Definition: A test ϕ of Θ_0 against Θ_1 is unbiased level α if it has level α and, for every $\theta \in \Theta_1$ we have

$$\pi(\theta) \geq \alpha\,.$$

When testing a point null hypothesis like $\mu = \mu_0$ this requires that the power function be minimized at μ_0 which will mean that if π is differentiable then

$$\pi'(\mu_0) = 0$$

Example: $N(\mu,1)$: data $X=(X_1,\ldots,X_n)$. If ϕ is any test function then

$$\pi'(\mu) = \frac{\partial}{\partial\mu}\int \phi(x) f(x,\mu) dx$$

Differentiate under the integral and use

$$\frac{\partial f(x,\mu)}{\partial\mu} = \sum (x_i-\mu) f(x,\mu)$$

to get the condition

$$\int \phi(x)\bar{x} f(x,\mu_0) dx = \mu_0\alpha_0$$

Minimize $\beta(\mu)$ subject to two constraints

$$E_{\mu_0}(\phi(X)) = \alpha_0$$

and

$$E_{\mu_0}(\bar{X}\phi(X)) = \mu_0\alpha_0.$$

Fix two values $\lambda_1 > 0$ and λ_2 and minimize

$$\lambda_1\alpha + \lambda_2 E_{\mu_0}[(\bar{X} - \mu_0)\phi(X)] + \beta$$

The quantity in question is just

$$\int [\phi(x) f_0(x)(\lambda_1 + \lambda_2(\bar{x} - \mu_0)) \\ + (1 - \phi(x)) f_1(x)] dx\,.$$

As before this is minimized by

$$\phi(x) = \begin{cases} 1 & \frac{f_1(x)}{f_0(x)} > \lambda_1 + \lambda_2(\bar{x} - \mu_0) \\ 0 & \frac{f_1(x)}{f_0(x)} < \lambda_1 + \lambda_2(\bar{x} - \mu_0) \end{cases}$$

The likelihood ratio f_1/f_0 is simply

$$\exp\{n(\mu_1 - \mu_0)\bar{X} + n(\mu_0^2 - \mu_1^2)/2\}$$

and this exceeds the linear function

$$\lambda_1 + \lambda_2(\bar{X} - \mu_0)$$

for all $\bar{X}$ sufficiently large or small. That is,

$$\lambda_1\alpha + \lambda_2 E_{\mu_0}[(\bar{X} - \mu_0)\phi(X)] + \beta$$

is minimized by a rejection region of the form

$$\{\bar{X} > K_U\} \cup \{\bar{X} < K_L\}$$

Satisfy constraints: adjust K_U and K_L to get level α and $\pi'(\mu_0) = 0$. 2nd condition shows rejection region symmetric about μ_0 so test rejects for

$$\sqrt{n}|\bar{X} - \mu_0| > z_{\alpha/2}$$

Mimic Neyman Pearson lemma proof to check that if λ_1 and λ_2 are adjusted so that the unconstrained problem has the rejection region given then the resulting test minimizes β subject to the two constraints.

A test ϕ^* is a Uniformly Most Powerful Unbiased level α_0 test if

1. ϕ^* has level $\alpha \le \alpha_0$.

2. ϕ^* is unbiased.

3. If ϕ has level $\alpha \le \alpha_0$ and ϕ is unbiased then for every $\theta \in \Theta_1$ we have

$$E_\theta(\phi(X)) \le E_\theta(\phi^*(X))$$

Conclusion: The two sided z test which rejects if

$$|Z| > z_{\alpha/2}$$

where

$$Z = n^{1/2}(\bar{X} - \mu_0)$$

is the uniformly most powerful unbiased test of $\mu = \mu_0$ against the two sided alternative $\mu \ne \mu_0$.

Nuisance Parameters

The t-test is UMPU.

Suppose $X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$. Test $\mu = \mu_0$ or $\mu \le \mu_0$ against $\mu > \mu_0$. Parameter space is two dimensional; boundary between the null and alternative is

$$\{(\mu, \sigma); \mu = \mu_0, \sigma > 0\}$$

If a test has $\pi(\mu, \sigma) \le \alpha$ for all $\mu \le \mu_0$ and $\pi(\mu, \sigma) \ge \alpha$ for all $\mu > \mu_0$ then $\pi(\mu_0, \sigma) = \alpha$ for all σ because the power function of any test must be continuous. (Uses dominated convergence theorem; power function is an integral.)

Think of $\{(\mu, \sigma); \mu = \mu_0\}$ as parameter space for a model. For this parameter space

$$S = \sum (X_i - \mu_0)^2$$

is complete and sufficient. Remember definitions of both completeness and sufficiency depend on the parameter space.

Suppose $\phi(\sum X_i, S)$ is an unbiased level α test. Then we have

$$E_{\mu_0,\sigma}(\phi(\sum X_i, S)) = \alpha$$

for all σ. Condition on S and get

$$E_{\mu_0,\sigma}[E(\phi(\sum X_i, S)|S)] = \alpha$$

for all σ. Sufficiency guarantees that

$$g(S) = E(\phi(\sum X_i, S)|S)$$

is a statistic and completeness that

$$g(S) \equiv \alpha$$

Now let us fix a single value of σ and a $\mu_1 > \mu_0$. To make our notation simpler I take $\mu_0 = 0$. Our observations above permit us to condition on $S = s$. Given $S = s$ we have a level α test which is a function of $\bar{X}$.

If we maximize the conditional power of this test for each s then we will maximize its power. What is the conditional model given $S = s$? That is, what is the conditional distribution of $\bar{X}$ given $S = s$? The answer is that the joint density of $\bar{X}, S$ is of the form

$$f_{\bar{X},S}(t,s) = h(s,t)\,\text{exp}\{\theta_1 t + \theta_2 s + c(\theta_1,\theta_2)\}$$

where $\theta_1 = n\mu/\sigma^2$ and $\theta_2 = -1/\sigma^2$.

This makes the conditional density of $\bar{X}$ given $S = s$ of the form

$$f_{\bar{X}|s}(t|s) = h(s,t)\,\text{exp}\{\theta_1 t + c^*(\theta_1, s)\}$$

Note disappearance of θ_2 and null is $\theta_1 = 0$. This permits application of NP lemma to the conditional family to prove that UMP unbiased test has form

$$\phi(\bar{X}, S) = 1(\bar{X} > K(S))$$

where $K(S)$ chosen to make conditional level α. The function $x \mapsto x/\sqrt{a - x^2}$ is increasing in x for each a so that we can rewrite ϕ in the form

$$\phi(\bar{X}, S) = 1(n^{1/2}\bar{X}/\sqrt{n[S/n - \bar{X}^2]/(n-1)} > K^*(S))$$

for some K^*. The quantity

$$T = \frac{n^{1/2}\bar{X}}{\sqrt{n[S/n - \bar{X}^2]/(n-1)}}$$

is the usual t statistic and is exactly independent of S (see Theorem 6.1.5 on page 262 in Casella and Berger). This guarantees that

$$K^*(S) = t_{n-1,\alpha}$$

and makes our UMPU test the usual t test.

Optimal tests

- A good test has $\pi(\theta)$ large on the alternative and small on the null.

- For one sided one parameter families with MLR a UMP test exists.

- For two sided or multiparameter families the best to be hoped for is UMP Unbiased or Invariant or Similar.

- Good tests are found as follows:
  1. Use the NP lemma to determine a good rejection region for a simple alternative.
  2. Try to express that region in terms of a statistic whose definition does not depend on the specific alternative.
  3. If this fails impose an additional criterion such as unbiasedness. Then mimic the NP lemma and again try to simplify the rejection region.