STAT 804: Lecture 6

The Multivariate Normal Distribution

Definition: $Z \in R^1 \sim N(0,1)$ iff

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

Definition: $Z \in R^p \sim MVN_p(0,I)$ iff $Z = (Z_1,\ldots,Z_p)'$ (a column vector for later use) with the Z_i independent and each $Z_i \sim N(0,1)$.

In this case,

$$\begin{aligned} f_Z(z_1,\ldots,z_p) &= \prod \frac{1}{\sqrt{2\pi}} e^{-z_i^2/2} \\ &= (2\pi)^{-p/2} \exp\{-z'z/2\}; \end{aligned}$$

superscript $'$ denotes matrix transpose.

Definition: $X \in R^p$ has a multivariate normal distribution if it has the same distribution as $AZ + \mu$ for some $\mu \in R^p$, some $p \times q$ matrix of constants A and $Z \sim MVN_q(0,I)$.

Lemma: The matrix A can be taken to be square with no loss of generality.

Proof: The simplest proof involves multivariate characteristic functions:

$$\phi_X(t) = \mathrm{E}(e^{it'X})$$

You check that for $Z \sim MVN_p(0, I)$ we have

$$\phi_Z(t) = e^{-t't/2}$$

Then

$$\begin{aligned}\phi_{AZ+\mu}(u) &= \mathrm{E}(e^{iu'(AZ+\mu)}) \\ &= e^{iu'\mu}\phi_Z(A'u) \\ &= e^{iu'\mu - u'AA'u/2}\end{aligned}$$

Result depends only on μ and $\Sigma = AA'$.

So: distribution is same for any two A giving same AA'.

Given $A : p \times q$ use Cholesky decomposition to find square B with $BB' = AA'$. I omit a proof.

A singular: X does not have a density.

A invertible: derive MVN density by change of variables:

$$X = AZ + \mu \Leftrightarrow Z = A^{-1}(X - \mu)$$

We get

$$f_X(x) = (2\pi)^{-p/2}\text{det}(\Sigma)^{-1/2} \times \exp\{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)\}$$

where $\Sigma = AA'$.

Properties of the MVN distribution

1. If $X \sim MVN(\mu, \Sigma)$ then

$$MX + b \sim MVN(A\mu + b, A\Sigma A')$$

follows easily from our definition.

2 All margins are multivariate normal: if

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

then $X \sim MVN(\mu, \Sigma)$ implies that $X_1 \sim MVN(\mu_1, \Sigma_{11})$. (Special case of the previous property.)

3 All conditionals are normal: the conditional distribution of X_1 given $X_2 = x_2$ is MVN:

$$\text{E}(X_1 | X_2 = x_2) = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$$

and

$$\text{Var}(X_1 | X_2 = x_2) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}.$$

Proof: algebra with definition of the conditional density.

Note: meaningful even if Σ_{22} is singular.

Partial Correlations

If X is an $AR(p)$ process then

$$X_t = \sum_{j=1}^{p} a_j X_{t-j} + \epsilon_t$$

If you hold $X_{t-1}, \ldots, X_{t-p}$ fixed then X_t is ϵ_t plus a fixed constant.

Since ϵ_t is independent of X_s for all $s < t$ we see that given $X_{t-1}, \ldots, X_{t-p}$ the variables X_t and X_{t-r} for $r > p$ are independent.

This would imply

$$\text{Cov}(X_t, X_{t-q} | X_{t-1}, \ldots, X_{t-p}) = 0$$

for all $q > p$.

Gaussian X: results above on conditional distributions guarantee that we can calculate this **partial autocovariance** from the variance covariance matrix of $(X_t, X_{t-q}, X_{t-1}, \ldots, X_{t-p})$.

Partition this vector into a first piece with 2 entries and a second piece with p entries.

Conditional covariance of (X_t, X_{t-q}) given the others is

$$\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

In this formula Σ_{11} is the 2 by 2 unconditional covariance matrix of (X_t, X_{t-q}), namely,

$$\begin{bmatrix} C(0) & C(q) \\ C(q) & C(0) \end{bmatrix}$$

Matrix Σ_{22}: $p \times p$ Toeplitz matrix with $C(0)$ down diagonal, $C(1)$ down first sub and super diagonal, and so on.

The matrix $\Sigma_{12} = \Sigma_{21}'$ is the $2 \times p$ matrix

$$\begin{bmatrix} C(1) & \cdots & C(p) \\ C(q-1) & \cdots & C(q-p) \end{bmatrix}$$

Calculate resulting 2×2 matrix; pick out off diagonal element; this is the partial autocovariance.

Even for non-Gaussian data use same arithmetic with autocovariance to define partial autocovariance.

