STAT 804: Spring 2006

Assignment 1

1. Let ϵ_t be a Gaussian white noise process. Define

$$X_t = \epsilon_{t-2} + 4\epsilon_{t-1} + 6\epsilon_t + 4\epsilon_{t+1} + \epsilon_{t+2}.$$

   Compute and plot the autocovariance function of X.

2. Suppose that X_t is strictly stationary.

   (a) If g is some function from R^{p+1} to R show that

$$Y_t = g(X_t, X_{t-1}, \ldots, X_{t-p})$$

   is strictly stationary.

   (b) What property must g have to guarantee the analogous result with strictly stationary replaced by 2^{nd} order stationary? [Note: I expect a sufficient condition on g; you need not try to prove the condition is necessary.]

3. Suppose that ϵ_t are iid and have mean 0 with finite variance. Verify that $X_t = \epsilon_t\epsilon_{t-1}$ is stationary and that it is wide sense white noise.

4. Suppose X_t is a stationary Gaussian series with mean μ_X and autocovariance $R_X(k)$, $k = 0, \pm 1, \ldots$. Show that $Y_t = \exp(X_t)$ is stationary and find its mean and autocovariance.

5. Suppose that

$$X_t = a_1 X_{t-1} + a_2 X_{t-2} + \epsilon_t$$

   where ϵ_t is an iid mean 0 sequence with variance σ_ϵ^2. Compute the autocovariance function and plot the results for $\rho_1 = 0.2$ and $\rho_2 = 0.1$. (NOTE: I mean ρ_i and NOT a_i here.) I have shown in class that the roots of a certain polynomial must have modulus more than 1 for there to be a stationary solution X for this difference equation. Translate the conditions on the roots $1/\alpha_1, 1/\alpha_2$ to get conditions on the coefficients a_1, a_2 and plot in the a_1, a_2 plane the region for which this process can be rewritten as a causal filter applied to the noise process ϵ_t.

6. Suppose that ϵ_t is an iid mean 0 variance σ_ϵ^2 sequence and that $a_t; t = 0, \pm 1, \pm 2, \ldots$ are constants. Define

$$X_t = \sum a_s \epsilon_{t-s}$$

   (a) Derive the autocovariance of the process X.

   (b) Show that $\sum a_s^2 < \infty$ implies

$$\lim_{N\to\infty} E[(X_t - \sum_{-N}^{N} a_s \epsilon_{t-s})^2] = 0$$

   This condition shows that the infinite sum defining X converges "in the sense of mean square". It is possible to prove that this means that X can be defined properly. [Note: I don't expect much rigour in this calculation. Mathematically, you can't just define X_t as this question supposes since the sum is infinite. A rigourous treatment — WHICH I DO NOT EXPECT — asks you to prove that the condition $\sum a_s^2 < \infty$ implies that the sequence $S_N \equiv \sum_{-N}^{N} a_s \epsilon_{t-s}$ is a Cauchy sequence in L^2. Then you have to know that this implies the existence of a limit in L^2 (technically, the point is that L^2 is a Banach space). Then you have to prove that the calculation you made in the first part of the question is mathematically justified.]

7. Given a stationary mean 0 series X_t with autocorrelation ρ_k, $k = 0, \pm 1, \ldots$ and a fixed lag D find the value of A which minimizes the mean squared error

$$E[(X_{t+d} - AX_t)^2]$$

   and for the minimizing A evaluate the mean squared error in terms of the autocorrelation and the variance of X_t.

8. The semivariogram of a stationary process X is

$$\gamma_X(m) = \frac{1}{2} E[(X_{t+m} - X_t)^2] .$$

   (Without the 1/2 it's called the variogram.) Evaluate γ in terms of the autocovariance of X.

9. A process X_t follows a an ARCH(1) model if the conditional distribution of X_{t+1} given $X_t, X_{t-1}, \cdots$ is normal with mean 0 and variance $a+bX_t^2$. Develop conditions on the parameters a and b for there to exist a stationary series with this behaviour. (There is an analogy to what I did for AR(1) processes.) Hint: Let $Z_{t+1} = X_{t+1}/(a + bX_t^2)$. What is the conditional distribution of Z_{t+1} given all previous Xs? What is the joint distribution of all the Zs? Can you define the Xs from the Zs?

DUE: Monday 23 January.