## Model Identification

Goal: develop tools to permit us to choose a model for a given series $X$.

Idea: attempting to fit an $ARMA(p,q)$; first step is to learn how to choose $p$ and $q$.

We try to get small values of these orders.

Efforts focused on cases with either $p$ or $q$ equal to 0.

Use autocorrelation or autocovariance function to do model identification.

## Some Theoretical Autocovariances

**Moving Averages**: Addition of a constant never affects a covariance, so take mean equal to 0.

Look at

$$X_t = \epsilon_t + \sum_1^p b_j \epsilon_{t-j}$$

Using $b_0 = 1$ we find

$$\begin{aligned} C_X(h) &= \text{Cov}(X_t, X_{t+h}) \\ &= \text{Cov}(\sum_{j=0}^p b_j \epsilon_{t-j}, \sum_{k=0}^p b_k \epsilon_{t+h-k}) \\ &= \sum_{j=0}^p \sum_{k=0}^p b_j b_k \text{Cov}(\epsilon_{t-j}, \epsilon_{t+h-k}) \end{aligned}$$

Each covariance is 0 unless $t-j = t+h-k$ or $k = j+h$. This gives

$$\begin{aligned} C_X(h) &= \sigma^2 \sum_{j=0}^{p} \sum_{k=0}^{p} b_j b_k 1(k = j+h) \\ &= \sigma^2 \sum_{j=0}^{p} b_j b_{j+h} 1(0 \le j+h \le p) \\ &= \sigma^2 \sum_{j=0}^{p-h} b_j b_{j+h} \end{aligned}$$

Notice that if $h > p$ (or $h < -p$) then we get $C_X(h) = 0$.

**Jargon**: We call $h$ the lag and say that for an $MA(p)$ process the autocovariance function is 0 at lags larger than $p$.

**To identify an $MA(p)$ look at a graph of an estimate $\widehat{C}(h)$ and look for a lag where it suddenly decreases to (nearly) 0.**

**Autoregressive Processes**: WLOG $\mu = 0$.

First do $p = 1$: $X_t = \rho X_{t-1} + \epsilon_t$. Then

$$\begin{aligned} C_X(h) &= \text{Cov}(X_t, X_{t+h}) \\ &= \text{Cov}(X_t, \rho X_{t+h-1} + \epsilon_{t+h}) \\ &= \rho \text{Cov}(X_t, X_{t+h-1}) + \text{Cov}(X_t, \epsilon_{t+h}) \end{aligned}$$

For $h > 0$ $\text{Cov}(X_t, \epsilon_{t+h}) = 0$. This gives

$$\begin{aligned} C_X(h) &= \rho C_X(h-1) \\ &= \rho^2 C_X(h-2) \\ &\vdots \\ &= \rho^h C_X(0) \end{aligned}$$

This gives

$$\rho_X(h) = \rho_X(1)^h = \rho^h$$

Also recall $C_X(0) = \sigma^2/(1-\rho^2)$.

Notice that $\rho_X(h)$ decreases geometrically to 0 but is never actually 0.

**Remark**: If $\rho$ is small so that $\rho^2$ is very small then an $AR(1)$ process is approximately the same as an $MA(1)$ process: we nearly have $X_t = \epsilon_t + \rho \epsilon_{t-1}$.

# Model identification

**Model identification** for time series $X$: select values of $p, q$ so that the $ARMA(p, q)$ process gives a reasonable fit to data.

Most important tool: plot of estimated autocorrelation function (ACF) of $X$.

Before we discuss doing this with real data we explore what plots of the ACF of various $ARMA(p, q)$ plots should look like (in the absence of estimation error).

For an $MA(p)$ process we found that

$$C_X(h) = \begin{cases} \sigma^2 \sum_{j=0}^{p-|h|} b_j b_{j+|h|} & |h| \le p \\ 0 & \text{otherwise} \end{cases}$$

Important *qualitative* feature: vanishes if $|h| > p$.

For an $AR(1)$ process $X_t - \mu = \rho(X_{t-1} - \mu) + \epsilon_t$ the autocorrelation function is

$$\rho_X(h) = \rho^{|h|}$$

qualitative feature: decreases geometrically.

To derive ACF for general $AR(p)$ we mimic the technique for $p = 1$.

If $X_t = \sum_1^p a_j X_{t-j} + \epsilon_t$ then

$$\begin{aligned} C_X(h) &= \text{Cov}(X_0, X_h) \\ &= \sum_{j=1}^{p} a_j \text{Cov}(X_0, X_{h-j}) + \text{Cov}(X_0, \epsilon_h) \\ &= \sum_{j=1}^{p} a_j C_X(h-j) \end{aligned}$$

for $h > 0$.

Divide through by $C_X(0)$.

Remember that $\rho_X(h) = C_X(h)/C_X(0)$ and $\rho_X(-k) = \rho_X(k)$: see that the above recursions for $h = 1, \ldots, p$ are $p$ linear equations in the $p$ unknowns $\rho_X(1), \ldots, \rho_X(p)$.