Define partial autocorrelation function:

$$PACF(h) = \text{Corr}(X_t, X_{t-h} | X_{t-1}, \ldots, X_{t-h+1})$$

If $h = 1$ then there are no Xs between X_t and X_{t-h} so there is nothing to condition on. This makes $PACF(1) = ACF(1)$.

Qualitative idea: plot sample partial autocorrelation function (replace C_X by $\widehat{C}_X$ in defn of $PACF$). For an $AR(p)$ process this sample plot ought to drop suddenly to 0 for $h > p$.

$AR(1)$ **example**

Calculate $PACF(2)$ for mean 0 $AR(1)$ process. Have $C_X(h) = C_X(0)\rho^{|h|}$. Let

$$\left[\begin{array}{c} Y \\ Z \end{array}\right] = \left[\begin{array}{c} X_t \\ X_{t-2} \\ X_{t-1} \end{array}\right]$$

with $Y = (X_t, X_{t-2})'$ and $Z = X_{t-1}$. Then $(Y, Z)'$ has a $MVN(0, \Sigma)$ distribution with

$$\Sigma = C_X(0) \left[\begin{array}{ccc} 1 & \rho^2 & \rho \\ \rho^2 & 1 & \rho \\ \rho & \rho & 1 \end{array}\right]$$

Following the partitioning into Y and Z we find

$$\Sigma_{11} = \Sigma = C_X(0) \left[\begin{array}{cc} 1 & \rho^2 \\ \rho^2 & 1 \end{array}\right]$$

$$\Sigma_{12} = C_X(0) \left[\begin{array}{c} \rho \\ \rho \end{array}\right]$$

and

$$\Sigma_{22} = C_X(0)\,[1]$$

Get

$$
\begin{aligned}
\text{Var}(X_t, X_{t-2} | X_{t-1}) \\
&= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \\
&= C_X(0) \begin{bmatrix} 1 & \rho^2 \\ \rho^2 & 1 \end{bmatrix} - C_X(0) \begin{bmatrix} \rho^2 & \rho^2 \\ \rho^2 & \rho^2 \end{bmatrix} \\
&= C_X(0) \begin{bmatrix} 1-\rho^2 & 0 \\ 0 & 1-\rho^2 \end{bmatrix}
\end{aligned}
$$

Now to compute the correlation we divide the conditional covariance by the product of the two conditional SDs, that is:

$$
\begin{aligned}
PACF(2) &= \frac{\text{Cov}(X_t, X_{t-2} | X_{t-1})}{\sqrt{\text{Var}(X_t | X_{t-1}) \text{Var}(X_{t-2} | X_{t-1})}} \\
&= \frac{0}{\sqrt{(1-\rho^2)(1-\rho^2)}} \\
&= 0
\end{aligned}
$$

Problem: predict X_t knowing $X_{t-1},\ldots,X_{t-K}$.

Choose predictor $\hat{X}_t = f(X_{t-1},\ldots,X_{t-K})$ to minimize mean squared prediction error:

$$\mathrm{E}[(X_t - \hat{X}_t)^2]$$

Solution: take

$$\hat{X}_t = \mathrm{E}(X_t | X_{t-1},\ldots,X_{t-K})$$

For multivariate normal data this predictor is the usual regression solution:

$$\mathrm{E}(X_t | X_{t-1},\ldots,X_{t-K}) = \mu + A \begin{bmatrix} X_{t-1} - \mu \\ \vdots \\ X_{t-k} - \mu \end{bmatrix}$$

where the matrix A is $\Sigma_{12}\Sigma_{22}^{-1}$ in the notation of the earlier notes. This can be written in the form

$$\hat{X}_t = \mu + \sum_1^k \alpha_i (X_{t-i} - \mu)$$

for suitable constants α. Let $Y_t = X_t - \mu$.

The α_i minimize

$$\begin{aligned} \mathsf{E}[(Y_t - \sum \alpha_i Y_{t-i})^2] \\ &= \mathsf{E}(Y_t^2) - 2\sum \alpha_i \mathsf{E}(Y_t Y_{t-i}) \\ &\quad + \sum_{ij} \alpha_i \alpha_j \mathsf{E}(Y_{t-i} Y_{t-j}) \\ &= C(0) - 2\sum \alpha_i C(i) + \sum_{ij} \alpha_i \alpha_j C(i-j) \end{aligned}$$

Take the derivative with respect to α_m and set the resulting k quantities equal to 0 giving

$$C(m) = \sum_j \alpha_j C(m-j)$$

or, dividing by $C(0)$

$$\rho(m) = \sum_j \alpha_j \rho(m-j)$$

(These are the Yule Walker equations again!) It is a fact that the solution α_m is

$$\text{Corr}(X_t, X_{t-m} | X_{t-1}, \ldots, X_{t-k} \text{ but not } X_{t-m})$$

so that α_k (the last α) is $PACF(k)$.

