Transforms of Stochastic Processes

Apply these ideas with f being stochastic process X. Several difficulties:

- X is not periodic.
- X often only discrete time function; data always discrete time.
- Even for continuous time X, Fourier transform integral typically doesn't converge:

$$X \not\to 0 \text{ as } t \to \pm\infty.$$

Discrete X leads to study of discrete approximation to integral:

$$\sum_{t=0}^{T-1} X_t \exp(i2\pi\omega t)$$

This object has real part

$$\sum_{t=0}^{T-1} X_t \cos(2\pi\omega t)$$

and imaginary part

$$\sum_{t=0}^{T-1} X_t \sin(2\pi\omega t).$$

So: apart from means not being 0 studying sample covariance with sines and cosines at frequency ω.

Statistical properties and interpretation?

Suppose X mean 0 stationary time series, autocovariance function C.

Def'n: *discrete Fourier transform* of X is

$$\hat{X}(\omega) = \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} X_t \exp(i2\pi\omega t)\,.$$

Division by $\sqrt{T}$ motivated by recognition that sum of T terms typically has standard deviation on order of $\sqrt{T}$.

So expect SD of $\hat{X}$ will have reasonable limit as $T \to \infty$.

First compute moments of $\hat{X}$.

Moments of complex valued $\hat{X}$?

One way: view $\hat{X}$ as vector with two components, the real and imaginary parts.

Gives $\hat{X}$ a mean and a 2 by 2 variance covariance matrix.

Also of interest: expected modulus squared of $\hat{X}$,:

$$E[|\hat{X}(\omega)|^2] = E[\hat{X}(\omega)\overline{\hat{X}(\omega)}]$$

where $\bar{z}$ is the complex conjugate of z.

(If $z = x + iy$ with x and y real then $\bar{z} = x - iy$.)

Since the Xs have mean 0 we see that

$$E\hat{X}(\omega) = 0$$

Note expected value of complex valued random variable is computed by finding expected value of real and imaginary parts.

Then

$$E[|\hat{X}(\omega)|^2] = \frac{1}{T}\sum_{s=0}^{T-1}\sum_{t=0}^{T-1} \exp(i2\pi\omega(s-t))E(X_s X_t)$$

Expected values are $C(s-t)$.

Gather together all terms involving $C(0)$, all those involving $C(1)$, etc.:

$$\begin{aligned}\mathsf{E}[|\hat{X}(\omega)|^2] = \frac{1}{T}\}TC(0) \\ + (T-1)(e^{i2\pi\omega} + e^{-i2\pi\omega})C(1) + \cdots\Big\}\end{aligned}$$

which simplifies to

$$\begin{aligned}C(0) + (1-1/T)C(1)(e^{i2\pi\omega} + e^{-i2\pi\omega}) \\ + (1-2/T)C(2)(e^{i4\pi\omega} + e^{-i4\pi\omega})\cdots\end{aligned}$$

As $T \to \infty$ coefficient of $C(k)$ converges to 1.

Use $C(k) = C(-k)$ to see

$$\lim_{T\to\infty} \mathsf{E}[|\hat{X}(\omega)|^2] = \sum_{-\infty}^{\infty} C(k)\exp(i2\pi\omega k)\,.$$

Def'n: *Spectral density*, or *power spectrum*, of X:

$$f_X(\omega) = \sum_{-\infty}^{\infty} C(k)\exp(i2\pi\omega k)\,.$$

Interpretations of spectral density and discrete Fourier transform:

- Discrete Fourier transform is rerepresentation of the data: can recover data from transform by inverse transform:

$$
\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{k=0}^{T-1}\exp\left(\frac{-i2\pi kt}{T}\right)\hat{X}(k/T) \\
&\quad = \frac{1}{T}\sum_{k=0}^{T-1}\exp\left(\frac{-i2\pi kt}{T}\right)\sum_{s=0}^{T-1}\exp\left(\frac{i2\pi ks}{T}\right)X_s \\
&\quad = \frac{1}{T}\sum_{s=0}^{T-1}X_s\sum_{k=0}^{T-1}\exp\left(\frac{i2\pi k(s-t)}{T}\right)
\end{aligned}
$$

For $s = t$ sum over k is T

For $s \neq t$ sum can be done as geometric series — get 0.

So inside sum just picks out term $s = t$ giving X_t as the inverse transform.

- So DFT decomposes X into trigonometric functions of various frequencies: $\hat{X}(k/T)$ is weight on component at frequency k/T.