Called the Yule Walker equations.

For instance, when $p = 2$ we get

$$C_X(2) = a_1 C_X(1) + a_2 C_X(0)$$
$$C_X(1) = a_1 C_X(0) + a_2 C_X(-1)$$

which becomes, after division by $C_X(0)$

$$\rho_X(2) = a_1 \rho_X(1) + a_2$$
$$\rho_X(1) = a_1 + a_2 \rho_X(1)$$

Can use generating functions to get explicit formulas for the $\rho(h)$.

Here simply observe: two equations in two unknowns to solve.

The second equation shows that

$$\rho(1) = \frac{a_1}{1 - a_2}$$

Not possible if $a_2 = 1$ (unless $a_1 = 0$)

Not a correlation for some other $(a_1, a_2)$ pairs (see homework).

The first equation then gives

$$\rho(2) = \frac{a_1^2 + a_2(1-a_2)}{1-a_2}$$

Note: can calculate $\rho(h)$ recursively from $\rho(1)$ and $\rho(2)$ for $h \geq 3$ via Yule Walker.

Look at characteristic polynomial $\phi(x)$:

When $a_2 = 1$ we have

$$\phi(x) = 1 - a_1 x - x^2 = (1-\alpha_1 x)(1-\alpha_2 x)$$

where $1/\alpha_i, i = 1, 2$ are the two roots.

Multiplying out: $\alpha_1\alpha_2 = -1$ so either:

One of two has modulus more than 1 (root $1/\alpha_i$ has modulus less than 1) or

Both have modulus 1.

Both roots real so would be $\pm 1$.

Since $\alpha_1 + \alpha_2 = a_1$ (again from multiplying it out and examining the coefficient of $x$) we would then know $a_1 = 0$. In either case there is no stationary solution.

**Qualitative features**: can prove solutions of Yule-Walker equations decay to 0 at a geometric rate: $|\rho_X(h)| \le a^{|h|}$ for some $a \in (0,1)$. However, for general $p$ they are not too simple.

## Periodic Processes

If $Z_1, Z_2$ are iid $N(0, \sigma^2)$ then we saw

$$X_t = Z_1 \cos(\omega t) + Z_2 \sin(\omega t)$$

is strictly stationary, mean 0, autocorrelation $\rho(h) = \cos(\omega h)$: perfectly periodic.

## Linear Superposition

If $X$ and $Y$ are jointly stationary then $Z = aX + bY$ is stationary and

$$\begin{aligned} C_Z(h) = a^2 C_X(h) &+ b^2 C_Y(h) \\ &+ ab(C_{XY}(h) + C_{YX}(h)) \end{aligned}$$

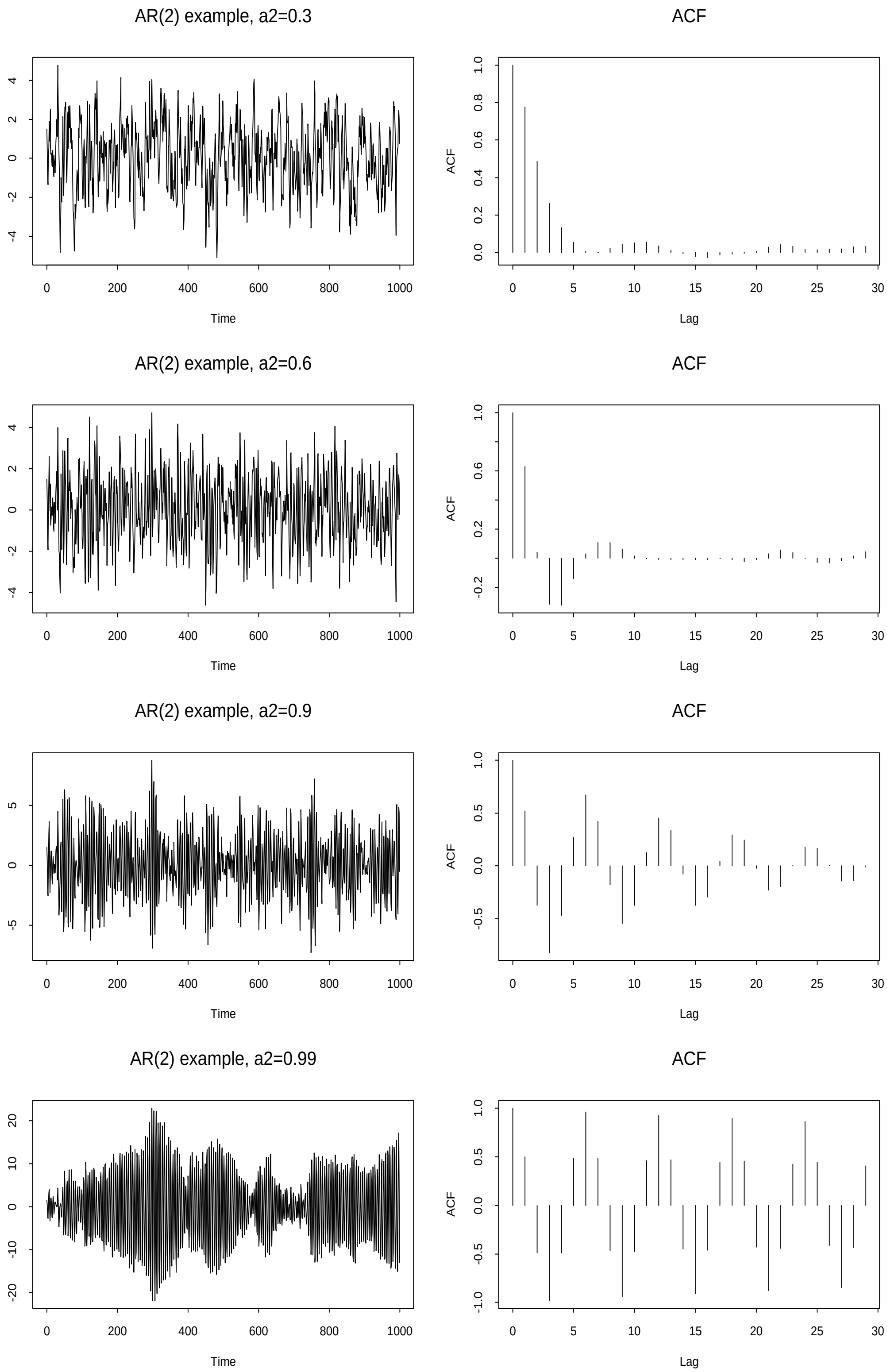
Could hope to recognize periodic component in series by looking for a periodic component in plotted ACF.