Remark: This makes it look as if you would compute the first 40 values of the PACF by solving 40 different problems and picking out the last α in each one but in fact there is an explicit recursive algorithm to compute these things.

Estimation of PACF

To estimate the PACF you can either estimate the ACF and do the arithmetic above with estimates instead of theoretical values or minimize a sample version of the mean squared prediction error:

Mean Squared Prediction Error is

$$\mathsf{E}\left\{[(X_t - \mu - \sum_{j=1}^{k} \alpha_j (X_{t-j} - \mu)]^2\right\}$$

Estimate μ by $\bar{X}$; replace expected value with average over the data.

So: minimize

$$\sum_{t=k}^{T-1} \left\{[(X_t - \bar{X} - \sum_{j=1}^{k} \alpha_j (X_{t-j} - \bar{X})]^2\right\} / T$$

Least squares problem regressing vector

$$Y = \begin{bmatrix} X_k - \bar{X} \\ \vdots \\ X_{T-1} - \bar{X} \end{bmatrix}$$

on design matrix

$$Z = \begin{bmatrix} X_{t-1} - \bar{X} & \cdots & X_0 - \bar{X} \\ \vdots & \vdots & \vdots \\ X_{T-2} - \bar{X} & \cdots & X_{T-k-1} - \bar{X} \end{bmatrix}$$

The estimate of $PACF(k)$ is the kth entry in

$$\hat{\alpha} = (Z'Z)^{-1} Z'Y \, .$$

Some example plots

Plots of 5 series, fitted ACFs and fitted PACFs. The series are:

1. A generated $MA(2)$ series:

$$X_t = \epsilon_t + 0.8\epsilon_{t-1} - 0.9\epsilon_{t-2}$$

2. A generated $AR(2)$ series

$$X_t = \epsilon_t + X_{t-1} - 0.99X_{t-2}$$

3. A generated $AR(3)$ series:

$$X_t = \epsilon_t + 0.8X_{t-1} - X_{t-2}/3 + .8X_{t-3}/\sqrt{3}$$

4. Monthly sunspot counts over a 200 year period.

5. Annual rain fall measurements in New York City.

Here is the SPlus code I used to make the following plots. You should note the use of the function `acf` which calculates and plots ACF and PACF.

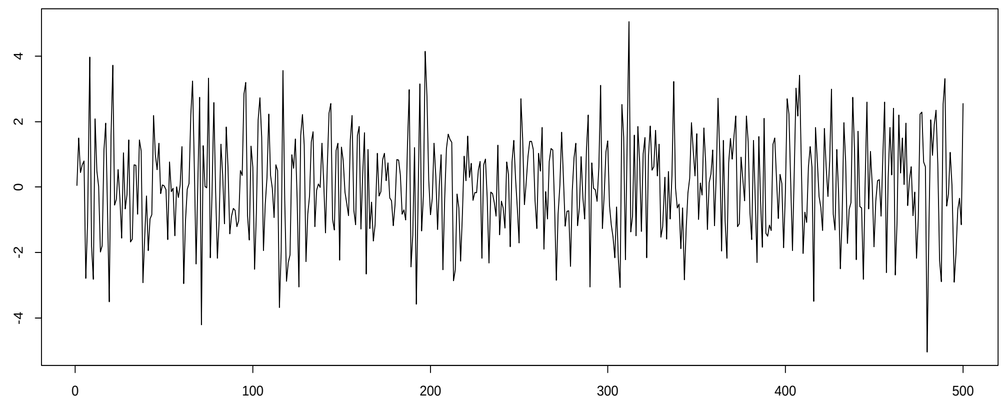
```
#
# Comments begin with #
#
# Begin by generating the series.
# AR series generated are NOT stationary.
# But: generate 10000 values from
# recurrence relation; use last 500.
# Asymptotic stationarity mean
# last 500 are pretty stationary.
#
n<- 10000
ep <- rnorm(n)
```

```
ma2 <-ep[3:502]+0.8*ep[2:501]-0.9*ep[1:500]
ar2 <- rep(0,n)
ar2[1:2] <- ep[1:2]
for(i in 3:n) {ar2[i] <- ep[i]
              + ar2[i-1] -0.99*ar2[i-2]}
ar2 <- ar2[(n-499):n]
ar2 <- ts(ar2)
ar3 <- rep(0,n)
ar3[1:3] <- ep[1:3]
for(i in 4:n){ar3[i] <- ep[i] + 0.8*ar3[i-1]
          -ar3[i-2]/3 +(0.8/1.712)*ar3[i-3]}
ar3 <- ar3[(n-499):n]
ar3 <- ts(ar3)
#
# The next line turns on a graphics device
#    -- in this case the graph will be
# made in postscript in file called ma2.ps.
# It will come out in portrait,
#   not landscape, format.
#
postscript(file="ma2.ps",horizontal=F)
```

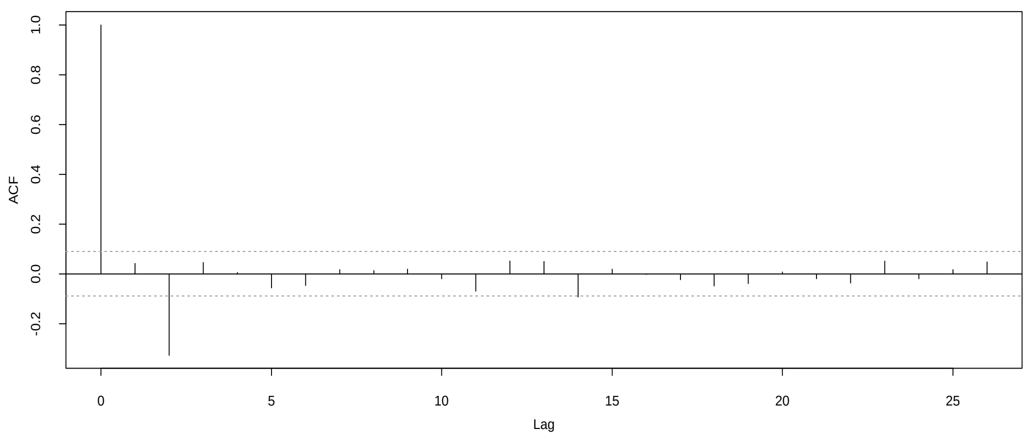
```
#
# The next line says to put 3 pictures
# in a single column on the plot
#
par(mfcol=c(3,1))
tsplot(ma2,main="MA(2) series")
acf(ma2)
acf(ma2,type="partial")
#
# When you finish a picture you turn
# off the graphics device
# with the next line.
#
dev.off()
#
#
#
postscript(file="ar2.ps",horizontal=F)
par(mfcol=c(3,1))
tsplot(ar2,main="AR(2) series")
acf(ar2)
acf(ar2,type="partial")
dev.off()
```

```
#
#
postscript(file="ar3.ps",horizontal=F)
par(mfcol=c(3,1))
tsplot(ar3,main="AR(3) series")
acf(ar3)
acf(ar3,type="partial")
dev.off()
#
#
postscript(file="sunspots.ps",horizontal=F)
par(mfcol=c(3,1))
tsplot(sunspots,main="Sunspots series")
acf(sunspots,lag.max=480)
acf(sunspots,lag.max=480,type="partial")
dev.off()
#
#
postscript(file="rain.nyc1.ps",horizontal=F)
par(mfcol=c(3,1))
tsplot(rain.nyc1,
            main="New York Rain Series")
acf(rain.nyc1)
acf(rain.nyc1,type="partial")
dev.off()
```

