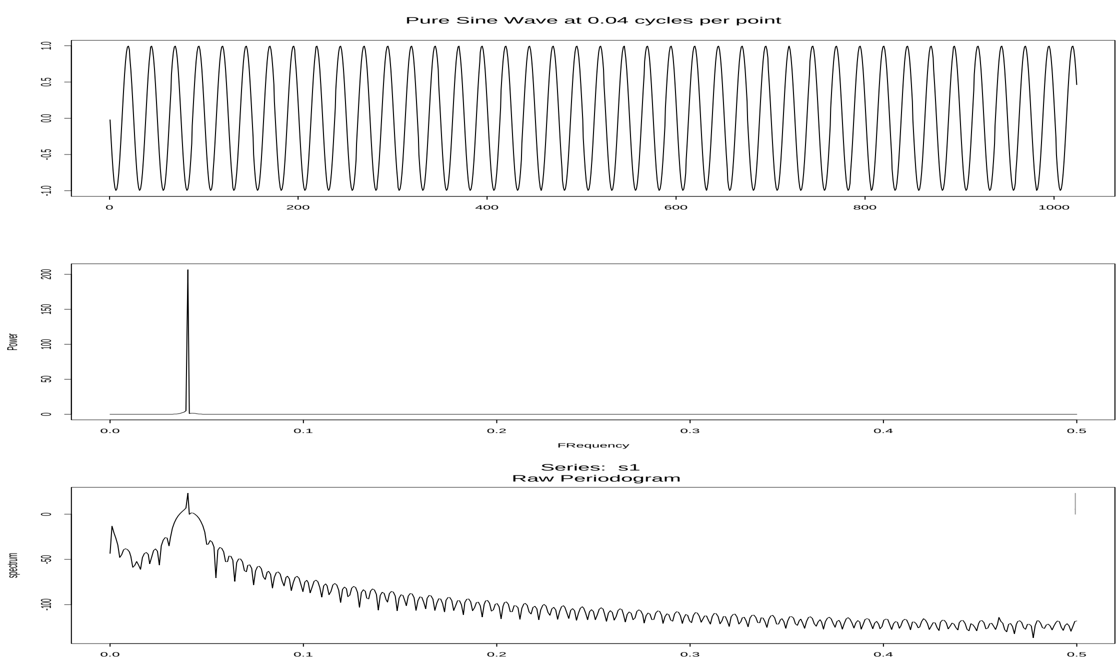
Periodogram of various generated series which have exact sinusoidal components.

First a pure sine wave with no noise.

Middle panel: periodogram.

