STAT 801: Mathematical Statistics

Confidence Sets

Definition: A level β confidence set for a parameter $\phi(\theta)$ is a random subset, C, of the set of possible values of ϕ such that for each θ

$$P_\theta(\phi(\theta) \in C) \ge \beta$$

Confidence sets are very closely connected with hypothesis tests:

From confidence sets to tests

Suppose C is a level $\beta = 1 - \alpha$ confidence set for ϕ. Then we may convert C to a family of hypothesis tests. To test $\phi = \phi_0$: reject if $\phi_0 \notin C$. This test has level α.

From tests to confidence sets

Conversely, suppose that for each ϕ_0 we have available a level α test of $\phi = \phi_0$ who rejection region is say R_{ϕ_0}. Define $C = \{\phi_0 : \phi = \phi_0 \text{ is not rejected}\}$; this set C is a level $1 - \alpha$ confidence set for ϕ.
Example: The usual t test gives rise in this way to the usual t confidence intervals

$$\bar{X} \pm t_{n-1,\alpha/2} \frac{s}{\sqrt{n}}.$$

Conversely μ_0 is in the usual confidence interval if and only if the t-statistic for testing $\mu = \mu_0$ is smaller than the corresponding t critical value.

Confidence sets from Pivots

Definition: A **pivot** (or pivotal quantity) is a function $g(\theta, X)$ whose distribution is the same for all θ. (The θ in pivot is same θ as being used to calculate the distribution of $g(\theta, X)$.

We can use pivots to generate confidence sets as follows: Pick a set A in the space of possible values for g. Let $\beta = P_\theta(g(\theta, X) \in A)$; since g is pivotal β is the same for all θ. Now given data X solve the relation

$$g(\theta, X) \in A$$

to get

$$\theta \in C(X, A) .$$

Then $C(X, A)$ is a level β confidence set.
Example: In the $N(\mu, \sigma^2)$ model the quantity $(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$ is a pivot. It leads to confidence intervals for σ as follows. Given $\beta = 1 - \alpha$ consider the two points

$$\chi^2_{n-1,1-\alpha/2} \text{ and } \chi^2_{n-1,\alpha/2}.$$

Then

$$P(\chi^2_{n-1,1-\alpha/2} \le (n-1)s^2/\sigma^2 \le \chi^2_{n-1,\alpha/2}) = \beta$$

for all μ, σ. Now solve this relation to get a set of values for σ:

$$P(\frac{(n-1)^{1/2}s}{\chi_{n-1,\alpha/2}} \le \sigma \le \frac{(n-1)^{1/2}s}{\chi_{n-1,1-\alpha/2}}) = \beta;$$

thus the interval

$$\left[\frac{(n-1)^{1/2}s}{\chi_{n-1,\alpha/2}}, \frac{(n-1)^{1/2}s}{\chi_{n-1,1-\alpha/2}}\right]$$

is a level $\beta = 1-\alpha$ confidence interval.

In the same model we also have

$$P(\chi^2_{n-1,1-\alpha} \le (n-1)s^2/\sigma^2) = \beta$$

which can be solved to get

$$P(\sigma \le \frac{(n-1)^{1/2}s}{\chi_{n-1,1-\alpha}}) = \beta$$

This gives a level $1-\alpha$ interval

$$(0, (n-1)^{1/2}s/\chi_{n-1,1-\alpha}) .$$

The right hand end of this interval is usually called a confidence upper bound.

In general the interval from

$$(n-1)^{1/2}s/\chi_{n-1,\alpha_1} \text{ to } (n-1)^{1/2}s/\chi_{n-1,1-\alpha_2}$$

has level $\beta = 1-\alpha_1-\alpha_2$. For fixed β it is possible to minimize the length of the resulting interval numerically — this procedure is rarely used. See the homework for an example.