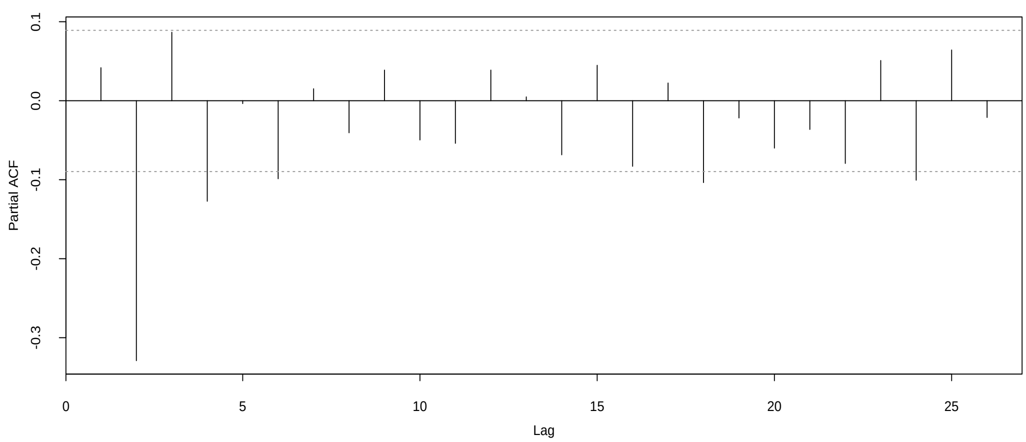
MA(3) series



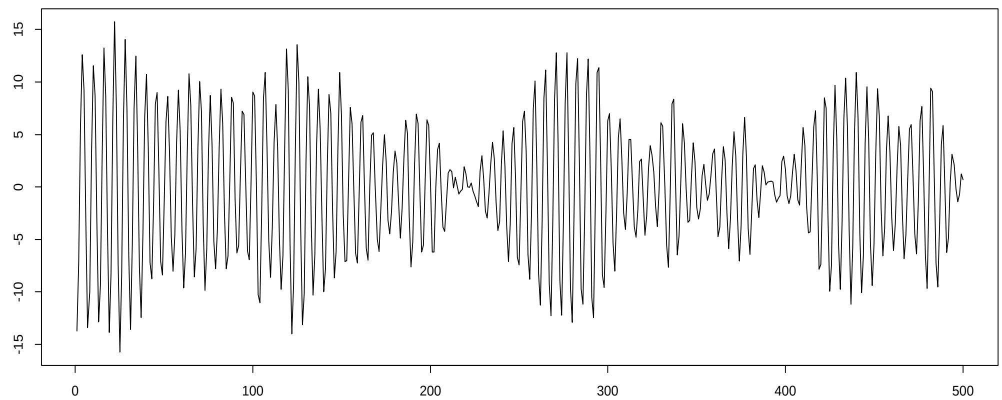
Series : ma2



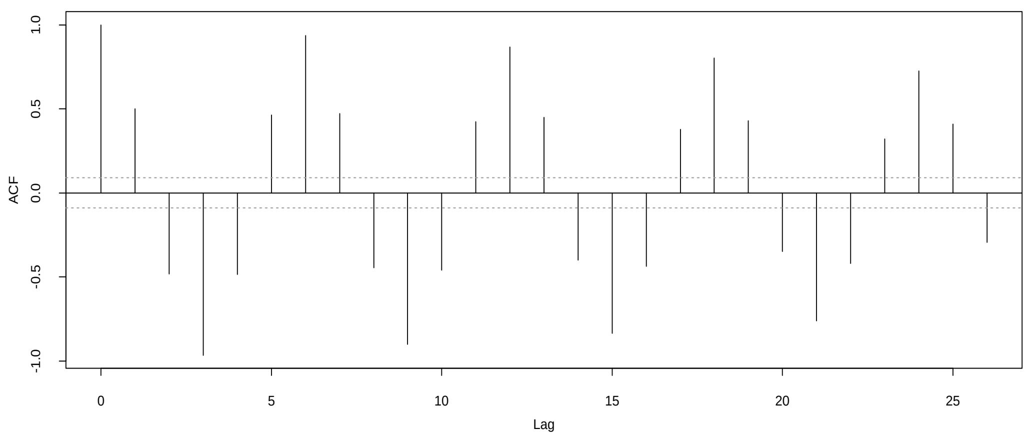
Series : ma2



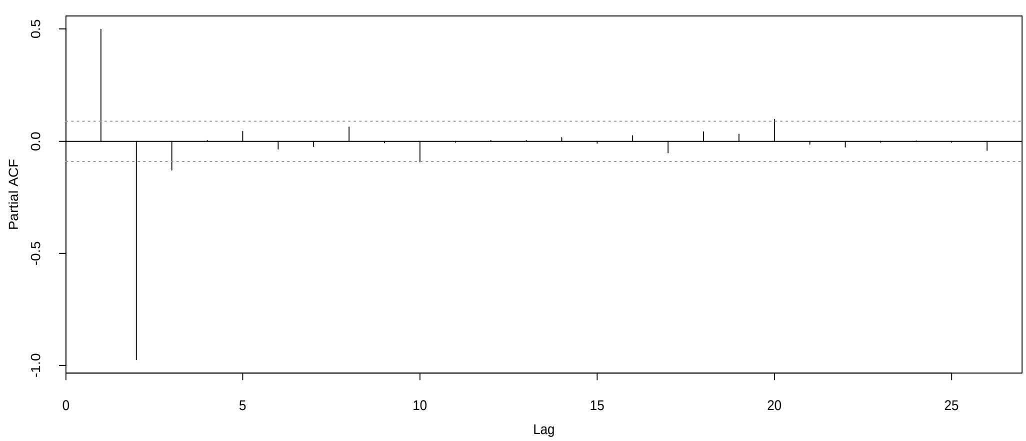
AR(2) series



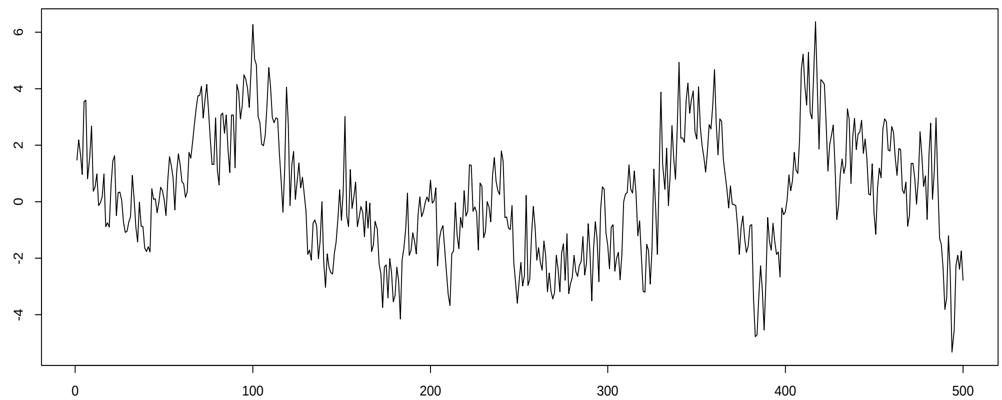
Series : ar2



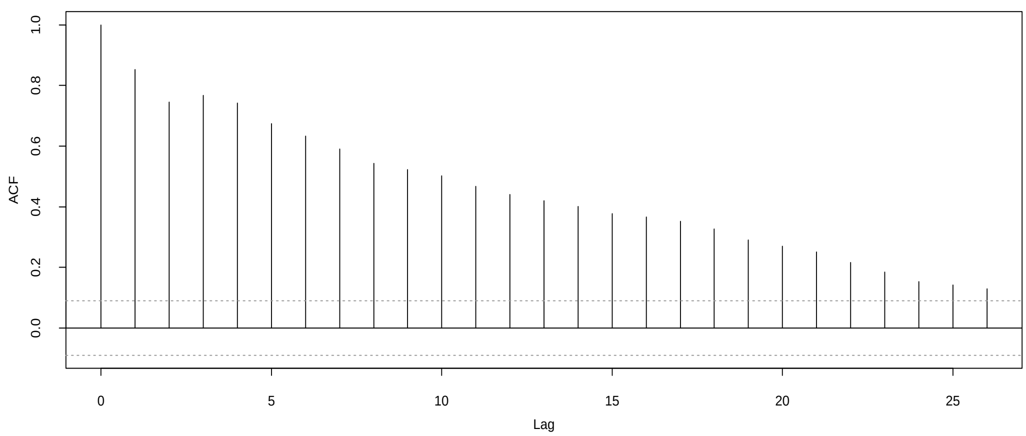