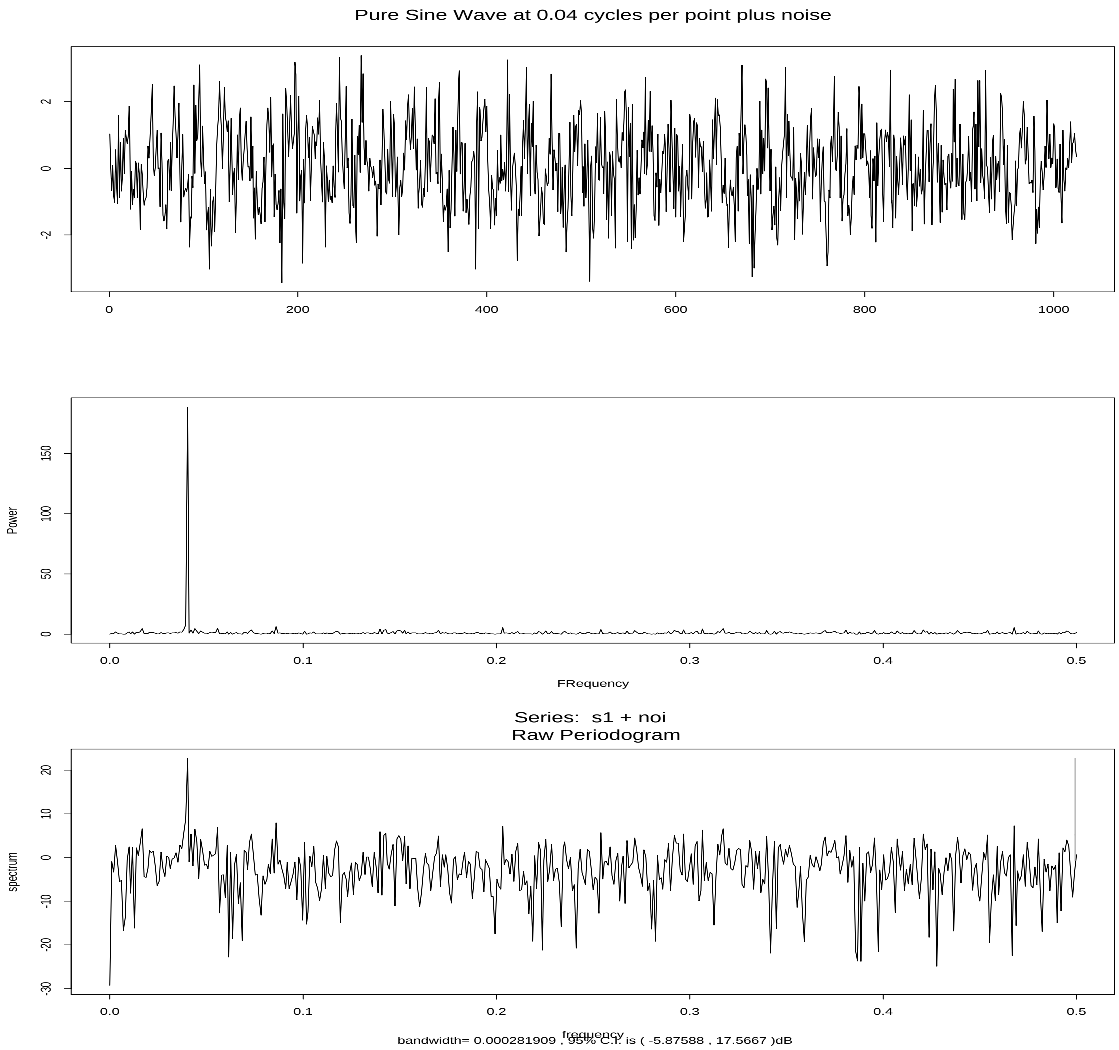
Lower panel: $\log_{10}(|\hat{X}(\omega)|) * 10$.

Apparent waves: round off error $\log(\approx 0)$.