- Spectral density is limit of variance of that weight or an approximation to variance of component of X at frequency k/T.

- Spectral density is transform of ACF of X:

$$\int_0^1 f_X(\omega) \exp(-i2\pi\ell\omega)\, d\omega = C_X(\ell)\,.$$

- Since for any integer $t \neq 0, \pm T, \pm 2T, \ldots$

$$\sum_{k=0}^{T-1} \exp(i2\pi kt/T) = 0$$

we see $\hat{X}(k/T)$ is, apart from a factor of $\sqrt{T}$, a complex number:

  - real part is sample covariance between X and $\cos(2\pi kt/T)$

  - imaginary part is sample covariance between X and $\sin(2\pi kt/T)$.

- Compute covariance between X and

$$a\cos(2\pi kt/T) + b\sin(2\pi kt/T).$$

Choose a, b to maximize covariance subject to $a^2 + b^2 = 1$.

Resulting coefficients found by multiple regression of X_t on the cosine and sine.

Since

$$\sum_{k=0}^{T-1} \cos(2\pi kt/T)\sin(2\pi kt/T) = 0$$

can check covariance maximized by taking a and b proportional to real and imaginary parts of $\hat{X}(k/T)$ respectively.

Also: covariance with this linear combination is $|\hat{X}(k/T)|^2$.

Calculation requires t to be a non-zero integer.

In practice apply techniques to $X - \bar{X}$.

- Later: if series Y is a filtered version of series X then spectral densities have simple relation to one another in terms of some property of filter.

  Can use this fact to estimate the filter itself when this is unknown.

The Periodogram

Sample covariance between X and $\sin(2\pi\omega t + \phi)$ is

$$\frac{1}{T}\sum_{t=0}^{T-1} X_t \sin(2\pi\omega t+\phi) - \bar{X}\frac{1}{T}\sum_{t=0}^{T-1}\sin(2\pi\omega t+\phi)$$

Use identity $\sin(\theta) = (e^{i\theta} - e^{-i\theta})/(2i)$ and formulas for geometric sums to compute mean.

When $\omega = k/T$ for an integer k, not 0, we find that $\sum_{t=0}^{T-1} \sin(2\pi\omega t+\phi) = 0$.

So sample covariance is simply

$$\frac{1}{T}\sum_{t=0}^{T-1} X_t \sin(2\pi\omega t+\phi)\,.$$

For these special ω we can also compute

$$\sum_{t=0}^{T-1}\sin^2(2\pi\omega t+\phi) = T/2.$$

So correlation between X and $\sin(2\pi\omega t + \phi)$ is

$$\frac{\frac{1}{T}\sum_{t=0}^{T-1} X_t \sin(2\pi\omega t + \phi)}{s_x\sqrt{1/2}}$$

where s_x^2 is sample variance $\sum(X_t - \bar{X})^2/T$.

Adjust ϕ to maximize this correlation.

The sine can be rewritten as

$$\cos(\phi)\sin(2\pi\omega t) + \sin(\phi)\cos(2\pi\omega t)$$

so choose coefficients a and b to maximize correlation between X and

$$a\sin(2\pi\omega t) + b\cos(2\pi\omega t)$$

subject to the condition $a^2 + b^2 = 1$.

Correlations are scale invariant so drop condition on a and b and maximize the correlation between X and the linear combination of sine and cosine.

Problem solved by linear regression. Coefficients given by $(M^T M)^{-1} M^T X$:

M is T by 2 design matrix full of sines and cosines.

Get $M^T M = \frac{T}{2} I_{T\times T}$; regression coefficients are

$$a = \frac{2}{T}\sum_{t=0}^{T-1} X_t \sin(2\pi\omega t)$$

and

$$b = \frac{2}{T}\sum_{t=0}^{T-1} X_t \cos(2\pi\omega t)\,.$$

Covariance between X and best linear combination is

$$\frac{1}{T}\left\{a\sum_{t=0}^{T-1} X_t \sin(2\pi\omega t) + b\sum_{t=0}^{T-1} X_t \cos(2\pi\omega t)\right\} = (a^2+b^2)/2.$$

But in fact

$$a^2+b^2 = \left|\frac{1}{T}\sum_{t=0}^{T-1} X_t \exp(2\pi\omega t i)\right|^2$$