Series : ar2



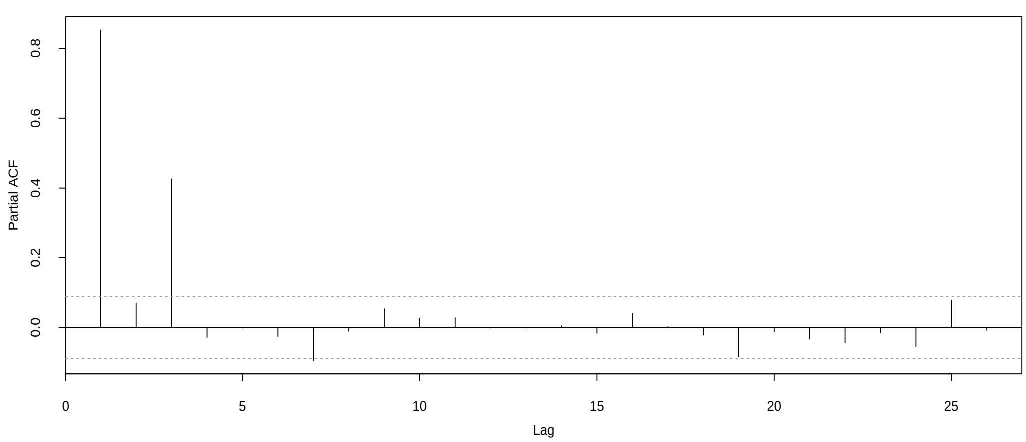
AR(3) series



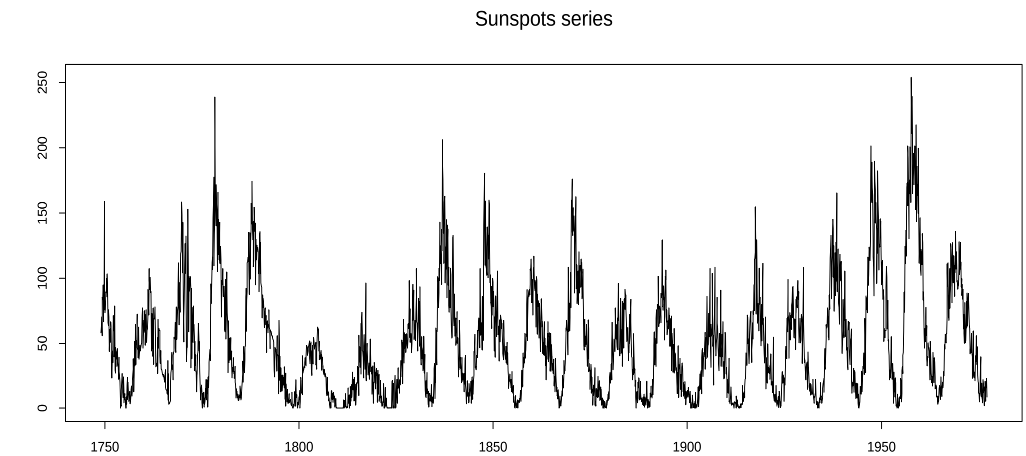
Series : ar3

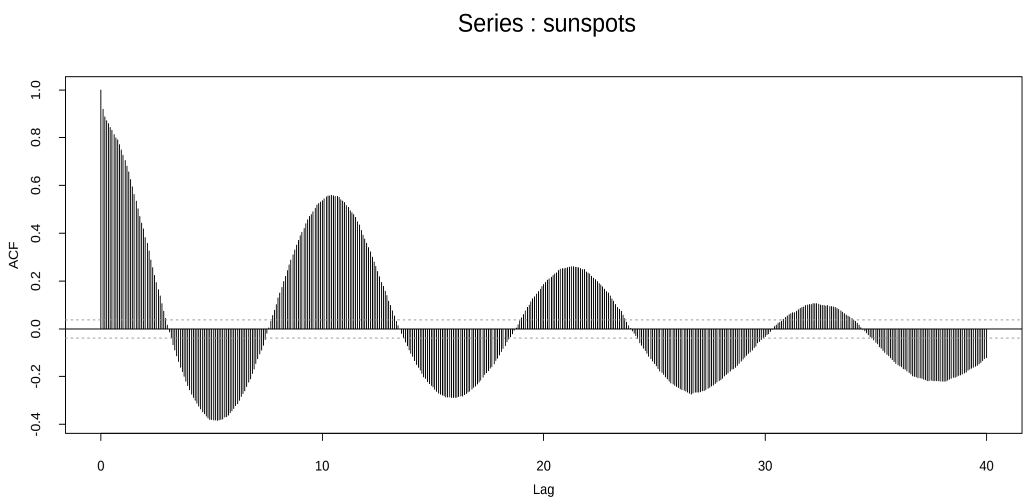


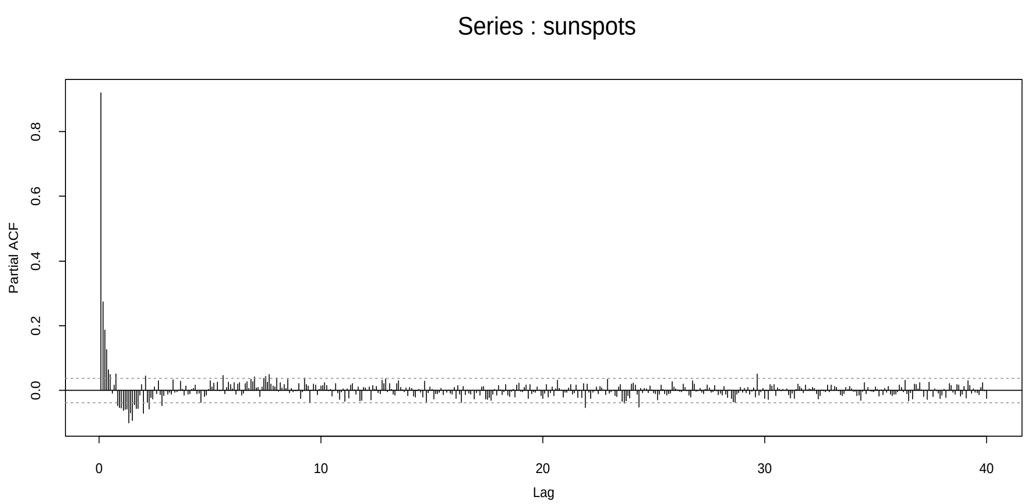
Series : ar3



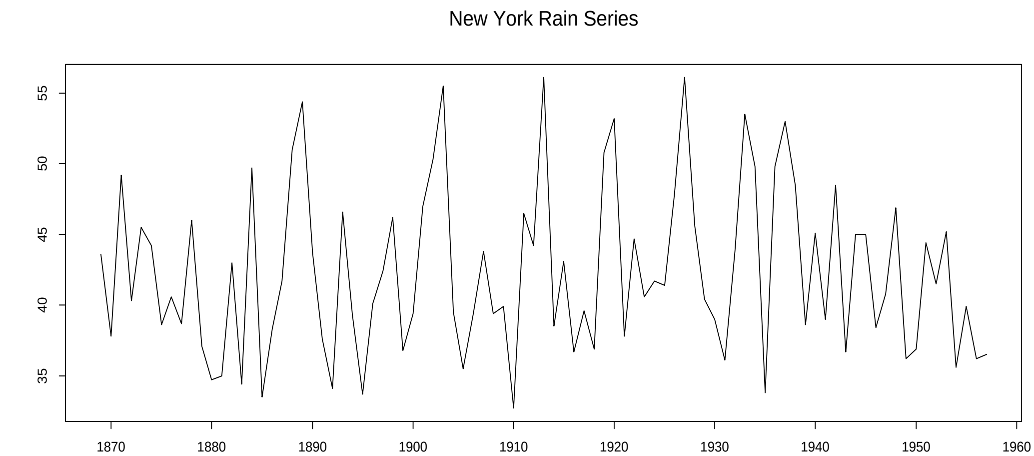
Sunspot data

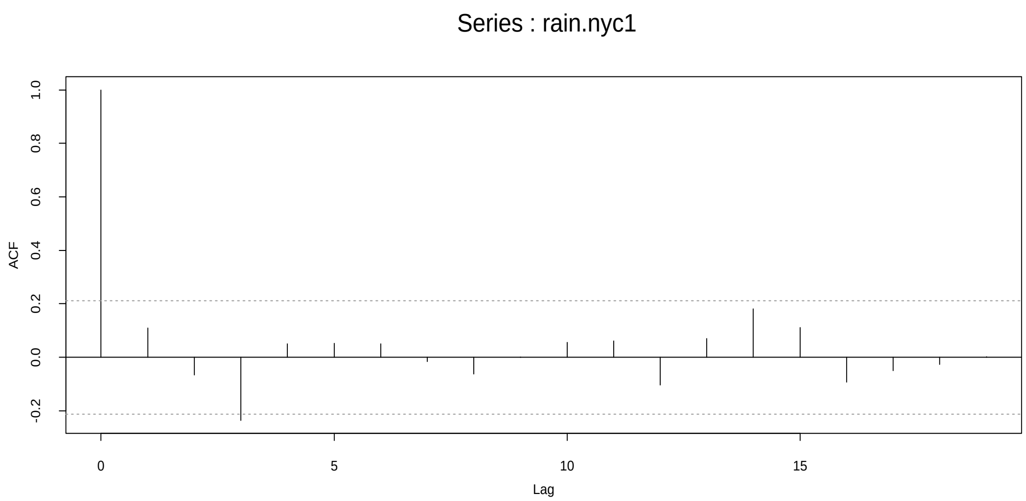