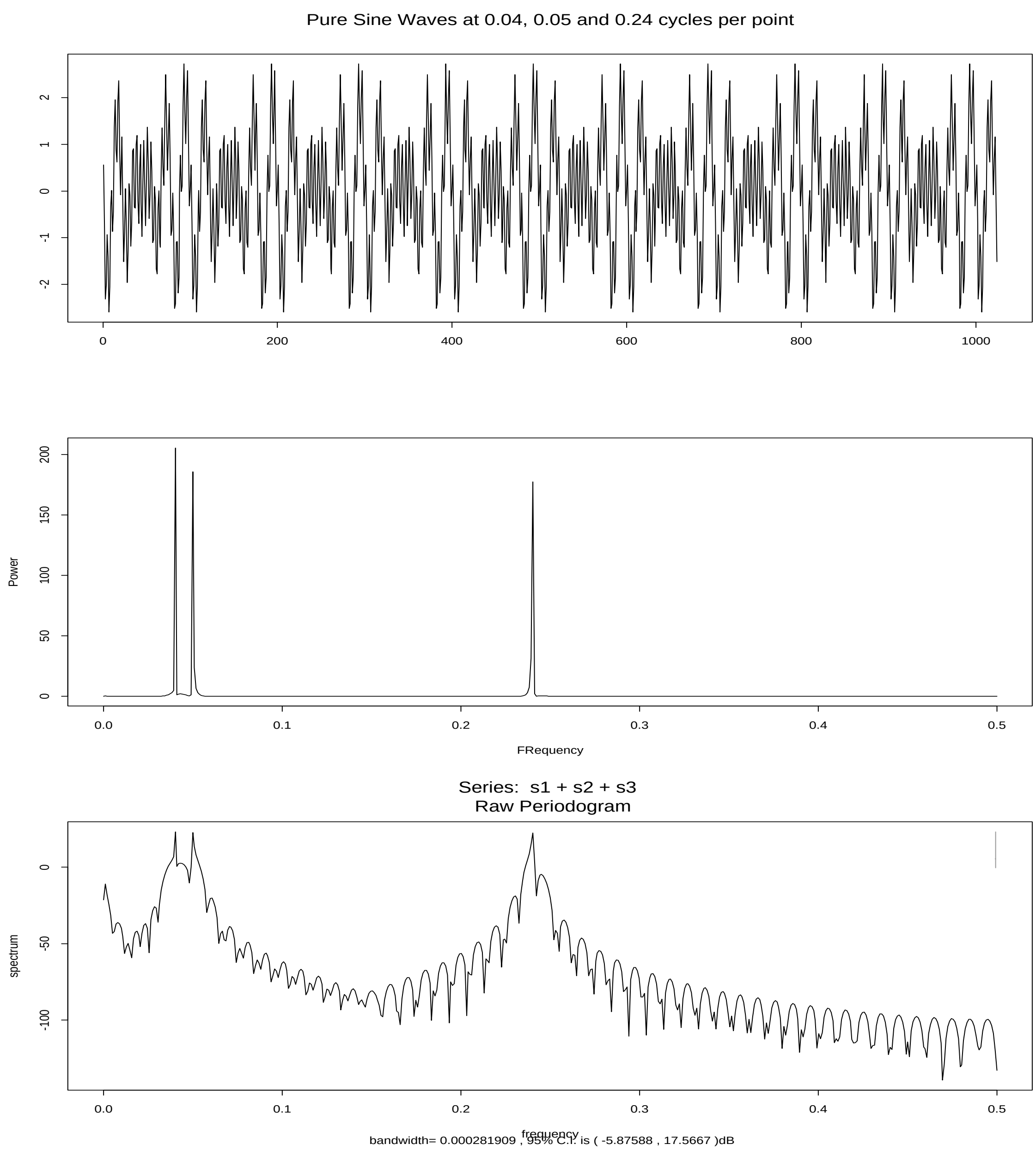


Same series plus N(0,1) white noise.

Note: much harder to see perfect sine wave in data but periodogram shows presence of sine wave quite clearly.

