STAT 804: 2006-01

Assignment 2

1. Consider the ARIMA(1,0,1) process

$$X_t - \phi X_{t-1} = \epsilon_t - \psi \epsilon_{t-1} \, .$$

Show that the autocorrelation function is

$$\rho(1) = \frac{(1 - \psi\phi)(\phi - \psi)}{1 + \psi^2 - 2\psi\phi}$$

and

$$\rho(k) = \phi^{k-1}\rho(1) \qquad k = 2, 3, \ldots$$

Plot the autocorrelation functions for the ARMA(1,1) process above, the AR(1) process with

$$X_t = \phi X_{t-1} + \epsilon_t$$

and the MA(1) process

$$X_t = \epsilon_t - \psi \epsilon_{t-1}$$

on the same plot when $\phi = 0.6$ and $\psi = -0.9$. Compute (numerically — you don't need to derive a general formula for the PACF) and plot the partial autocorrelation functions up to lag 30. Comment on the usefulness of these plots in distinguishing the three models. Explain what goes wrong when ϕ is close to ψ.

2. Suppose Φ is a Uniform$[0, 2\pi]$ random variable. Define

$$X_t = \cos(\omega t + \Phi) \, .$$

Show that X is weakly stationary. (In fact it is strongly stationary so show that if you can.) Compute the autocorrelation function of X.

3. Show that X of the previous question satisfies the AR(2) model

$$X_t = (2 - \lambda^2)X_{t-1} - X_{t-2}$$

for some value of λ. Show that the roots of the characteristic polynomial lie on the boundary of the unit circle in the complex plain. (Hint: show that $e^{i\theta}$ is a root if θ is chosen correctly. Do not spend too much time on this question; the point is to illustrate that AR(2) models can be found whose behaviour is much like a sinusoid.)

4. Suppose that X_t is an ARMA(1,1) process

$$X_t - \rho X_{t-1} = \epsilon_t - \theta \epsilon_{t-1}$$

(a) Suppose we mistakenly fit an AR(1) model (mean 0) to X using the Yule-Walker estimate

$$\hat{\rho} = \left(\sum_{1}^{T-1} X_t X_{t-1} \right) / \left(\sum_{0}^{T-1} X_t^2 \right)$$

In terms of θ, ρ and σ what is $\hat{\rho}$ close to? [That is, compute limits as $T \to \infty$.]

(b) If we use this AR(1) estimate $\hat{\rho}$ and calculate residuals using $\hat{\epsilon}_t = X_t - \hat{\rho} X_{t-1}$ what kind of time series is $\hat{\epsilon}$? What will plots of the Autocorrelation and Partial Autocorrelation functions of this residual series look like?