STAT 802

Assignment 1

1. Suppose $X^T = (X_1^T, X_2^T)$ has aMVN_{p+q} distribution with μ partitioned as $\mu^T = (\mu_1^T, \mu_2^T)$ and Variance covariance

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

   (a) Show that $\text{Cov}(X_1, X_2) = 0$ if and only if X_1 and X_2 are independent. Use my definition of MVN; you are not allowed to assume that X has a density. A mathematically careful argument may rely on the fact that if $Y_1, \ldots, Y_r$ are independent and ϕ_i are (measurable) functions then $\phi_1(Y_1), \ldots, \phi_r(Y_r)$ are independent.

   (b) Show that whether or not Σ_{11} is singular, each column of Σ_{12} is in the column space of Σ_{11}.

   Hint: one way to do this is to use the following outline.

      i. Define P and Λ to be the matrix of orthonormal eigenvectors and the diagonal matrix of corresponding eigenvalues of Σ_{11} I showed in class. The columns of P are a basis of p dimensional space if X_1 has p components. You can write any vector v as a linear combination of the columns of P, that is, in the form $Px = v$.

      ii. Suppose v is a column of Σ_{12} and write $v = \sum b_i P_i$ where the P_i are the columns of P. You are supposed to find an x such that $\Sigma_{11}x = P\Lambda P^t x = v$. If $x = \sum a_i P_i$ then use the orthogonality of the P_i to derive a relationship between a_i, λ_i and b_i.

      iii. If $\lambda_i \neq 0$ you can solve this for a_i while if $\lambda_i = 0$ then the relationship in the previous part is true anyway. This gives you a formula for x as a linear combination of the P_i.

   (c) Show that there is a matrix A which is $p \times q$ such that $\Sigma_{12} = \Sigma_{11} A$.

   (d) Let h be a column of Σ_{12} and v_1 and v_2 be any vectors such that $\Sigma_{11} v_i = h$ for $i = 1, 2$. Show that $\Sigma_{21} v_1 = \Sigma_{21} v_2$.

   (e) Show that if $\Sigma_{11} A_i = \Sigma_{12}$ for $i = 1, 2$ then $\Sigma_{21} A_1 = \Sigma_{21} A_2$.

(f) Show that $a^T\Sigma_{11}a = 0$ implies $P(a^TX_1 = a^T\mu_1) = 1$ for any $a \in R^p$.

(g) Suppose x is such that $a^T\Sigma_{11}a = 0$ implies $a^Tx = a^T\mu$. Show that the conditional distribution of X_2 given $X_1 = x$ is well-defined and is multivariate normal with mean

$$\mu_1 + \Sigma_{21}a$$

and variance covariance

$$\Sigma_{22} - \Sigma_{21}A$$

where a is any solution of $\Sigma_{11}a = (x - \mu_1)$ and A is any solution of $\Sigma_{11}A = \Sigma_{12}$.

NOTE: Most facts about $MVN(\mu, \Sigma)$ variates X can be demonstrated by writing $X = AZ + \mu$ for well chosen A.

2. Fix $a \in R^p$. Minimize and maximize $f(x) = (a^Tx)^2$ subject to $x^Tx = 1$.

3. Fix $a \in R^p$. Minimize and maximize $f(x) = a^Tx$ subject to $x^TQx = 1$ for a given symmetric positive definite matrix Q.

4. Write out the spectral decomposition of the matrix

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and then find a symmetric square root.

5. Suppose $X = (X_1, X_2, X_3)$ has a MVN distribution with $\mu = 0$ and

$$\Sigma = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

Find the conditional distribution of X_1 given $X_2 = x_2$ and $X_3 = x_3$. For which values of x_2 and x_3 does this make sense?