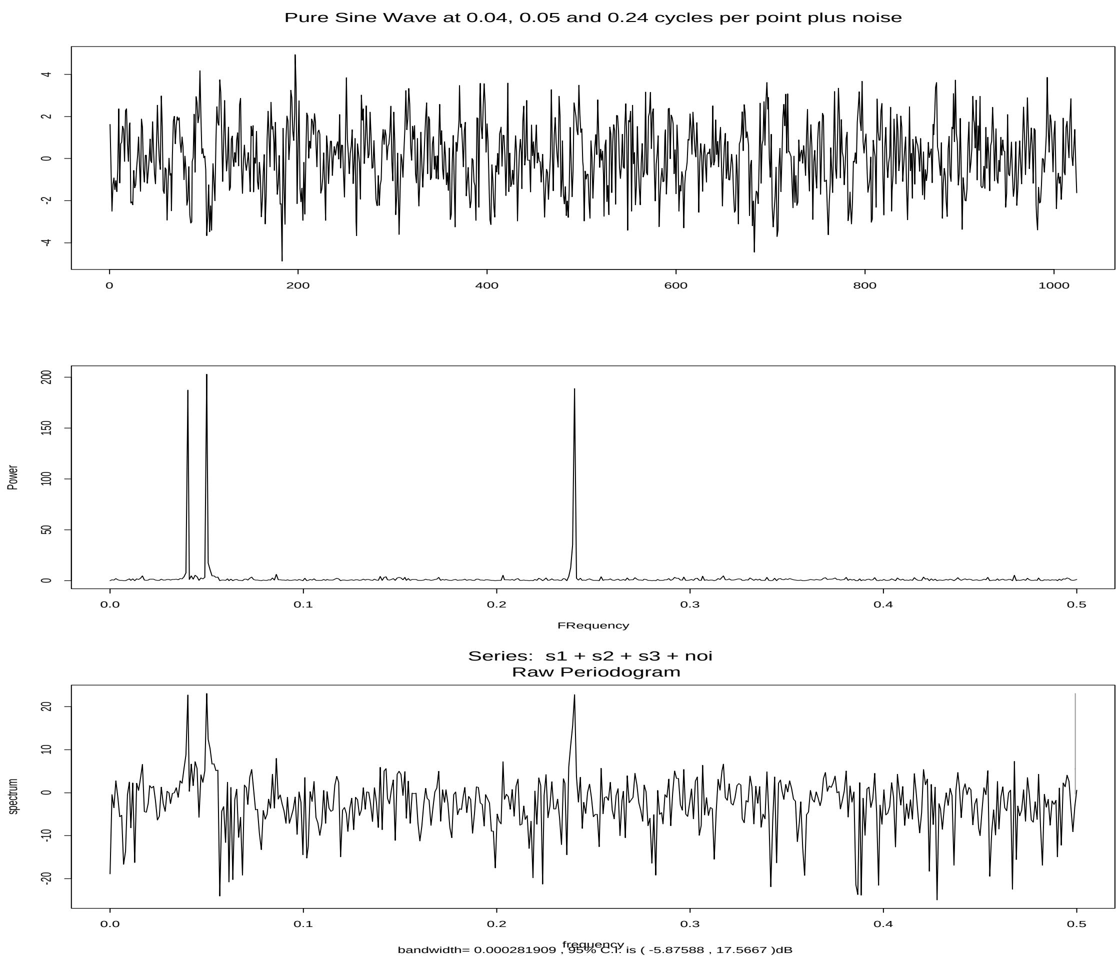


The sum of three sine waves.



Now add N(0,1) white noise. Periodogram still picks out each of component.

