Multiple time series

Given: two series Y and X.

Relationship between series?

Possible approaches:

- X deterministic: regress Y on X via generalized least squares: `arima.mle` in SPlus.

- X random; interested in impact of X on Y. Time series analogue of regression.

- X random; interested in joint behaviour. Multivariate ARIMA modelling.

Def'n: Two processes X and Y are *jointly (strictly) stationary* if

$$\mathcal{L}(X_t,\ldots,X_{t+h},Y_t,\ldots,Y_{t+h}) \\ = \mathcal{L}(X_0,\ldots,X_h,Y_0,\ldots,Y_h)$$

for all t and h.

Def'n: X and Y are *jointly second order stationary* if each is second order stationary and also

$$C_{XY}(h) \equiv \text{Cov}(X_t, Y_{t+h}) = \text{Cov}(X_0, Y_h)$$

for all t and h.

Notice that negative values of h give, in general, different covariances than positive values of h.

Regression: estimate impulse response

Assume: $\mathrm{E}(X) = \mathrm{E}(Y) = 0$.

Example to illustrate range of models:

$$Y_t = aX_t + bX_{t-2} + cX_{t+4} + \nu_t$$

where ν is "noise", not necessarily white noise.

Crucial: ν independent of X.

General model:

$$Y_t = \sum_{-\infty}^{\infty} a_s X_{t-s} + \nu_t$$

Problem: estimate impulse response function a.

Alternative description

Find best predictor of Y from X?

Minimize

$$\mathsf{E}\left[\{Y_t - f_t(\ldots, X_{t-1}, X_t, \ldots)\}^2\right]$$

over all functions f_t.

Joint stationarity implies:

$$f_{t+1}(BX) = f_t(X)$$

For Gaussian mean 0 series:

$$f_0(X) = \sum a_s X_{t-s}$$

for some coefficients a_s

So: minimize

$$\mathsf{E}\left\{\left(Y_t - \sum_s a_s X_{t-s}\right)^2\right\}$$

Expand out:

$$\mathsf{E}(Y_t^2) - 2\sum a_s \mathsf{E}(Y_t X_{t-s}) + \sum a_r a_s \mathsf{E}(X_{t-r} X_{t-s})$$

which is

$$\mathsf{E}(Y_t^2) - 2\sum a_s C_{XY}(s) + \sum a_r a_s C_X(r-s)$$

Derivative wrt a_k:

$$-2C_{XY}(k) + 2\sum_s a_s C_X(k-s)$$

Set to 0 to find minimum:

$$C_{XY}(k) = \sum_s a_s C_X(k-s)$$

Solve?

With data?

Use spectral methods:

Def'n: : convolution of two sequences a_t, b_t is

$$(a * b)_t = \sum_r a_r b_{t-r} = \sum b_r a_{t-r}$$

Fourier transform:

$$f_a(\omega) = \sum a_r e^{2\pi r\omega i}$$

$$\begin{aligned} f_{a*b}(\omega) &= \sum_{r=0}^{\infty} (a*b)_r e^{2\pi\omega r i} \\ &= \sum_s \sum_r a_s b_{r-s} e^{2\pi\omega(r-s+s)i} \\ &= \left(\sum a_s e^{2\pi\omega s i}\right)\left(\sum b_r e^{2\pi\omega r i}\right) \\ &= f_a(\omega) f_b(\omega) \end{aligned}$$

So

$$f_{C_{XY}}(\omega) = f_a(\omega) f_{C_X}(\omega)$$

Cross spectrum

$$f_{XY}(\omega) = f_{C_{XY}}(\omega) = \sum C_{XY}(r)e^{2\pi r\omega i}$$

Property:

$$f_{XY}(\omega) = f_{YX}(-\omega) = \overline{f_{XY}(\omega)}$$

because

$$C_{XY}(h) = C_{YX}(-h)$$

Conclusion: *Frequency response function*, A, of filter a is:

$$\begin{aligned} A(\omega) &= \sum a_r e^{2\pi r\omega i} \\ &= f_{XY}(\omega)/f_X(\omega) \end{aligned}$$

Estimate A from estimates of f_{XY} and f_X.

