Spectra of Some Basic Processes

First method: direct.

More powerful technique: indirect.

Direct from the definition

White Noise: Since $C_\epsilon(k) = 0$ for all $k \neq 0$:

$$f_\epsilon(\omega) \equiv C_\epsilon(0) = \sigma_\epsilon^2 .$$

MA(1): For $X_t = \epsilon_t - b\epsilon_{t-1}$:

Have $C_X(0) = \sigma^2(1+b^2)$, $C_X(\pm 1) = -b\sigma^2$ so

$$\begin{aligned} f_X(\omega) &= \sigma^2(1+b^2) \\ &\qquad -b\sigma^2(\exp(2\pi\omega i) + \exp(-2\pi\omega i)) \\ &= \sigma^2(1+b^2) - 2b\sigma^2\cos(2\pi\omega). \end{aligned}$$

AR(1): We have $C_X(k) = \rho^{|k|} C_X(0)$ and

$$
\begin{aligned}
&f_X(\omega)\\
=&C_X(0)\{1 + \sum_{k>0} \rho^k (e^{2\pi\omega k i} + e^{-2\pi\omega k i})\}\\
=&C_X(0)\{1\\
&\quad + \rho\left(\frac{e^{2\pi\omega i}}{1-\rho e^{2\pi\omega i}} + \frac{e^{-2\pi\omega i}}{1-\rho e^{-2\pi\omega i}}\right)\}\\
=&C_X(0)\left\{1 + \rho\frac{e^{2\pi\omega i} - \rho + e^{-2\pi\omega i} - \rho}{(1-\rho e^{2\pi\omega i})(1-\rho e^{-2\pi\omega i})}\right\}\\
=&C_X(0)\left[1 + \rho\frac{2\{\cos(2\pi\omega) - \rho\}}{1+\rho^2 - 2\rho\cos(2\pi\omega)}\right]\\
=&C_X(0)\frac{1-\rho^2}{1+\rho^2 - 2\rho\cos(2\pi\omega)}\\
=&\frac{\sigma_\epsilon^2}{1+\rho^2 - 2\rho\cos(2\pi\omega)}
\end{aligned}
$$

Using filters

Write mean 0 ARMA(p, q) process in MA form:

$$X_t = \sum_{s=0}^{\infty} a_s \epsilon_{t-s}.$$

Covariance of X is

$$C_X(h) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} a_r a_s \text{Cov}(\epsilon_{t+h-r}, \epsilon_{t-s})$$

Covariance simplifies for white noise

But in general $C(t+h-r-(t-s)) = C(h+s-r)$ is covariance in double sum.

Plug double sum into definition of f_X to get

$$f_X(\omega) = \sum_{h=-\infty}^{\infty} e^{2\pi\omega hi} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} a_r a_s C(h+s-r)$$

Write h in exponential in form $(h+s-r)+r-s$.

Bring sum over h to the inside to get

$$f_X(\omega) = \sum_{r=0}^{\infty} a_r e^{2\pi\omega r i} \sum_{s=0}^{\infty} a_s e^{-2\pi\omega s i}$$
$$\sum_{h=-\infty}^{\infty} \exp(2\pi\omega(h+s-r)i) C(h+s-r)$$

Substite $k = h+s-r$ in inside sum and define

$$A(\omega) = \sum_{r=0}^{\infty} a_r e^{2\pi\omega r i}$$

to see that

$$f_X(\omega) = A(\omega)\overline{A(\omega)} f_\epsilon(\omega)$$

or

$$f_X(\omega) = |A(\omega)|^2 f_\epsilon(\omega)\,.$$

Frequency response function: A (or $\bar{A}$).

Power transfer function: $|A|^2$

Gain is sometimes used for $|A|$.

The Spectrum of an ARMA(p, q)

An ARMA(p, q) process X satisfies

$$\sum_{s=0}^{p} a_s X_{t-s} = \sum_{r=0}^{q} b_r \epsilon_{t-r}$$

so that if Y is the process $\sum_{s=0}^{p} a_s X_{t-s}$ then

$$f_Y(\omega) = |A(\omega)|^2 f_X(\omega)$$

where $A(\omega) = \sum_{s=0}^{p} a_s \exp(2\pi\omega s i)$.

Also $Y_t = \sum_{r=0}^{q} b_r \epsilon_{t-r}$ so

$$f_Y(\omega) = |B(\omega)|^2 f_\epsilon(\omega)$$

where $B(\omega) = \sum_{s=0}^{q} b_s \exp(2\pi\omega s i)$. Hence

$$f_X(\omega) = \frac{|B(\omega)|^2}{|A(\omega)|^2} \sigma_\epsilon^2 .$$

Example: ARMA(1,1)

$$X_t - aX_{t-1} = \epsilon_t - b\epsilon_{t-1}$$

From MA(1) calculation:

$$|B(\omega)|^2 = 1 + b^2 - 2b\cos(2\pi\omega)$$

for filter on MA side and

$$|A(\omega)|^2 = 1 + a^2 - 2a\cos(2\pi\omega)$$

for filter on AR side.

So

$$f_X(\omega) = \frac{1 + b^2 - 2b\cos(2\pi\omega)}{1 + a^2 - 2a\cos(2\pi\omega)}\sigma_\epsilon^2.$$