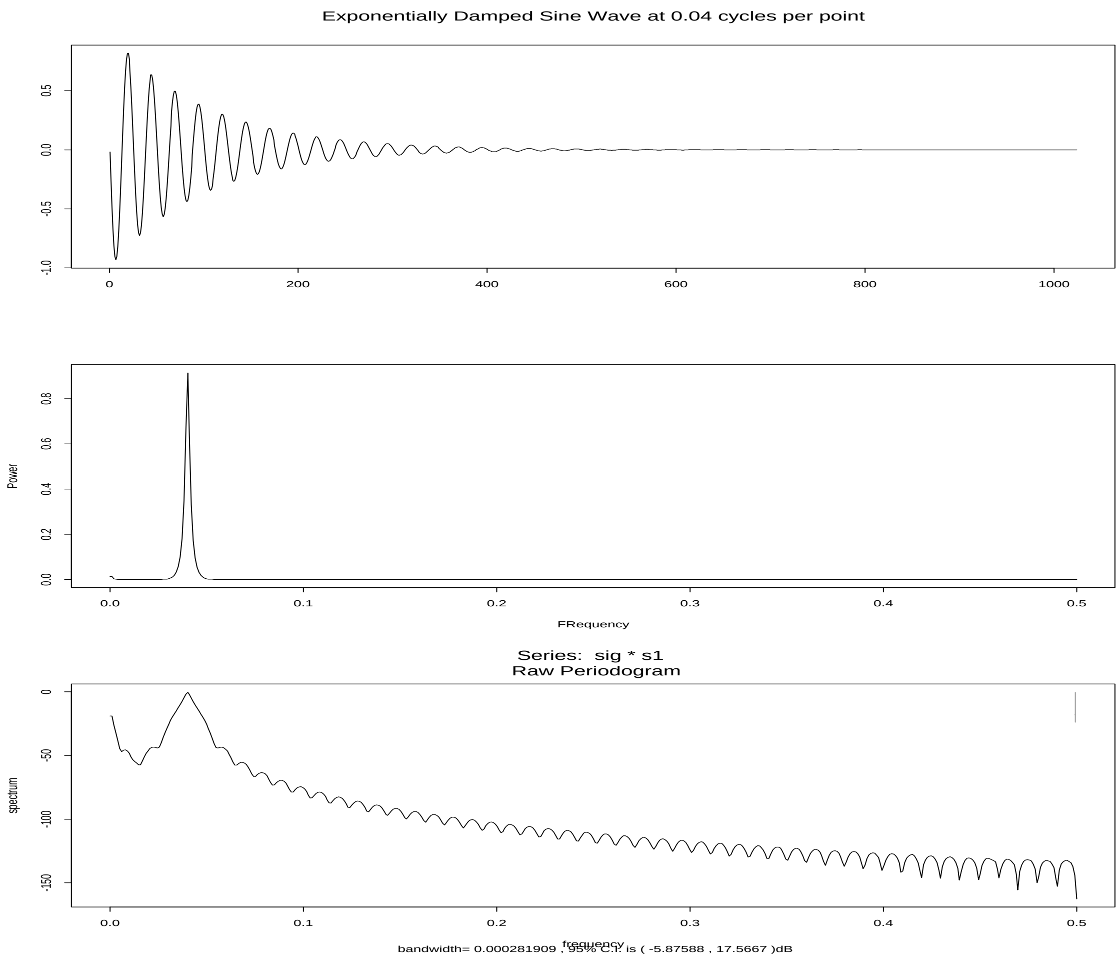
The sum of three sine waves.



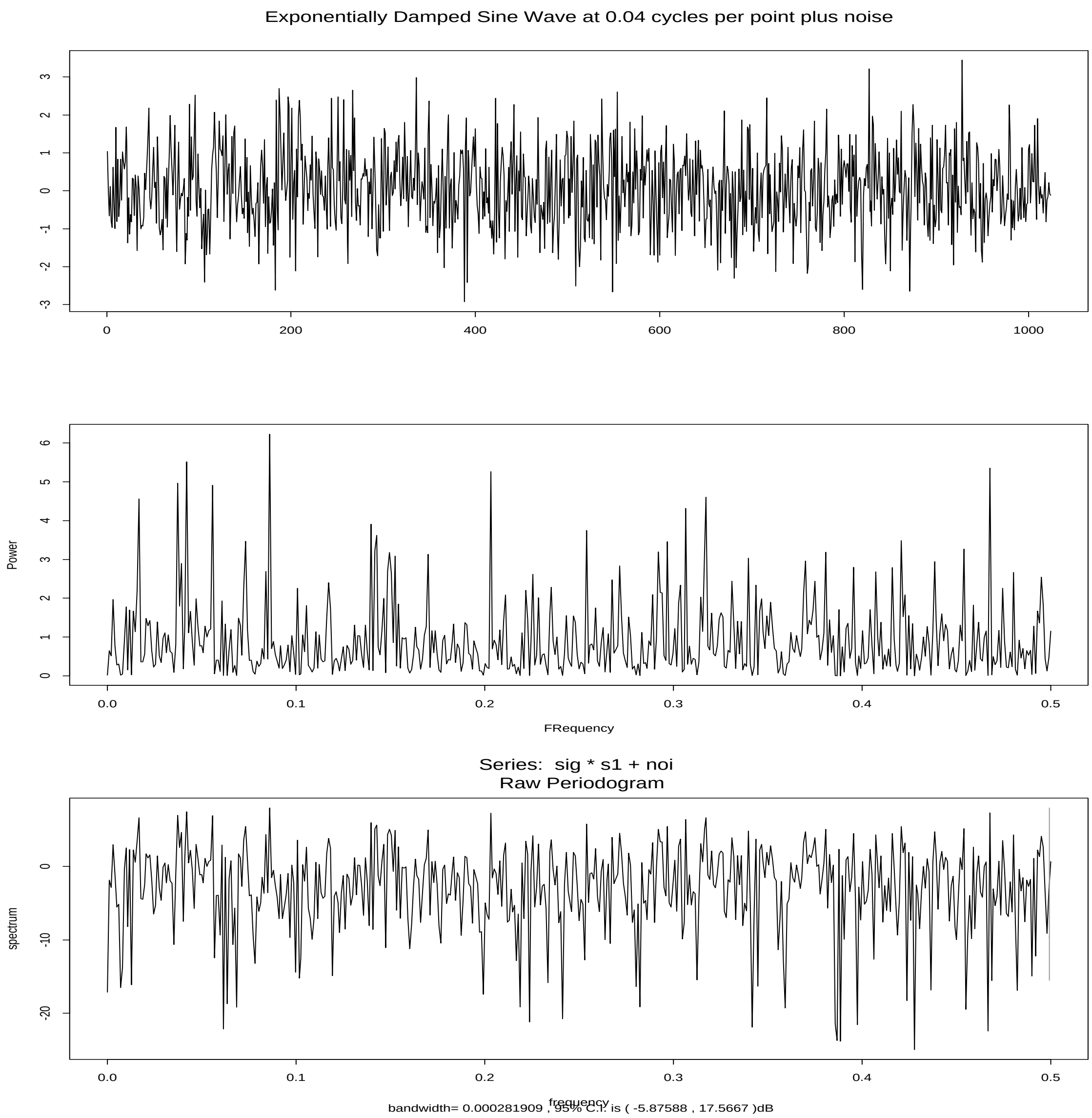
Multiply sine wave by damping exponential.

Signal gone ≈ quarter of way through series.

Periodogram peak still at 0.04 cycles per point.



With noise added can still see effect. But compare the scales on the middle plots between all these series.