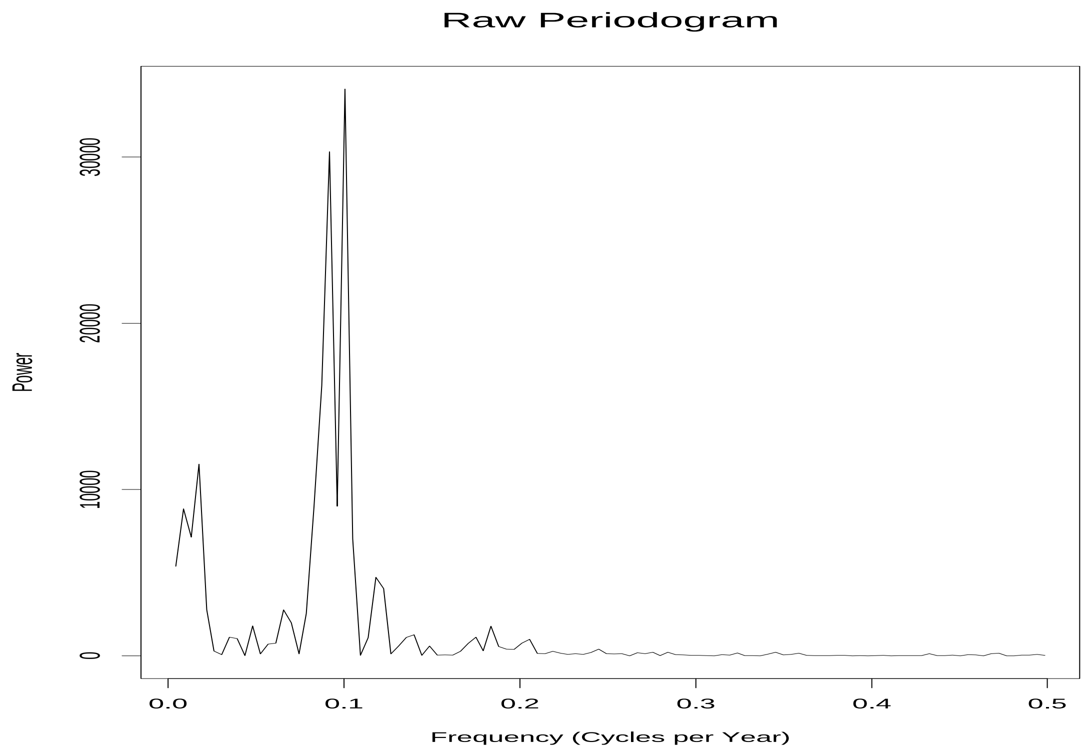
is modulus of DFT $\widehat{X}(\omega)$ divided by T.

Def'n: **Periodogram** is function

$$|\hat{X}(\omega)|^2$$

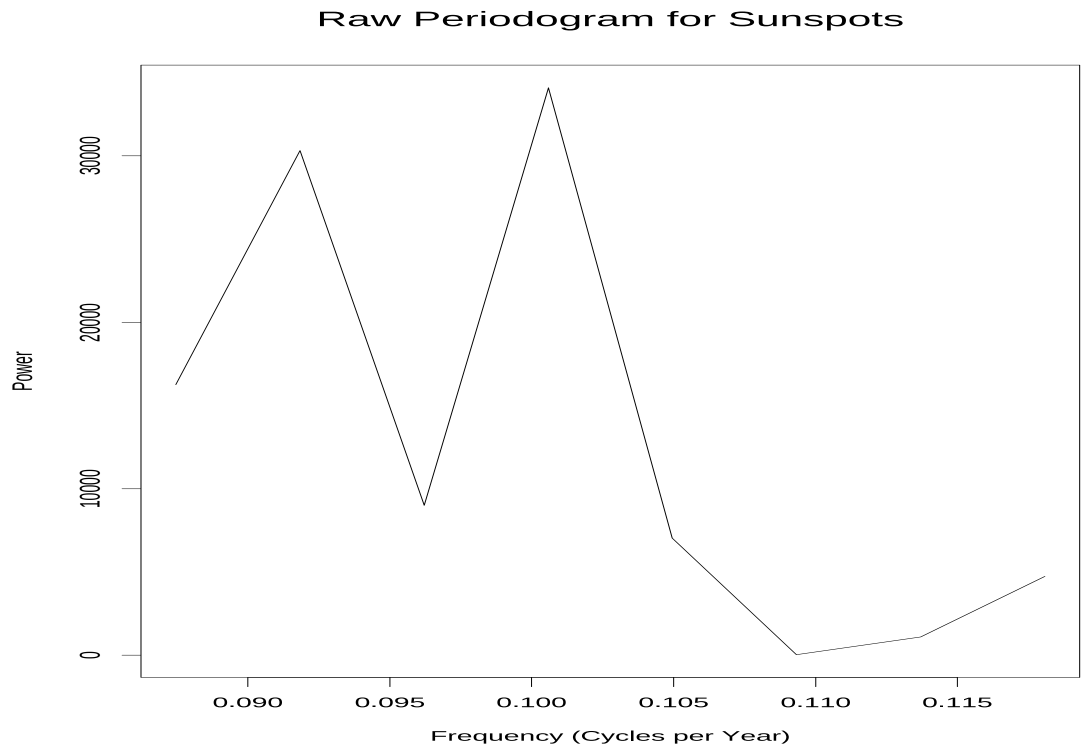
Some periodogram plots:

