Non stationary processes

If X is not stationary: transform X to find related stationary series.

In this course: two sorts of non-stationarity — non constant mean and integration.

Non constant mean: If $\mathrm{E}(X_t)$ is not constant we will hope to model $\mu_t = \mathrm{E}(X_t)$ using a small number of parameters and then model $Y_t = X_t - \mu_t$ as a stationary series.

Three common structures for μ_t: linear, polynomial and periodic.

Linear trend: Suppose

$$\mu_t = \alpha + \beta t$$

We seek to estimate α and β in order to analyze $Y_t = X_t - \alpha - \beta t$.

Method 1: regression (detrending). Regress X_t on t to get $\hat{\alpha}$ and $\hat{\beta}$ and analyze

$$\hat{Y}_t = X_t - \hat{\alpha} - \hat{\beta} t$$

We hope

$$\hat{Y} \approx Y$$

Method 2: differencing. Define

$$\begin{aligned} W_t &= X_t - X_{t-1} \\ &= [(I - B)X]_t \end{aligned}$$

Then

$$\begin{aligned} W_t &= [(\alpha + \beta t + Y_t) - (\alpha + \beta(t-1) + Y_{t-1})] \\ &= \beta + Y_t - Y_{t-1} \end{aligned}$$

Linear filter applied to series Y so stationary if Y is.

BUT, it might be stationary even if Y is not.

Suppose that ϵ_t is an iid mean 0 sequence. Then

$$Y_t = \sum_1^t \epsilon_j$$

is a **random walk**.

Notice that $Y_t - Y_{t-1} = \epsilon_t$ is stationary.

To see that Y_t, the random walk, is not stationary notice that $\text{Var}(Y_t) = t\text{Var}(\epsilon_1)$.

Random walk models common in Economics.

In physics: used in limit of very small time increments – leads to Brownian motion.

Defn: X satisfies ARIMA(p, d, q) model if

$$\phi(B)\nabla^d X = \psi(B)\epsilon$$

where

1. ϵ is white noise,

2. ϕ is a polynomial of degree p,

3. ψ is a polynomial of degree q,

4. $\nabla = I - B$ is the differencing operator.

Remark: If $X_t = \mu_t + Y_t$ where Y is stationary and μ is a polynomial of degree less than or equal to d then $\nabla^d X$ is stationary. (So a cubic shaped trend could be removed by differencing 3 times.)

WARNING: it is a common mistake in students' data analyses to over difference. When you difference a stationary $ARMA(p,q)$ you introduce a unit root in the defining polynomial – the result cannot be written as an infinite order moving average.

Detrending: Define a response vector

$$U = \begin{bmatrix} X_0 \\ \vdots \\ X_{T-1} \end{bmatrix}$$

and a design matrix by

$$V = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & T-1 \end{bmatrix}$$

Write

$$U = V\theta + Y$$

with

$$Y = \left[\begin{array}{c} Y_0 \\ \cdots \\ Y_{T-1} \end{array} \right]$$

and

$$\theta = \left[\begin{array}{c} \alpha \\ \beta \end{array} \right].$$

Then we estimate by

$$\hat{\theta} = (V^T V)^{-1} V^T U$$

Our detrended series is

$$\begin{aligned} \hat{Y} &= U - V\hat{\theta} \\ &= (I - V(V^T V)^{-1} V^T)U \end{aligned}$$

Since Cov$(Y) \neq \sigma^2 I$ ordinary least squares is not strictly appropriate.

In general in the model

$$U = V\theta + W$$

where W has mean 0 and variance covariance matrix Σ, the minimum variance linear estimator of θ is the generalized least squares estimate

$$\hat{\theta}_{GLS} = (V^T\Sigma^{-1}V)^{-1}V^T\Sigma^{-1}U\,.$$

This estimate is unbiased and has variance

$$(V^T\Sigma^{-1}V)^{-1}$$

For this model the ordinary least squares estimate is also unbiased and has variance

$$(V^TV)^{-1}V^T\Sigma V(V^TV)^{-1}$$

Problem in our context (almost always a problem): can only use $\hat{\theta}_{GLS}$ if you know Σ.

In our context you won't know Σ until you have removed a trend, selected a suitable $ARMA$ model and estimated the parameters.

Natural proposal: follow an iterative process:

1. Fit linear model by ordinary least squares.
2. Compute the residual series.
3. Identify a suitable $ARMA(p, q)$ model.
4. Fit the parameters of this model.
5. Using the estimates compute an estimated variance covariance of the residual process.
6. Refit linear model in 1 using Generalized Least Squares — covariance as estimated in 5, then go back to 2.

Repeat until estimates stop changing.

Folklore: There is evidence that the OLS estimator has a variance which is not too much different from GLS in common ARMA models.