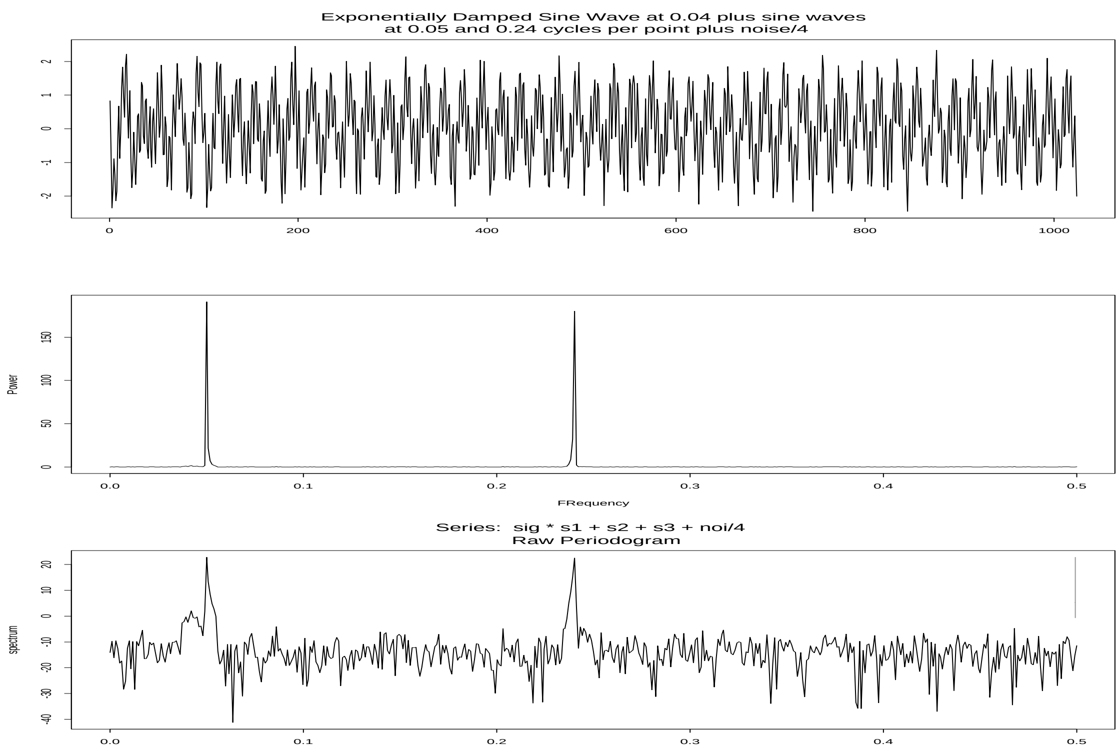


Exponentially damped sine wave plus two sine waves with N(0,1/16) noise.

Only two peaks visible in raw periodogram.

On logarithmic scale: hump on left of peak at 0.05 which is peak at 0.04.

Raw scale can make small secondary peaks invisible.



Properties of the Periodogram

The discrete Fourier transform

$$\hat{X}(\omega) = \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} X_t \exp(2\pi\omega t i)$$

is periodic with period 1 because all the exponentials have period 1. Moreover,

$$\hat{X}(1-\omega) = \frac{1}{\sqrt{T}} \sum_{t=0}^{T-1} X_t \exp(-2\pi\omega t i)\exp(2\pi t i) = \overline{\hat{X}(\omega)}$$

so periodogram satisfies

$$|\hat{X}(1-\omega)|^2 = |\hat{X}(\omega)|^2.$$

So periodogram symmetric around $\omega = 1/2$.

Called *Nyquist* or *folding* frequency.

(Value is always 1/2 in cycles per point; usually converted to cycles per time unit like year or day.)

Similarly power spectral density f_X given by

$$f_X(\omega) = \sum_{-\infty}^{\infty} C_X(h) \exp(2\pi h\omega i)$$

is periodic with period 1 and satisfies

$$f_X(-\omega) = f_X(\omega)$$

which is equivalent to

$$f_X(1-\omega) = f_X(\omega) \, .$$

Spectra of Some Basic Processes

First method: direct.

More powerful technique: indirect.

Direct from the definition

White Noise: Since $C_\epsilon(k) = 0$ for all $k \neq 0$:

$$f_\epsilon(\omega) \equiv C_\epsilon(0) = \sigma_\epsilon^2 .$$

MA(1): For $X_t = \epsilon_t - b\epsilon_{t-1}$:

Have $C_X(0) = \sigma^2(1+b^2)$, $C_X(\pm 1) = -b\sigma^2$ so

$$\begin{aligned} f_X(\omega) &= \sigma^2(1+b^2) \\ &\qquad -b\sigma^2(\exp(2\pi\omega i) + \exp(-2\pi\omega i)) \\ &= \sigma^2(1+b^2) - 2b\sigma^2\cos(2\pi\omega). \end{aligned}$$