Estimate f_X by smoothing periodogram.

Estimating the cross-spectrum

Use cross-periodogram:

$$\hat{f}_{XY}(\omega) = \hat{Y}(\omega)\overline{\hat{X}(\omega)}$$

where $\hat{Y}$ and $\hat{X}$ are discrete Fourier transforms.

Smooth to improve estimate.

Estimating the impulse response

Recall Fourier inversion

$$a_t = \int_0^1 A(\omega)e^{-2\pi t\omega i}d\omega$$

Replace A by estimate $\hat{A}$ to get $\hat{a}$.

Gain is sometimes used for $|A|$.

Write

$$A(\omega) = |A(\omega)|e^{i\theta(\omega)}$$

for some function θ with

$$-\pi < \theta(\omega) \leq \pi$$

Call θ the *phase shift* of the filter.

MSE of optimal filter?

$$\mathrm{E}\left\{\left(Y_t - \sum_s a_s X_{t-s}\right)^2\right\}$$
$$= C_Y(0) - 2\sum a_s C_{XY}(s) + \sum a_r a_s C_X(r-s)$$

Recall Fourier inversion formulas like

$$C_Y(h) = \int f_Y(\omega) e^{-2\pi h\omega i} d\omega$$

Put $h = 0$ get

$$C_Y(0) = \int f_Y(\omega) d\omega$$

Notice

$$\sum a_s C_{XY}(s) = \sum a_s C_{YX}(0-s)$$

But

$$f_{a*C_{YX}} = f_a f_{C_{YX}} = f_a f_{YX}$$

so by Fourier inversion

$$\sum a_s C_{YX}(0-s) = \int f_a(\omega) f_{YX}(\omega) d\omega$$

Also

$$\sum a_r a_s C_X(r-s) = \sum_r \int f_{a*C_X}(\omega) e^{-2\pi r\omega i} d\omega$$
$$= \int f_a(\omega) f_X(\omega) f_a(-\omega) d\omega$$

But f_a is just A and so

$$\sum a_r a_s C_X(r-s) = \int f_X(\omega)|A(\omega)|^2 d\omega$$

Assemble:

$$MSE = \int \{f_Y(\omega) - 2A(\omega) f_{YX}(\omega) + f_X(\omega)|A(\omega)|^2\} d\omega$$

Factor out f_Y, recall $A = f_{XY}/f_X$:

$$MSE = \int f_Y(\omega) \left\{ 1 - 2\frac{|f_{XY}(\omega)|^2}{f_X(\omega) f_Y(\omega)} + \frac{|f_{XY}(\omega)|^2}{f_X(\omega) f_Y(\omega)} \right\} d\omega$$

Def'n: Coherence between X and Y at frequency ω is

$$\gamma_{XY}(\omega) = \frac{|f_{XY}(\omega)|}{\sqrt{f_X(\omega) f_Y(\omega)}}$$

So

$$MSE = \int f_Y(\omega) \left\{ 1 - \gamma^2_{XY}(\omega) \right\} d\omega$$

Interpretation: γ is a correlation coefficient.

Note: suppose we use raw periodgrams to estimate cross-spectrum:

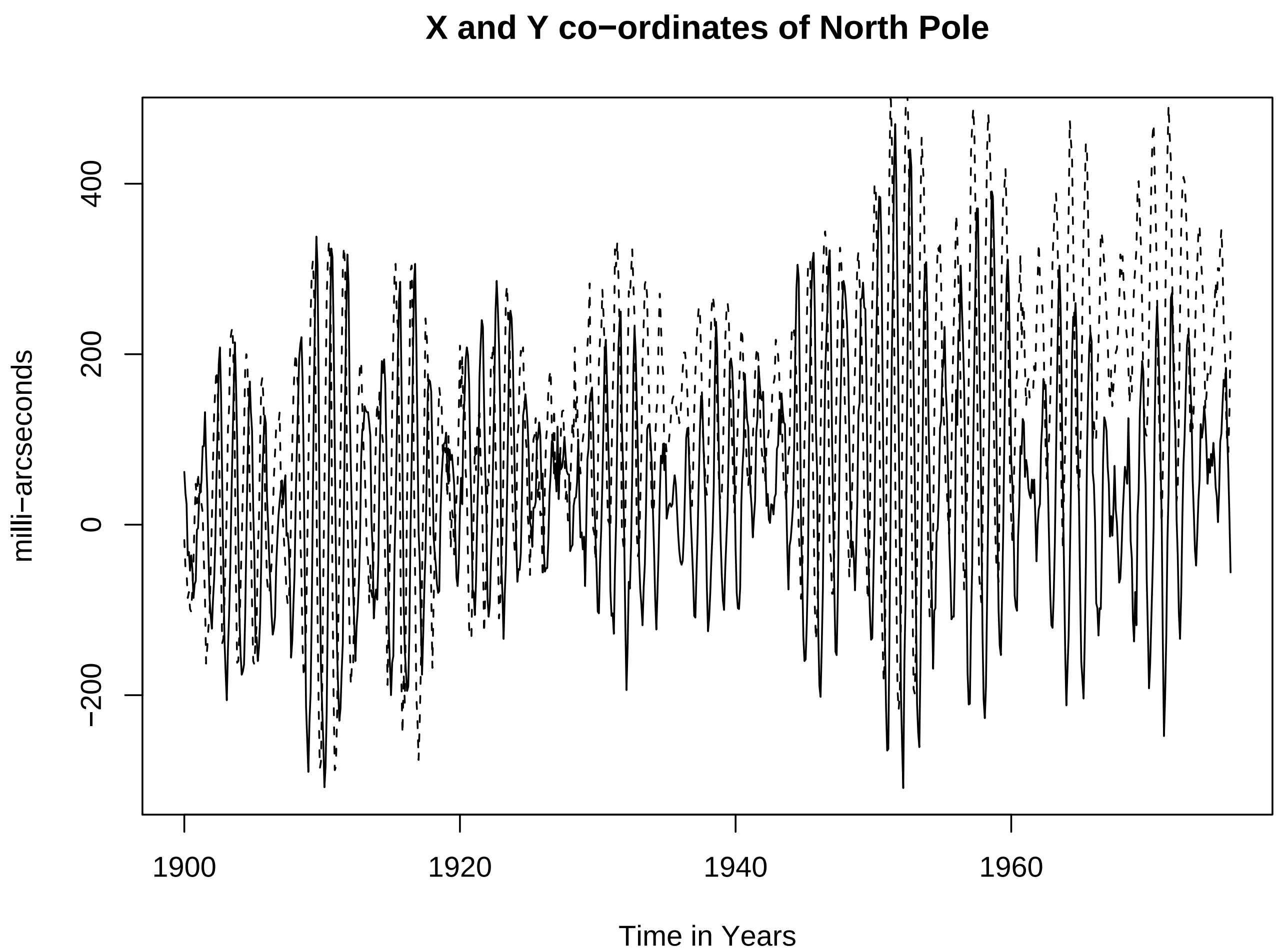
Then

$$\hat{\gamma}_{XY}(\omega) = \frac{|\hat{Y}(\omega)\overline{\hat{X}(\omega)}|}{|\hat{Y}(\omega)| \cdot |\hat{X}(\omega)|} = 1$$

Conclusion: must smooth to get useful estimates of γ.

A data example: study X and Y motions of pole together.

Plot of X and Y together:



Parametric plots of X and Y: