and conditional distributions.

**Def'n**: Events $A$ and $B$ are independent if

$$P(AB) = P(A)P(B)\,.$$

(Notation: $AB$ is the event that both $A$ and $B$ happen, also written $A \cap B$.)

**Def'n**: $A_i$, $i = 1, \ldots, p$ are **independent** if

$$P(A_{i_1} \cdots A_{i_r}) = \prod_{j=1}^{r} P(A_{i_j})$$

for any $1 \le i_1 < \cdots < i_r \le p$.

Example: $p = 3$

$$\begin{aligned} P(A_1A_2A_3) &= P(A_1)P(A_2)P(A_3) \\ P(A_1A_2) &= P(A_1)P(A_2) \\ P(A_1A_3) &= P(A_1)P(A_3) \\ P(A_2A_3) &= P(A_2)P(A_3) \end{aligned}$$

All these equations needed for independence!

**Example**: Toss a coin twice.

$$\begin{aligned} A_1 &= \{\text{first toss is a Head}\} \\ A_2 &= \{\text{second toss is a Head}\} \\ A_3 &= \{\text{first toss and second toss different}\} \end{aligned}$$

Then $P(A_i) = 1/2$ for each $i$ and for $i \neq j$

$$P(A_i \cap A_j) = \frac{1}{4}$$

but

$$P(A_1 \cap A_2 \cap A_3) = 0 \neq P(A_1)P(A_2)P(A_3)\,.$$

**Def'n**: $X$ and $Y$ are **independent** if

$$P(X \in A; Y \in B) = P(X \in A)P(Y \in B)$$

for all $A$ and $B$.

**Def'n**: Rvs $X_1, \ldots, X_p$ **independent**:

$$P(X_1 \in A_1, \cdots, X_p \in A_p) = \prod P(X_i \in A_i)$$

for any $A_1, \ldots, A_p$.

**Theorem**:

1. If $X$ and $Y$ are independent then for all $x, y$
$$F_{X,Y}(x,y) = F_X(x)F_Y(y)\,.$$
2. If $X$ and $Y$ are independent with joint density $f_{X,Y}(x,y)$ then $X$ and $Y$ have densities $f_X$ and $f_Y$, and
$$f_{X,Y}(x,y) = f_X(x)f_Y(y)\,.$$
3. If $X$ and $Y$ independent with marginal densities $f_X$ and $f_Y$ then $(X,Y)$ has joint density
$$f_{X,Y}(x,y) = f_X(x)f_Y(y)\,.$$
4. If $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ for **all** $x, y$ then $X$ and $Y$ are independent.
5. If $(X,Y)$ has density $f(x,y)$ and there exist $g(x)$ and $h(y)$ st $f(x,y) = g(x)h(y)$ for (almost) **all** $(x,y)$ then $X$ and $Y$ are independent with densities given by

$$f_X(x) = g(x)/\int_{-\infty}^{\infty} g(u)du$$

$$f_Y(y) = h(y)/\int_{-\infty}^{\infty} h(u)du\,.$$

**Proof**: See STAT 802

**Theorem**: If $X_1, \ldots, X_p$ are independent and $Y_i = g_i(X_i)$ then $Y_1, \ldots, Y_p$ are independent. Moreover, $(X_1, \ldots, X_q)$ and $(X_{q+1}, \ldots, X_p)$ are independent.

## Conditional probability

**Def'n**: $P(A|B) = P(AB)/P(B)$ if $P(B) \neq 0$.

**Def'n**: For discrete $X$ and $Y$ the conditional probability mass function of $Y$ given $X$ is

$$\begin{aligned} f_{Y|X}(y|x) &= P(Y = y|X = x) \\ &= f_{X,Y}(x,y)/f_X(x) \\ &= f_{X,Y}(x,y)/\sum_t f_{X,Y}(x,t) \end{aligned}$$

For absolutely continuous $X$ $P(X = x) = 0$ for all $x$. What is $P(A|X = x)$ or $f_{Y|X}(y|x)$? Solution: use limit

$$P(A|X = x) = \lim_{\delta x \to 0} P(A|x \leq X \leq x + \delta x)$$

If, e.g., $X, Y$ have joint density $f_{X,Y}$ then with $A = \{Y \leq y\}$ we have

$$\begin{aligned} &P(A|x \leq X \leq x + \delta x) \\ &\qquad = \frac{P(A \cap \{x \leq X \leq x + \delta x\})}{P(x \leq X \leq x + \delta x)} \\ &\qquad = \frac{\int_{-\infty}^{y} \int_{x}^{x+\delta x} f_{X,Y}(u,v)\,du\,dv}{\int_{x}^{x+\delta x} f_X(u)\,du} \end{aligned}$$

Divide top, bottom by $\delta x$; let $\delta x \to 0$. Denom converges to $f_X(x)$; numerator converges to

$$\int_{-\infty}^{y} f_{X,Y}(x,v)\,dv$$

Define conditional cdf of $Y$ given $X = x$:

$$P(Y \leq y|X = x) = \frac{\int_{-\infty}^{y} f_{X,Y}(x,v)\,dv}{f_X(x)}$$

Differentiate wrt $y$ to get def'n of conditional density of $Y$ given $X = x$:

$$f_{Y|X}(y|x) = f_{X,Y}(x,y)/f_X(x);$$

in words "conditional = joint/marginal".