STAT 804: 2004-01

Assignment 2

1. Consider the ARIMA(1,0,1) process

$$X_t - \phi X_{t-1} = \epsilon_t - \psi\epsilon_{t-1} \, .$$

   Show that the autocorrelation function is

$$\rho(1) = \frac{(1-\psi\phi)(\phi-\psi)}{1+\psi^2-2\psi\phi}$$

   and

$$\rho(k) = \phi^{k-1}\rho(1) \qquad k = 2, 3, \ldots$$

   Plot the autocorrelation functions for the ARMA(1,1) process above, the AR(1) process with

$$X_t = \phi X_{t-1} + \epsilon_t$$

   and the MA(1) process

$$X_t = \epsilon_t - \psi\epsilon_{t-1}$$

   on the same plot when $\phi = 0.6$ and $\theta = -0.9$. Compute and plot the partial autocorrelation functions up to lag 30. Comment on the usefulness of these plots in distinguishing the three models. Explain what goes wrong when ϕ is close to ψ.

2. Suppose Φ is a Uniform$[0, 2\pi]$ random variable. Define

$$X_t = \cos(\omega t + \Phi) \, .$$

   Show that X is weakly stationary. (In fact it is strongly stationary so show that if you can.) Compute the autocorrelation function of X.

3. Show that X of the previous question satisfies the AR(2) model

$$X_t = (2-\lambda^2)X_{t-1} - X_{t-2}$$

   for some value of λ. Show that the roots of the characteristic polynomial lie on the boundary of the unit circle in the complex plain. (Hint: show that $e^{i\theta}$ is a root if θ is chosen correctly. Do not spend too much time on this question; the point is to illustrate that AR(2) models can be found whose behaviour is much like a sinusoid.)

4. Suppose that X_t is an ARMA(1,1) process

$$X_t - \rho X_{t-1} = \epsilon_t - \theta\epsilon_{t-1}$$

   (a) Suppose we mistakenly fit an AR(1) model (mean 0) to X using the Yule-Walker estimate

$$\hat\rho = \left(\sum_1^{T-1} X_t X_{t-1}\right) / \left(\sum_0^{T-1} X_t^2\right)$$

   In terms of θ, ρ and σ what is $\hat\rho$ close to?

   (b) If we use this AR(1) estimate $\hat\rho$ and calculate residuals using $\hat\epsilon_t = X_t - \hat\rho X_{t-1}$ what kind of time series is $\hat\epsilon$? What will plots of the Autocorrelation and Partial Autocorrelation functions of this residual series look like?