STAT 804 Solutions

Assignment 2

1. Consider the ARIMA(1,0,1) process

$$X_t - \phi X_{t-1} = \epsilon_t - \psi \epsilon_{t-1} \, .$$

Show that the autocorrelation function is

$$\rho(1) = \frac{(1-\psi\phi)(\phi - \psi)}{1 + \psi^2 - 2\psi\phi}$$

and

$$\rho(k) = \phi^{k-1}\rho(1) \qquad k = 2, 3, \ldots$$

Plot the autocorrelation functions for the ARMA(1,1) process above, the AR(1) process with

$$X_t = \phi X_{t-1} + \epsilon_t$$

and the MA(1) process

$$X_t = \epsilon_t - \psi \epsilon_{t-1}$$

on the same plot when $\phi = 0.6$ and $\theta = -0.9$. Compute and plot the partial autocorrelation functions up to lag 30. Comment on the usefulness of these plots in distinguishing the three models. Explain what goes wrong when ϕ is close to ψ.

Solution: The most important part of this problem is that when $\phi = \psi$ the autocorrelation is identically 0. This means that $\phi = \psi$ gives simply white noise. In general in the ARMA(p, q) model

$$\phi(B)X = \psi(B)\epsilon$$

any common root of the polynomials ϕ and ψ gives a common factor on both sides of the model equation which can effectively be cancelled. In other words, if $\phi(x) = \psi(x) = 0$ for some particular x then we can write $\phi(B) = (1 - x^{-1}B)\phi^*(B)$ for a suitable ϕ^* and also $\psi(B) = (1 - x^{-1}B)\psi(B)$. In the model equation we can cancel the common factor $(1 - x^{-1}B)$ and reduce the model to an ARMA($p - 1, q - 1$).

A second important point is that the autocorrelation of an ARMA(1,1) decreases geometrically just like that of an AR(1) but only starting from lag 2 on.

2. Suppose Φ is a Uniform$[0, 2\pi]$ random variable. Define

$$X_t = \cos(\omega t + \Phi) \, .$$

Show that X is weakly stationary. (In fact it is strongly stationary so show that if you can.) Compute the autocorrelation function of X.

Solution:

First

$$\mathrm{E}(X_t) = \int_0^{2\pi} \cos(\omega t + \phi) \, d\phi = \sin(\omega t + \phi)|_0^{2\pi} = 0$$

Second

$$E(X_t X_{t+h}) = \int_0^{2\pi} \cos(\omega t + \phi)\cos(\omega(t+h)+\phi)\,d\phi$$

Expand each $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and you get integrals involving $\cos^2(\phi)$, $\sin^2(\phi)$ and $\sin(\phi)\cos(\phi)$. The latter integrates to 0 while each of the former integrates to 1/2. Thus

$$E(X_t X_{t+h}) = (\cos(\omega t)\cos(\omega(t+h)) + \sin(\omega t)\sin(\omega(t+h)))/2$$

and again using $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$ we get

$$E(X_t X_{t+h}) = \cos(\omega h)/2$$

Put $h = 0$ to get $C(0) = 1/2$ and divide to see that $\rho(h) = \cos(\omega h)$.

This shows that X is wide sense stationary.

If Z_1 and Z_2 are independent standard normals then

$$Y_t = Z_1 \cos(\omega t) + Z_2 \sin(\omega t)$$

has mean 0 and covariance

$$Cov(Y_t, Y_{t+h}) = \cos(\omega t)\cos(\omega(t+h)) + \sin(\omega t)\sin(\omega(t+h)) = \cos(\omega h)$$

so that Y is also weakly stationary. Since Y is Gaussian it is also strongly stationary. Now write (Z_1, Z_2) in polar co-ordinates with $R = \sqrt{Z_1^2 + Z_2^2}$ and $0 < \Phi \le 2\pi$ being the polar co-ordinates. Then you can check that

- Φ is uniform on $[0, 2\Pi]$.
- Y/R is strongly stationary because R is free of t and Y is strongly stationary.
- $Y/R = \cos(\omega t + \Phi)$.

This is a proof that $\cos(\omega t + \Phi)$ is strongly stationary.

3. Show that X of the previous question satisfies the AR(2) model

$$X_t = (2 - \lambda^2)X_{t-1} - X_{t-2}$$

for some value of λ. Show that the roots of the characteristic polynomial lie on the boundary of the unit circle in the complex plain. (Hint: show that $e^{i\theta}$ is a root if θ is chosen correctly. Do not spend too much time on this question; the point is to illustrate that AR(2) models can be found whose behaviour is much like a sinusoid.)

Solution: We have

$$\begin{aligned} X_{t+1} - (2-\lambda^2)X_t + X_{t-1} &= \cos(\omega t)\cos(\omega) - \sin(\omega t)\sin(\omega) \\ &\quad -(2-\lambda^2)\cos(\omega t) + (\cos(\omega t)\cos(\omega) +)\sin(\omega t)\sin(\omega)) \\ &= (2\cos(\omega) - (2-\lambda^2))\cos(\omega t) \end{aligned}$$

which is 0 provided $\lambda^2 = 2(1-\cos(\omega))$.

The characteristic polynomial is

$$1 - 2\cos(\omega)x + x^2$$

whose roots are

$$\frac{2\cos(\omega) \pm \sqrt{4\cos^2(\omega) - 4}}{2} = \cos(\omega) \pm i\sin(\omega)$$

which is $\exp(\pm i\omega)$ whose modulus is 1 for both choices of the sign.

4. Suppose that X_t is an ARMA(1,1) process

$$X_t - \rho X_{t-1} = \epsilon_t - \theta\epsilon_{t-1}$$

   (a) Suppose we mistakenly fit an AR(1) model (mean 0) to X using the Yule-Walker estimate

$$\hat{\rho} = \left(\sum_1^{T-1} X_t X_{t-1}\right) / \left(\sum_0^{T-1} X_t^2\right)$$

   In terms of θ, ρ and σ what is $\hat{\rho}$ close to?

   (b) If we use this AR(1) estimate $\hat{\rho}$ and calculate residuals using $\hat{\epsilon}_t = X_t - \hat{\rho}X_{t-1}$ what kind of time series is $\hat{\epsilon}$? What will plots of the Autocorrelation and Partial Autocorrelation functions of this residual series look like?

Solution: Let $\psi = (1-\rho\theta)(\rho-\theta)/(1+\theta^2-2\theta\rho)$ denote the autocovariance at lag 1. For large values of T we may write approximately

$$\hat{\epsilon}_t = X_t - \psi X_{t-1}$$

or $\hat{\epsilon} = (I-\psi B)X = (I-\psi B)(I-\rho B)^{-1}(I-\theta B)\epsilon$ or just

$$(I-\rho B)\hat{\epsilon} = (I-\psi B)(I-\theta B)\epsilon$$

which makes $\hat{\epsilon}$ an ARMA(1,2) process. By way of answer about the plots I was merely looking for the knowledge that the plots will match those of an ARMA(1,2) with autoregressive parameter ρ and MA parameters ψ and θ. The model identification problem may well be somewhat harder. It is a useful exercise to generate some data with `ar.sim` from an ARIMA(1,0,1) and try the model fitting process. Look at what happens if you fit an AR(1) and then look at the residuals; you don't see anything helpful in general.