STAT 450: Statistical Theory

Independence, conditional distributions

Def'n: Events A and B are independent if

$$P(AB) = P(A)P(B)\,.$$

(Notation: AB is the event that both A and B happen, also written $A \cap B$.)

Def'n: A_i, $i = 1, \ldots, p$ are **independent** if

$$P(A_{i_1} \cdots A_{i_r}) = \prod_{j=1}^{r} P(A_{i_j})$$

for any $1 \le i_1 < \cdots < i_r \le p$.

Example: $p = 3$

$$\begin{aligned} P(A_1A_2A_3) &= P(A_1)P(A_2)P(A_3) \\ P(A_1A_2) &= P(A_1)P(A_2) \\ P(A_1A_3) &= P(A_1)P(A_3) \\ P(A_2A_3) &= P(A_2)P(A_3) \end{aligned}$$

All these equations needed for independence!

Example: Toss a coin twice.

$$\begin{aligned} A_1 &= \{\text{first toss is a Head}\} \\ A_2 &= \{\text{second toss is a Head}\} \\ A_3 &= \{\text{first toss and second toss different}\} \end{aligned}$$

Then $P(A_i) = 1/2$ for each i and for $i \neq j$

$$P(A_i \cap A_j) = \frac{1}{4}$$

but

$$P(A_1 \cap A_2 \cap A_3) = 0 \neq P(A_1)P(A_2)P(A_3)\,.$$

Def'n: X and Y are **independent** if

$$P(X \in A; Y \in B) = P(X \in A)P(Y \in B)$$

for all A and B.

Def'n: Rvs $X_1, \ldots, X_p$ **independent**:

$$P(X_1 \in A_1, \cdots, X_p \in A_p) = \prod P(X_i \in A_i)$$

for any $A_1, \ldots, A_p$.

Theorem:

1. If X_1 and X_p are independent then for all x, y
$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$
2. If X and Y are independent with joint density $f_{X,Y}(x,y)$ then X and Y have densities f_X and f_Y, and
$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$
3. If X and Y independent with marginal densities f_X and f_Y then (X,Y) has joint density
$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$
5. If (X,Y) has density $f(x,y)$ and there exist $g(x)$ and $h(y)$ st $f(x,y) = g(x)h(y)$ for (almost) **all** (x,y) then X and Y are independent with densities given by
$$f_X(x) = g(x) / \int_{-\infty}^{\infty} g(u)du$$
$$f_Y(y) = h(y) / \int_{-\infty}^{\infty} h(u)du\,.$$
4. If $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ for **all** x, y then X and Y are independent.

Proof: See STAT 802

Theorem: If $X_1, \ldots, X_p$ are independent and $Y_i = g_i(X_i)$ then $Y_1, \ldots, Y_p$ are independent. Moreover, $(X_1, \ldots, X_q)$ and $(X_{q+1}, \ldots, X_p)$ are independent.

Conditional probability

Def'n: $P(A|B) = P(AB)/P(B)$ if $P(B) \neq 0$.

Def'n: For discrete X and Y the conditional probability mass function of Y given X is

$$\begin{aligned} f_{Y|X}(y|x) &= P(Y = y|X = x) \\ &= f_{X,Y}(x,y)/f_X(x) \\ &= f_{X,Y}(x,y)/\sum_t f_{X,Y}(x,t) \end{aligned}$$

For absolutely continuous X $P(X = x) = 0$ for all x. What is $P(A|X = x)$ or $f_{Y|X}(y|x)$? Solution: use limit

$$P(A|X = x) = \lim_{\delta x \to 0} P(A|x \leq X \leq x + \delta x)$$

If, e.g., X, Y have joint density $f_{X,Y}$ then with $A = \{Y \leq y\}$ we have

$$\begin{aligned} &P(A|x \leq X \leq x + \delta x) \\ &\quad = \frac{P(A \cap \{x \leq X \leq x + \delta x\})}{P(x \leq X \leq x + \delta x)} \\ &\quad = \frac{\int_{-\infty}^{y} \int_x^{x+\delta x} f_{X,Y}(u,v)dudv}{\int_x^{x+\delta x} f_X(u)du} \end{aligned}$$

Divide top, bottom by δx; let $\delta x \to 0$. Denom converges to $f_X(x)$; numerator converges to

$$\int_{-\infty}^{y} f_{X,Y}(x,v)dv$$

Define conditional cdf of Y given $X = x$:

$$P(Y \leq y|X = x) = \frac{\int_{-\infty}^{y} f_{X,Y}(x,v)dv}{f_X(x)}$$

Differentiate wrt y to get def'n of conditional density of Y given $X = x$:

$$f_{Y|X}(y|x) = f_{X,Y}(x,y)/f_X(x)\,;$$

in words "conditional = joint/marginal".