## Periodic versus AR processes

In fact you can make AR processes which behave very much like periodic processes. Consider the process

$$X_t = X_{t-1} - aX_{t-2} + \epsilon_t$$

Here are graphs of trajectories and autocorrelations for $a = 0.3, 0.6, 0.9$ and $0.99$.


AR(2) example, a2=0.3
ACF
AR(2) example, a2=0.6
ACF
AR(2) example, a2=0.9
ACF
AR(2) example, a2=0.99
ACF
Time
Lag

Observe: slow decay of waves in autocovariances, particularly for $a$ near 1.

When $a = 1$ characteristic polynomial is $1 - x + x^2$ which has roots

$$\frac{1 \pm \sqrt{-3}}{2}$$

Both these roots have modulus 1 so there is no stationary trajectory with $a = 1$. The point is that some $AR$ processes have nearly periodic components.

To get more insight consider the differential equation describing a sine wave:

$$\frac{d^2}{dx^2} f(x) = -\omega^2 f(x)\,;$$

solution is $f(x) = a \sin(\omega x + \phi)$. Replace derivative by differences: get approximation

$$\frac{d^2}{dx^2} f(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

so that

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \approx -\omega^2 f(x)$$

Take $h = 1$ and reorganize to get

$$f(x+1) = (2 - \omega^2) f(x) - f(x-1)$$

If we add noise, change notation to $t = x + 1$ and replace the letter $f$ by $X$ we get

$$X_t = (2 - \omega^2) X_{t-1} - X_{t-2} + \epsilon_t$$

Formalism only; no stationary solution exists.

But, $AR(2)$ processes are at least analogous to solutions of second order differential equations with added noise.

## Estimates of $C$ and $\rho$

In order to identify suitable $ARMA$ models using data we need estimates of $C$ and $\rho$. If $\mu = 0$ is known then

$$\begin{aligned} C_X(h) &= \text{Cov}(X_0, X_h) = \text{Cov}(X_1, X_{h+1}) = \cdots \\ &= \text{E}(X_0 X_h) = \text{E}(X_1 X_{h+1}) = \cdots . \end{aligned}$$

We would then be motivated to use

$$\widehat{C}(h) = \sum_{0}^{T-1-h} X_t X_{t+h}/T.$$

Average products over all pairs which are $h$ time units apart. When $\mu$ is unknown often simply use $\hat{\mu} = \bar{X}$; take

$$\widehat{C}(h) = \sum_{0}^{T-1-h} (X_t - \hat{\mu})(X_{t+h} - \hat{\mu})/T$$

Alternative: only $T - h$ terms in the sum

$$\widehat{C}(h) = \sum_{0}^{T-1-h} (X_t - \hat{\mu})(X_{t+h} - \hat{\mu})/(T - h).$$

so use

$$\hat{\rho}(h) = \frac{\widehat{C}(h)}{\widehat{C}(0)}.$$

(Note, however, that when $T - h$ is used in the divisor it is technically possible to get a $\hat{\rho}$ value which exceeds 1.)