Convergence in Distribution

Undergraduate version of central limit theorem: if $X_1, \ldots, X_n$ are iid from a population with mean μ and standard deviation σ then $n^{1/2}(\bar{X} - \mu)/\sigma$ has approximately a normal distribution.

Also Binomial(n, p) random variable has approximately a $N(np, np(1-p))$ distribution.

Precise meaning of statements like "X and Y have approximately the same distribution"?

Desired meaning: X and Y have nearly the same cdf.

But care needed.

Q1) If n is a large number is the $N(0,1/n)$ distribution close to the distribution of $X \equiv 0$?

Q2) Is $N(0,1/n)$ close to the $N(1/n,1/n)$ distribution?

Q3) Is $N(0,1/n)$ close to $N(1/\sqrt{n},1/n)$ distribution?

Q4) If $X_n \equiv 2^{-n}$ is the distribution of X_n close to that of $X \equiv 0$?

Answers depend on how close close needs to be so it's a matter of definition.

In practice the usual sort of approximation we want to make is to say that some random variable X, say, has nearly some continuous distribution, like $N(0,1)$.

So: want to know probabilities like $P(X > x)$ are nearly $P(N(0,1) > x)$.

Real difficulty: case of discrete random variables or infinite dimensions: not done in this course.

Mathematicians' meaning of close:

Either they can provide an upper bound on the distance between the two things or they are talking about taking a limit.

In this course we take limits.

Definition: A sequence of random variables X_n converges in distribution to a random variable X if

$$E(g(X_n)) \to E(g(X))$$

for every bounded continuous function g.

Theorem 1 *The following are equivalent:*

1. *X_n converges in distribution to X.*

2. *$P(X_n \le x) \to P(X \le x)$ for each x such that $P(X = x) = 0$.*

3. *The limit of the characteristic functions of X_n is the characteristic function of X:*

   $$E(e^{itX_n}) \to E(e^{itX})$$

   for every real t.

These are all implied by

$$M_{X_n}(t) \to M_X(t) < \infty$$

for all $|t| \le \epsilon$ for some positive ϵ.

Now let's go back to the questions I asked:

- $X_n \sim N(0, 1/n)$ and $X = 0$. Then

$$P(X_n \le x) \to \begin{cases} 1 & x > 0 \\ 0 & x < 0 \\ 1/2 & x = 0 \end{cases}$$

Now the limit is the cdf of $X = 0$ except for $x = 0$ and the cdf of X is not continuous at $x = 0$ so yes, X_n converges to X in distribution.

- I asked if $X_n \sim N(1/n, 1/n)$ had a distribution close to that of $Y_n \sim N(0, 1/n)$. The definition I gave really requires me to answer by finding a limit X and proving that both X_n and Y_n converge to X in distribution. Take $X = 0$. Then

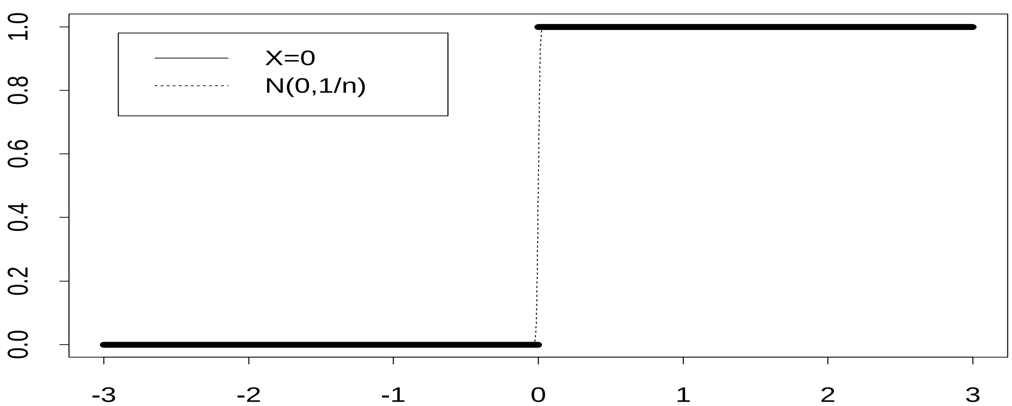
$$E(e^{tX_n}) = e^{t/n + t^2/(2n)} \to 1 = E(e^{tX})$$

and

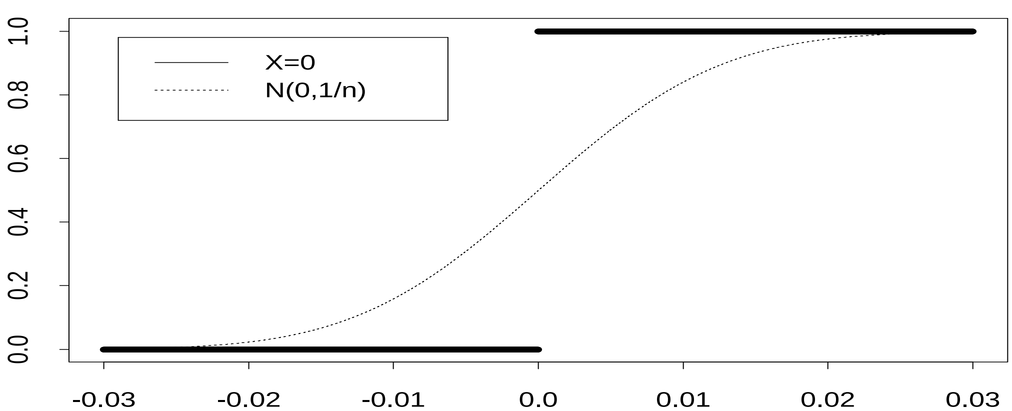
$$E(e^{tY_n}) = e^{t^2/(2n)} \to 1$$

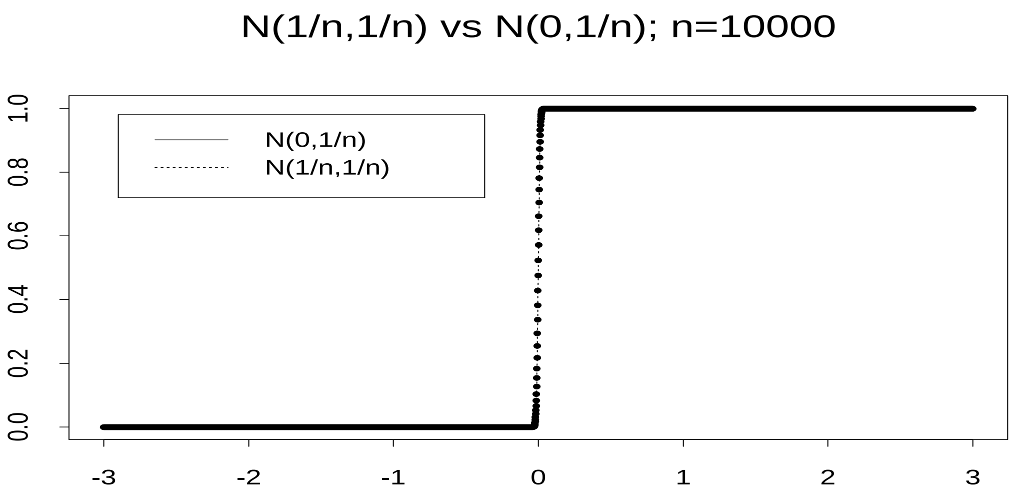
so that both X_n and Y_n have the same limit in distribution.

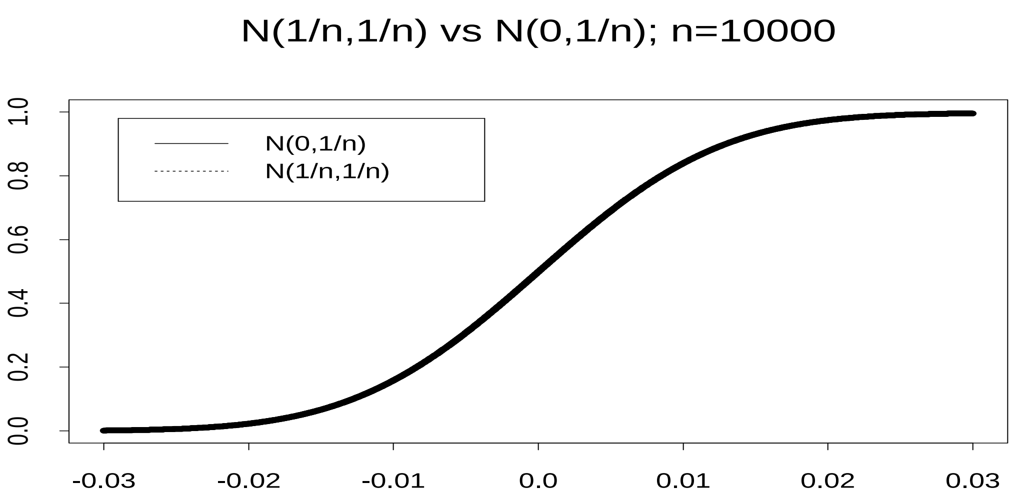
N(0,1/n) vs X=0; n=10000



N(0,1/n) vs X=0; n=10000




N(1/n,1/n) vs N(0,1/n); n=10000
N(0,1/n)
N(1/n,1/n)
1.0
0.8
0.6
0.4
0.2
0.0
-3
-2
-1
0
1
2
3


N(1/n,1/n) vs N(0,1/n); n=10000
N(0,1/n)
N(1/n,1/n)
1.0
0.8
0.6
0.4
0.2
0.0
-0.03
-0.02
-0.01
0.0
0.01
0.02
0.03

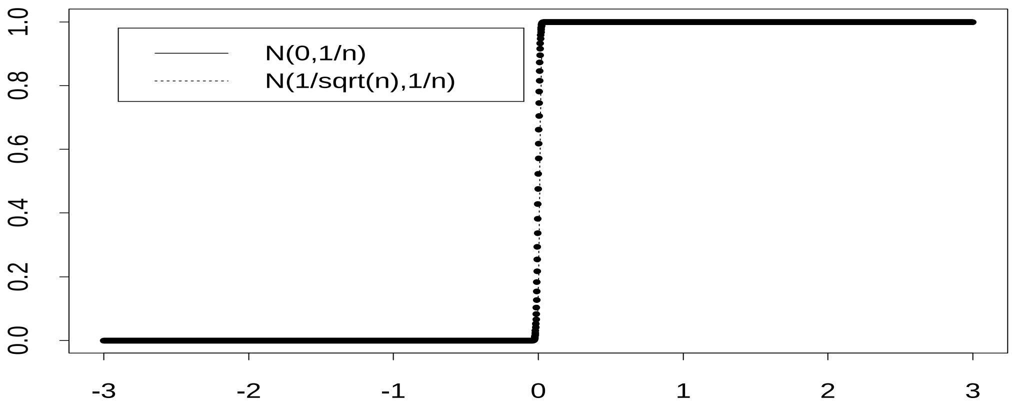
- Multiply both X_n and Y_n by $n^{1/2}$ and let $X \sim N(0,1)$. Then $\sqrt{n}X_n \sim N(n^{-1/2}, 1)$ and $\sqrt{n}Y_n \sim N(0,1)$. Use characteristic functions to prove that both $\sqrt{n}X_n$ and $\sqrt{n}Y_n$ converge to $N(0,1)$ in distribution.

- If you now let $X_n \sim N(n^{-1/2}, 1/n)$ and $Y_n \sim N(0, 1/n)$ then again both X_n and Y_n converge to 0 in distribution.