- $|\hat{X}|$ vs frequency for sunspots minus mean



Notice peak at frequency slightly below 0.1 cycles per year as well as peak at frequency close to 0.03.

Plot only for frequencies from 1/12 to 1/8 which should include the largest peak.

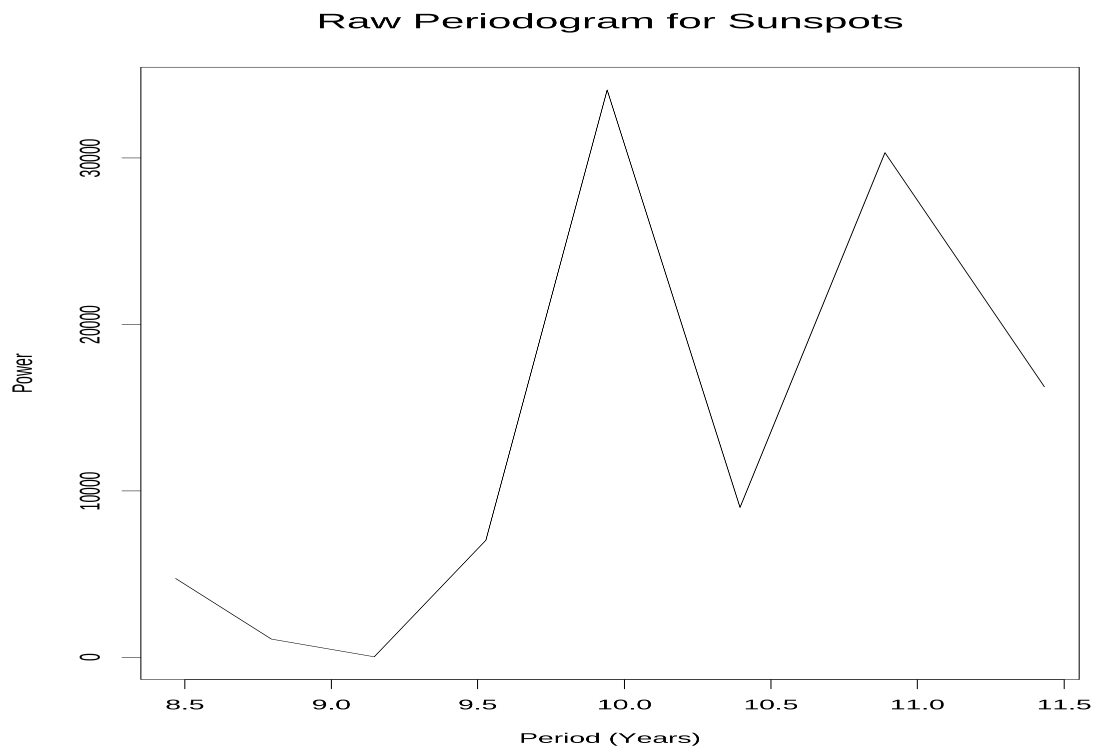


Notice: picture clearly piecewise linear.

Actually using DFT: computes sample spectrum only at frequencies of form k/T (in cycles per point) for integer values of K.

There are only about 10 points on this plot.

Same plot against period ($= 1/\omega$) shows peaks just below 10 years and just below 11.