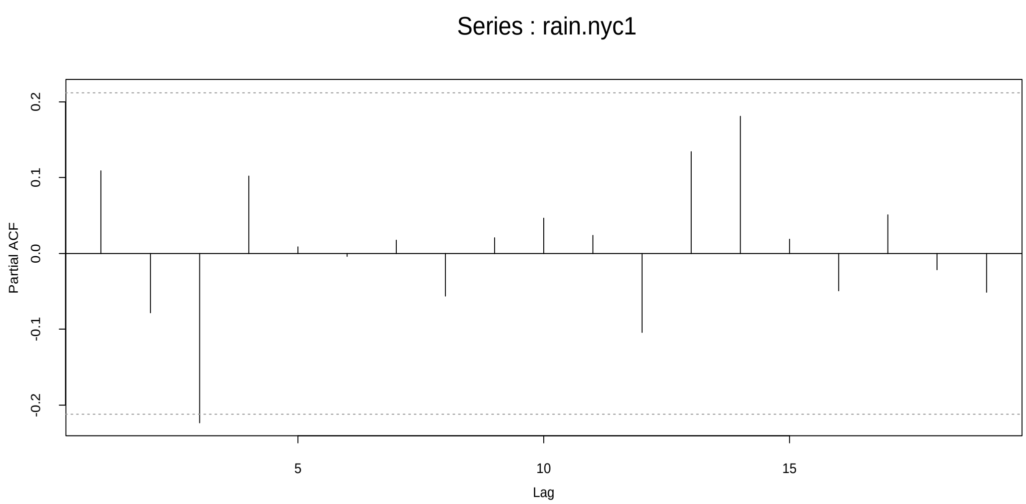




New York City Rainfall







You should notice:

- The MA series has an ACF which is near 0 for lags over 2 as predicted theoretically.

- The $AR(p)$ series have complicated ACF's but the PACFs vanish at lags more than p.

- The sunspot data has significant autocorrelation over many years but the PACF is pretty small after say 2 years or so. Notice that SPlus has labelled the lags in years so that the number 2 on the x axis corresponds to about 24 months. An $AR(p)$ model with p around 24 would then be a possibility.

- The New York rainfall series looks quite a bit like white noise.