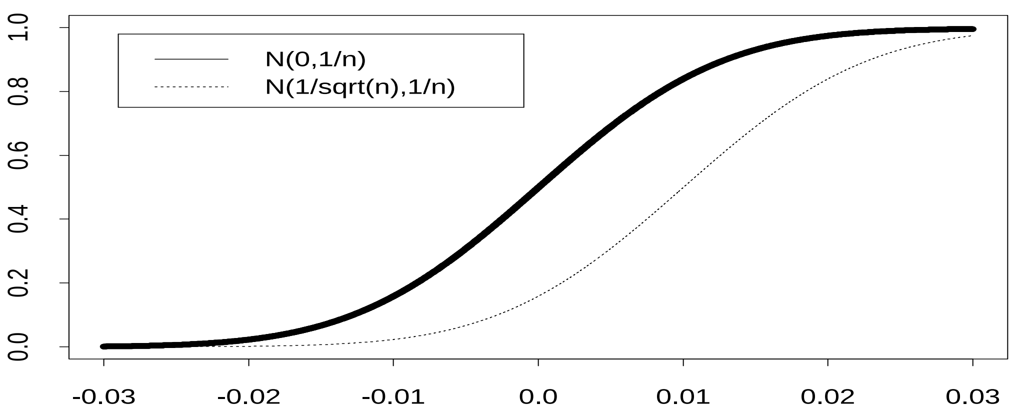
- If you multiply X_n and Y_n in the previous point by $n^{1/2}$ then $n^{1/2}X_n \sim N(1,1)$ and $n^{1/2}Y_n \sim N(0,1)$ so that $n^{1/2}X_n$ and $n^{1/2}Y_n$ are **not** close together in distribution.

- You can check that $2^{-n} \to 0$ in distribution.

N(1/sqrt(n),1/n) vs N(0,1/n); n=10000



N(1/sqrt(n),1/n) vs N(0,1/n); n=10000



Summary: to derive approximate distributions:

Show sequence of rvs X_n converges to some X.

The limit distribution (i.e. dstbon of X) should be non-trivial, like say $N(0,1)$.

Don't say: X_n is approximately $N(1/n, 1/n)$.

Do say: $n^{1/2}(X_n - 1/n)$ converges to $N(0,1)$ in distribution.

The Central Limit Theorem

If $X_1, X_2, \cdots$ are iid with mean 0 and variance 1 then $n^{1/2}\bar{X}$ converges in distribution to $N(0,1)$. That is,

$$P(n^{1/2}\bar{X} \le x) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy\,.$$

Proof: As before

$$E(e^{itn^{1/2}\bar{X}}) \to e^{-t^2/2}\,.$$

This is the characteristic function of $N(0,1)$ so we are done by our theorem.

Edgeworth expansions

In fact if $\gamma = E(X^3)$ then

$$\phi(t) \approx 1 - t^2/2 - i\gamma t^3/6 + \cdots$$

keeping one more term. Then

$$\log(\phi(t)) = \log(1 + u)$$

where

$$u = -t^2/2 - i\gamma t^3/6 + \cdots .$$

Use $\log(1 + u) = u - u^2/2 + \cdots$ to get

$$\log(\phi(t)) \approx \\ [-t^2/2 - i\gamma t^3/6 + \cdots] \\ - [\cdots]^2/2 + \cdots$$

which rearranged is

$$\log(\phi(t)) \approx -t^2/2 - i\gamma t^3/6 + \cdots .$$

Now apply this calculation to

$$\log(\phi_T(t)) \approx -t^2/2 - iE(T^3)t^3/6 + \cdots .$$

Remember $E(T^3) = \gamma/\sqrt{n}$ and exponentiate to get

$$\phi_T(t) \approx e^{-t^2/2} \exp\{-i\gamma t^3/(6\sqrt{n}) + \cdots\} .$$

You can do a Taylor expansion of the second exponential around 0 because of the square root of n and get

$$\phi_T(t) \approx e^{-t^2/2}(1 - i\gamma t^3/(6\sqrt{n}))$$

neglecting higher order terms. This approximation to the characteristic function of T can be inverted to get an **Edgeworth** approximation to the density (or distribution) of T which looks like

$$f_T(x) \approx \frac{1}{\sqrt{2\pi}} e^{-x^2/2}[1 - \gamma(x^3 - 3x)/(6\sqrt{n}) + \cdots] .$$

Remarks:

1. The error using the central limit theorem to approximate a density or a probability is proportional to $n^{-1/2}$.

2. This is improved to n^{-1} for symmetric densities for which $\gamma = 0$.

3. These expansions are **asymptotic**. This means that the series indicated by $\cdots$ usually does **not** converge. When $n = 25$ it may help to take the second term but get worse if you include the third or fourth or more.

4. You can integrate the expansion above for the density to get an approximation for the cdf.

Multivariate convergence in distribution

Definition: $X_n \in R^p$ converges in distribution to $X \in R^p$ if

$$E(g(X_n)) \to E(g(X))$$

for each bounded continuous real valued function g on R^p.

This is equivalent to either of

Cramér Wold Device: $a^t X_n$ converges in distribution to $a^t X$ for each $a \in R^p$.

or

Convergence of characteristic functions:

$$E(e^{ia^t X_n}) \to E(e^{ia^t X})$$

for each $a \in R^p$.

Extensions of the CLT

1. $Y_1, Y_2, \cdots$ iid in R^p, mean μ, variance covariance Σ then $n^{1/2}(\bar{Y} - \mu)$ converges in distribution to $MVN(0, \Sigma)$.

2. Lyapunov CLT: for each n $X_{n1}, \ldots, X_{nn}$ independent rvs with
$$E(X_{ni}) = 0$$
$$\text{Var}(\sum_i X_{ni}) = 1$$
$$\sum E(|X_{ni}|^3) \to 0$$
then $\sum_i X_{ni}$ converges to $N(0,1)$.

3. Lindeberg CLT: 1st two conds of Lyapunov and
$$\sum E(X_{ni}^2 1(|X_{ni}| > \epsilon)) \to 0$$
each $\epsilon > 0$. Then $\sum_i X_{ni}$ converges in distribution to $N(0,1)$. (Lyapunov's condition implies Lindeberg's.)

4. Non-independent rvs: m-dependent CLT, martingale CLT, CLT for mixing processes.

5. Not sums: Slutsky's theorem, δ method.

Slutsky's Theorem: If X_n converges in distribution to X and Y_n converges in distribution (or in probability) to c, a constant, then $X_n + Y_n$ converges in distribution to $X + c$. More generally, if $f(x, y)$ is continuous then $f(X_n, Y_n) \Rightarrow f(X, c)$.

Warning: the hypothesis that the limit of Y_n be constant is essential.

Definition: We say Y_n converges to Y in probability if $\forall \epsilon > 0$:

$$P(|Y_n - Y| > \epsilon) \to 0\,.$$

Fact: for Y constant convergence in distribution and in probability are the same.

Always convergence in probability implies convergence in distribution.

Both are weaker than almost sure convergence:

Definition: We say Y_n converges to Y almost surely if

$$P(\{\omega \in \Omega : \lim_{n\to\infty} Y_n(\omega) = Y(\omega)\}) = 1\,.$$

The delta method: Suppose:

- Sequence Y_n of rvs converges to some y, a constant.

- $X_n = a_n(Y_n - y)$ then X_n converges in distribution to some random variable X.

- f is differentiable ftn on range of Y_n.

Then $a_n(f(Y_n) - f(y))$ converges in distribution to $f'(y)X$.

If $X_n \in R^p$ and $f : R^p \mapsto R^q$ then f' is $q \times p$ matrix of first derivatives of components of f.

Example: Suppose $X_1, \ldots, X_n$ are a sample from a population with mean μ, variance σ^2, and third and fourth central moments μ_3 and μ_4. Then

$$n^{1/2}(s^2 - \sigma^2) \Rightarrow N(0, \mu_4 - \sigma^4)$$

where $\Rightarrow$ is notation for convergence in distribution. For simplicity I define $s^2 = \overline{X^2} - \bar{X}^2$.

How to apply δ method:

1) Write statistic as a function of averages:

Define

$$W_i = \left[\begin{array}{c} X_i^2 \\ X_i \end{array} \right].$$

See that

$$\bar{W}_n = \left[\begin{array}{c} \overline{X^2} \\ \bar{X} \end{array} \right]$$

Define

$$f(x_1, x_2) = x_1 - x_2^2$$

See that $s^2 = f(\bar{W}_n)$.

2) Compute mean of your averages:

$$\mu_W \equiv \mathrm{E}(\bar{W}_n) = \left[\begin{array}{c} \mathrm{E}(X_i^2) \\ \mathrm{E}(X_i) \end{array} \right] = \left[\begin{array}{c} \mu^2 + \sigma^2 \\ \mu \end{array} \right].$$

3) In δ method theorem take $Y_n = \bar{W}_n$ and $y = \mathrm{E}(Y_n)$.

4) Take $a_n = n^{1/2}$.

5) Use central limit theorem:

$$n^{1/2}(Y_n - y) \Rightarrow MVN(0, \Sigma)$$

where $\Sigma = \text{Var}(W_i)$.

6) To compute Σ take expected value of

$$(W - \mu_W)(W - \mu_W)^t$$

There are 4 entries in this matrix. Top left entry is

$$(X^2 - \mu^2 - \sigma^2)^2$$

This has expectation:

$$\text{E}\left\{(X^2 - \mu^2 - \sigma^2)^2\right\} = \text{E}(X^4) - (\mu^2 + \sigma^2)^2.$$

Using binomial expansion:

$$\begin{aligned}\mathsf{E}(X^4) &= \mathsf{E}\{(X-\mu+\mu)^4\} \\ &= \mu_4 + 4\mu\mu_3 + 6\mu^2\sigma^2 \\ &\quad + 4\mu^3\mathsf{E}(X-\mu) + \mu^4 \,.\end{aligned}$$

So

$$\Sigma_{11} = \mu_4 - \sigma^4 + 4\mu\mu_3 + 4\mu^2\sigma^2$$

Top right entry is expectation of

$$(X^2 - \mu^2 - \sigma^2)(X-\mu)$$

which is

$$\mathsf{E}(X^3) - \mu\mathsf{E}(X^2)$$

Similar to 4th moment get

$$\mu_3 + 2\mu\sigma^2$$

Lower right entry is σ^2.

So

$$\Sigma = \begin{bmatrix} \mu_4 - \sigma^4 + 4\mu\mu_3 + 4\mu^2\sigma^2 & \mu_3 + 2\mu\sigma^2 \\ \mu_3 + 2\mu\sigma^2 & \sigma^2 \end{bmatrix}$$

7) Compute derivative (gradient) of f: has components $(1, -2x_2)$. Evaluate at $y = (\mu^2 + \sigma^2, \mu)$ to get

$$a^t = (1, -2\mu) .$$

This leads to

$$n^{1/2}(s^2 - \sigma^2) \approx n^{1/2}[1, -2\mu] \left[\begin{array}{c} \overline{X^2} - (\mu^2 + \sigma^2) \\ \bar{X} - \mu \end{array} \right]$$

which converges in distribution to

$$(1, -2\mu) MVN(0, \Sigma) .$$

This rv is $N(0, a^t \Sigma a) = N(0, \mu_4 - \sigma^4)$.

Alternative approach worth pursuing. Suppose c is constant.

Define $X_i^* = X_i - c$.

Then: sample variance of X_i^* is same as sample variance of X_i.

Notice all central moments of X_i^* same as for X_i. Conclusion: no loss in $\mu = 0$. In this case:

$$a^t = (1, 0)$$

and

$$\Sigma = \left[\begin{array}{cc} \mu_4 - \sigma^4 & \mu_3 \\ \mu_3 & \sigma^2 \end{array} \right] .$$

Notice that

$$a^t \Sigma = [\mu_4 - \sigma^4, \mu_3]$$

and

$$a^t \Sigma a = \mu_4 - \sigma^4 .$$

Special case: if population is $N(\mu, \sigma^2)$ then $\mu_3 = 0$ and $\mu_4 = 3\sigma^4$. Our calculation has

$$n^{1/2}(s^2 - \sigma^2) \Rightarrow N(0, 2\sigma^4)$$

You can divide through by σ^2 and get

$$n^{1/2}(\frac{s^2}{\sigma^2} - 1) \Rightarrow N(0, 2)$$

In fact ns^2/σ^2 has a χ^2_{n-1} distribution and so the usual central limit theorem shows that

$$(n-1)^{-1/2}[ns^2/\sigma^2 - (n-1)] \Rightarrow N(0, 2)$$

(using mean of χ^2_1 is 1 and variance is 2).

Factor out n to get

$$\sqrt{\frac{n}{n-1}} n^{1/2}(s^2/\sigma^2 - 1) + (n-1)^{-1/2} \Rightarrow N(0, 2)$$

which is δ method calculation except for some constants.

Difference is unimportant: Slutsky's theorem.