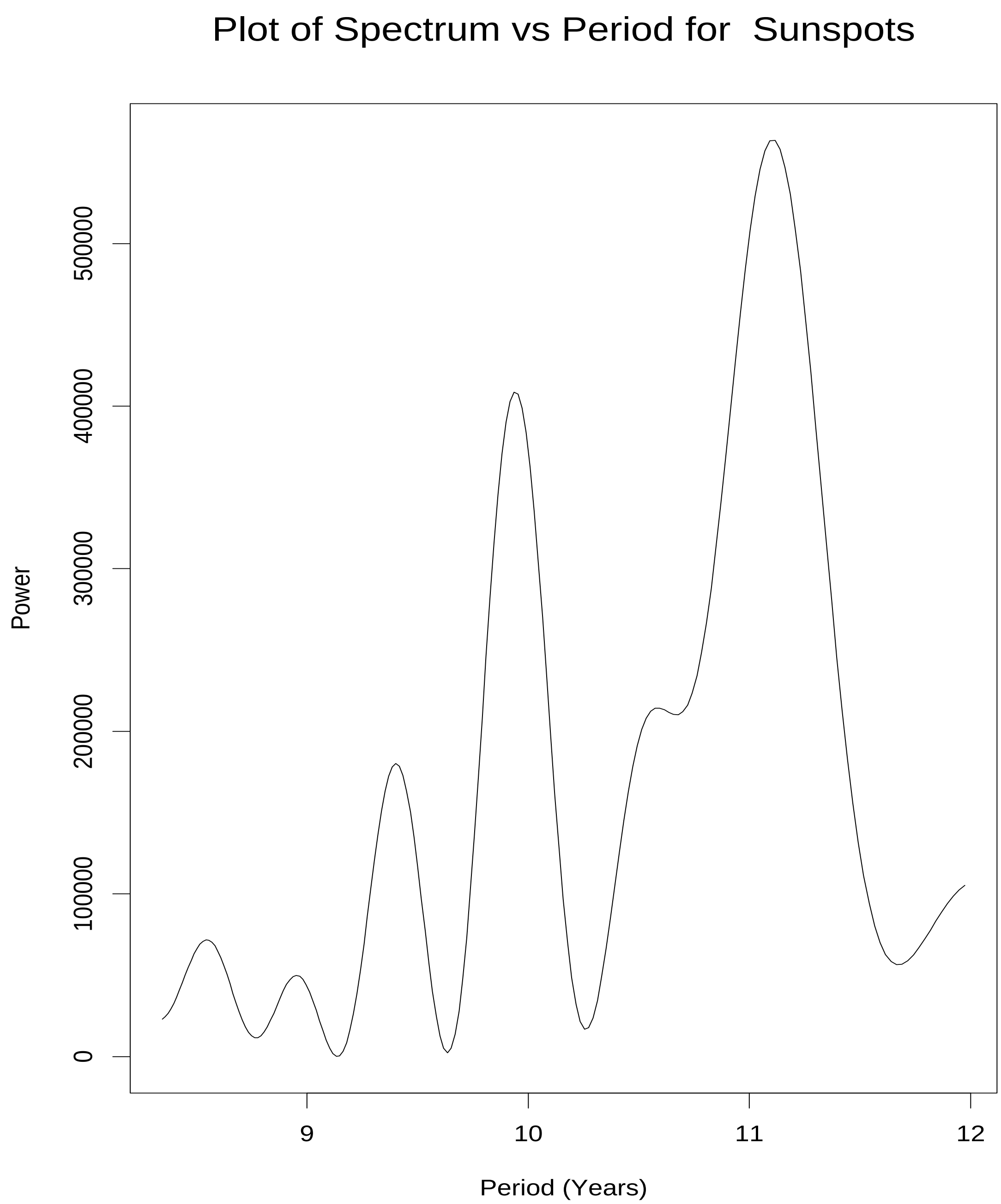


DFT can be computed very quickly at special frequencies but to see structure clearly near a peak need to compute $\hat{X}(\omega)$ for a denser grid of ω.

Use S-Plus function

```
transform<- function(x, a, b, n = 100)
{
    f <- seq(a, b, length = n)
    nn <- 1:length(x)
    args <- outer(f, nn, "*") * 2 * pi
    cosines <- t(cos(t(args)) * x)
    sines <- t(sin(t(args)) * x)
    one <- rep(1, length(x))
    ((cosines %*% one)^2
       + (sines %*% one)^2)/length(x)
}
```

to compute lots of values for periods between 8 and 12 years.

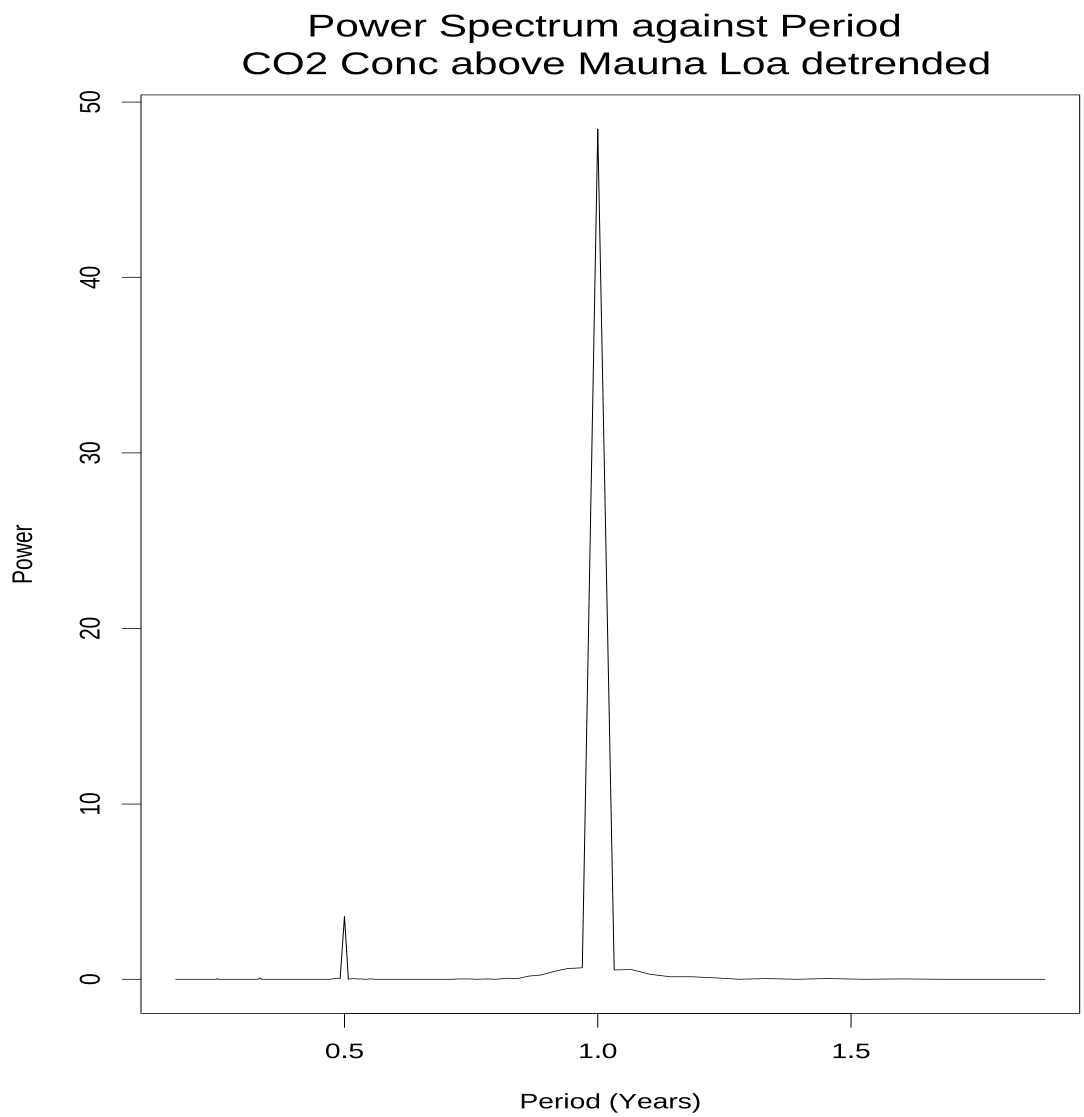

Plot of Spectrum vs Period for Sunspots
Power
0
100000
200000
300000
400000
500000
9
10
11
12
Period (Years)

Periodogram for CO_2 above Mauna Loa: Linear trend removed by linear regression.

Note peaks at periods of 1 year and 6 months.

Peaks show clear annual cycle.

Annual cycle not simple sine wave – contains overtones: components whose frequency is integer multiple of basic frequency of 1 cycle per year.


Power Spectrum against Period
CO2 Conc above Mauna Loa detrended
Power
0
10
20
30
40
50
0.5
1.0
1.5
Period (Years)

Now a detail of this image: