Hypothesis Testing and Decision Theory

Decision analysis of hypothesis testing takes $D = \{0, 1\}$ and

$$L(d, \theta) = 1(\text{make an error})$$

or more generally $L(0, \theta) = \ell_1 1(\theta \in \Theta_1)$ and $L(1, \theta) = \ell_2 1(\theta \in \Theta_0)$ for two positive constants ℓ_1 and ℓ_2. We make the decision space convex by allowing a decision to be a probability measure on D. Any such measure can be specified by $\delta = P(\text{reject})$ so $\mathcal{D} = [0, 1]$. The loss function of $\delta \in [0, 1]$ is

$$L(\delta, \theta) = (1 - \delta)\ell_1 1(\theta \in \Theta_1) + \delta \ell_0 1(\theta \in \Theta_0)\,.$$

Simple hypotheses: Prior is $\pi_0 > 0$ and $\pi_1 > 0$ with $\pi_0 + \pi_1 = 1$.

Procedure: map from sample space to $\mathcal{D}$ – a test function.

Risk function of procedure $\phi(X)$ is a pair of numbers:

$$R_\phi(\theta_0) = E_0(L(\delta, \theta_0))$$

and

$$R_\phi(\theta_1) = E_1(L(\delta, \theta_1))$$

We find

$$R_\phi(\theta_0) = \ell_0 E_0(\phi(X)) = \ell_0 \alpha$$

and

$$R_\phi(\theta_1) = \ell_1 E_1(1 - \phi(X)) = \ell_1 \beta$$

The Bayes risk of ϕ is

$$\pi_0\ell_0\alpha + \pi_1\ell_1\beta$$

We saw in the hypothesis testing section that this is minimized by

$$\phi(X) = 1(f_1(X)/f_0(X) > \pi_0\ell_0/(\pi_1\ell_1))$$

which is a likelihood ratio test. These tests are Bayes and admissible. The risk is constant if $\beta\ell_1 = \alpha\ell_0$; you can use this to find the minimax test in this context.