AR(1): We have $C_X(k) = \rho^{|k|} C_X(0)$ and

$$
\begin{aligned}
& f_X(\omega) \\
=& C_X(0)\{1 + \sum_{k>0} \rho^k (e^{2\pi\omega k i} + e^{-2\pi\omega k i})\} \\
=& C_X(0)\{1 \\
& + \rho\left(\frac{e^{2\pi\omega i}}{1-\rho e^{2\pi\omega i}} + \frac{e^{-2\pi\omega i}}{1-\rho e^{-2\pi\omega i}}\right)\} \\
=& C_X(0)\left\{1 + \rho\frac{e^{2\pi\omega i} - \rho + e^{-2\pi\omega i} - \rho}{(1-\rho e^{2\pi\omega i})(1-\rho e^{-2\pi\omega i})}\right\} \\
=& C_X(0)\left[1 + \rho\frac{2\{\cos(2\pi\omega) - \rho\}}{1+\rho^2 - 2\rho\cos(2\pi\omega)}\right] \\
=& C_X(0)\frac{1-\rho^2}{1+\rho^2 - 2\rho\cos(2\pi\omega)} \\
=& \frac{\sigma_\epsilon^2}{1+\rho^2 - 2\rho\cos(2\pi\omega)}
\end{aligned}
$$

Using filters

Write mean 0 ARMA(p, q) process in MA form:

$$X_t = \sum_{s=0}^{\infty} a_s \epsilon_{t-s}.$$

Covariance of X is

$$C_X(h) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} a_r a_s \text{Cov}(\epsilon_{t+h-r}, \epsilon_{t-s})$$

Covariance simplifies for white noise

But in general $C(t+h-r-(t-s)) = C(h+s-r)$ is covariance in double sum.

Plug double sum into definition of f_X to get

$$f_X(\omega) = \sum_{h=-\infty}^{\infty} e^{2\pi\omega hi} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} a_r a_s C(h+s-r)$$

Write h in exponential in form $(h+s-r)+r-s$.

Bring sum over h to the inside to get

$$f_X(\omega) = \sum_{r=0}^{\infty} a_r e^{2\pi\omega r i} \sum_{s=0}^{\infty} a_s e^{-2\pi\omega s i}$$
$$\sum_{h=-\infty}^{\infty} \exp(2\pi\omega(h+s-r)i)C(h+s-r)$$

Substite $k = h+s-r$ in inside sum and define

$$A(\omega) = \sum_{r=0}^{\infty} a_r e^{2\pi\omega r i}$$

to see that

$$f_X(\omega) = A(\omega)\overline{A(\omega)}f_\epsilon(\omega)$$

or

$$f_X(\omega) = |A(\omega)|^2 f_\epsilon(\omega)\,.$$

Frequency response function: A (or $\bar{A}$).

Power transfer function: $|A|^2$

Gain is sometimes used for $|A|$.

The Spectrum of an ARMA(p, q)

An ARMA(p, q) process X satisfies

$$\sum_{s=0}^{p} a_s X_{t-s} = \sum_{r=0}^{q} b_r \epsilon_{t-r}$$

so that if Y is the process $\sum_{s=0}^{p} a_s X_{t-s}$ then

$$f_Y(\omega) = |A(\omega)|^2 f_X(\omega)$$

where $A(\omega) = \sum_{s=0}^{p} a_s \exp(2\pi\omega s i)$.

Also $Y_t = \sum_{r=0}^{q} b_r \epsilon_{t-r}$ so

$$f_Y(\omega) = |B(\omega)|^2 f_\epsilon(\omega)$$

where $B(\omega) = \sum_{s=0}^{q} b_s \exp(2\pi\omega s i)$. Hence

$$f_X(\omega) = \frac{|B(\omega)|^2}{|A(\omega)|^2}.$$

Example: ARMA(1,1)

$$X_t - aX_{t-1} = \epsilon_t - b\epsilon_{t-1}$$

From MA(1) calculation:

$$|B(\omega)|^2 = 1 + b^2 - 2b\cos(2\pi\omega)$$

for filter on MA side and

$$|A(\omega)|^2 = 1 + a^2 - 2a\cos(2\pi\omega)$$

for filter on AR side.

So

$$f_X(\omega) = \frac{1 + b^2 - 2b\cos(2\pi\omega)}{1 + a^2 - 2a\cos(2\pi\omega)}.$$