Stationarity

Goal: find assumptions on a discrete time process which will permit us to make reasonable estimates of the parameters.

Intuition: need some notion of replication.

Defn: Stochastic process $X_t; t = 0, \pm 1, \ldots$ is *stationary* if joint distribution of $X_t, \cdots, X_{t+k}$ is same as joint distribution of $X_0, \cdots, X_k$ for all t and all k. (Often we call this *strictly* stationary.)

Defn: Stochastic process $X_t; t = 0, \pm 1, \ldots$ is *weakly* (or *second order*) stationary if

$$\mathrm{E}(X_t) \equiv \mu$$

for all t (that is the mean does not depend on t) and

$$\mathrm{Cov}(X_t, X_{t+h}) = \mathrm{Cov}(X_0, X_h) \equiv C_X(h)$$

is a function of h only (and does not depend on t).

Remark:

1. X finite variance, strictly stationary implies X weakly stationary.

2. X second order stationary and Gaussian implies X strictly stationary.

Defn: X is *Gaussian* if for each $t_1, \ldots, t_k$ the vector $(X_{t_1}, \ldots, X_{t_k})'$ has a Multivariate Normal Distribution

Defn: The process X has a **stationary** covariance if:

$$\begin{aligned}\text{Cov}(X_t, X_s) &= \text{Cov}(X_{t+1}, X_{s+1}) \\ &= \text{Cov}(X_{t+2}, X_{s+2}) \cdots\end{aligned}$$

If so then for all t and h we find

$$\begin{aligned}\text{Cov}(X_t, X_{t+h}) &= \text{Cov}(X_0, X_h) \\ &\equiv C_X(h)\end{aligned}$$

Call C_X autocovariance function of X.

Notice: $\mathbf{\Sigma}$ has

$C(0)$ down the diagonal

$C(1)$ down the first sub and super diagonals

$C(2)$ down the next sub and super diagonals and so on.

Such a matrix is called a Toeplitz matrix.